

Graphs



EECS3101 E:
Design and Analysis of Algorithms
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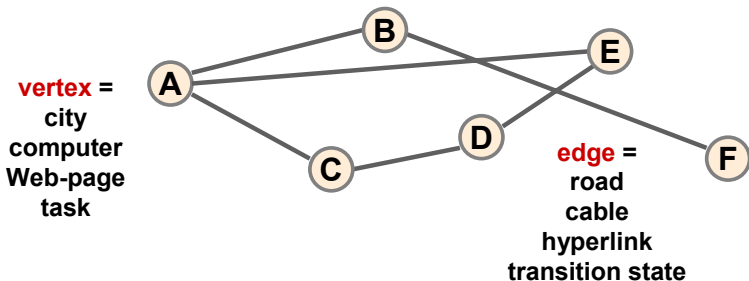
Learning Outcomes of this Lecture

This module is designed to help you understand:

- Vocabulary of the **Graph** ADT
- **Properties** of Graphs
- **Algorithms** on Graphs
 - Traversals: **Depth-First Search** vs. **Breadth-First Search**
 - Topological Sort
 - Minimum Spanning Trees (MST)
 - Dijkstra's Shortest Path Algorithm
- **Proving** Properties of Graphs
- **Implementing** Graphs in Java

Graphs: Definition

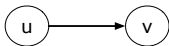
A **graph** $G = (V, E)$ represents *relations* that exist between pairs of objects.



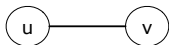
- A set V of **objects: vertices (nodes)**
- A set E of **connections** between objects: **edges (arcs)**
 - Each **edge** (from E) is an **ordered pair** of **vertices** (from V).
- e.g., $G = (\{A, B, C, D, E, F\}, \{(A, B), (A, C), (A, E), (C, D), (D, E), (B, F)\})$

Directed vs. Undirected Edges

- An **edge** (u, v) connects two **vertices** u and v in the graph.
- Edge** (u, v) is **directed** if it indicates the direction of travel.



- Vertex u is the **origin**.
 - Vertex v is the **destination**.
 - $(u, v) \neq (v, u)$
- Edge** (u, v) is **undirected** if it does not indicate a direction.



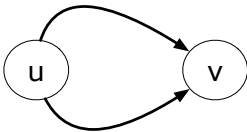
- $(u, v) = (v, u)$
- 1 undirected edge $(u, v) \equiv$ 2 directed edges (u, v) and (v, u) .
- Directions** of **edges** represent dependency, order, or flow.

Self vs. Parallel Edges

- An edge (u, u) , either directed or undirected, is called a **self-edge** (or a **self-loop**).



- Edges that have the same two end vertices are **parallel edges** or **multiple edges**.



e.g., In a flight network graph, there are more than one airlines flying between two Seoul and Vancouver.

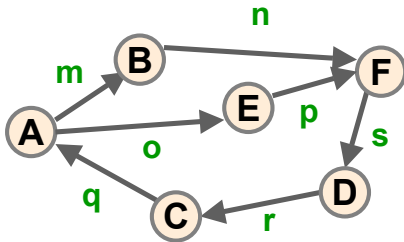
- A **simple graph** has no **self-loops** and **parallel edges**.

Vertices

Given an **edge** (u, v) :

- Vertices u and v are its two **End vertices** (**Endpoints**).
- The two end vertices u and v is said to be **adjacent**.
- Edge (u, v) is **incident on** the two end vertices u and v .
- When edge (u, v) is directed:
 - u is **origin** and v is **destination**
 - Edge (u, v) is an **outgoing edge** of the origin u
 - Edge (u, v) is an **incoming edge** of the destination v
- The **degree** of a vertex v is the number of edges **incident on** v .

Exercise (1)

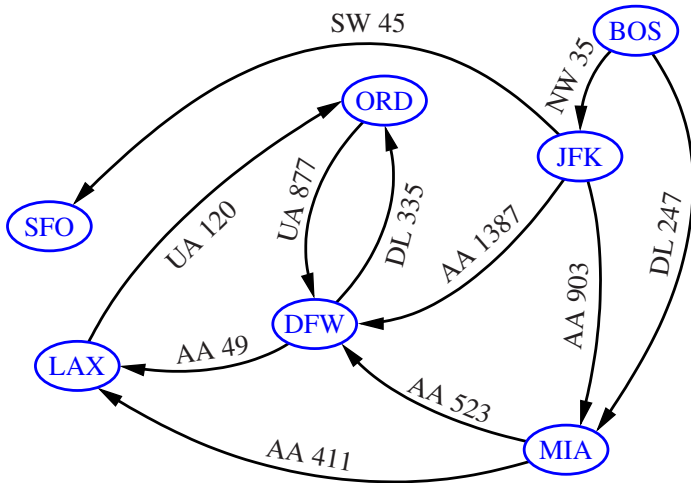


- *End vertices* of edge *m*? [A, B]
- *Outgoing edges* of vertex *A*? [m, o]
- *Incoming edges* of vertex *A*? [q]
- Edges *incident* on vertex *A*? [m, o, q]
- *Degree* of vertex *A*? [3]

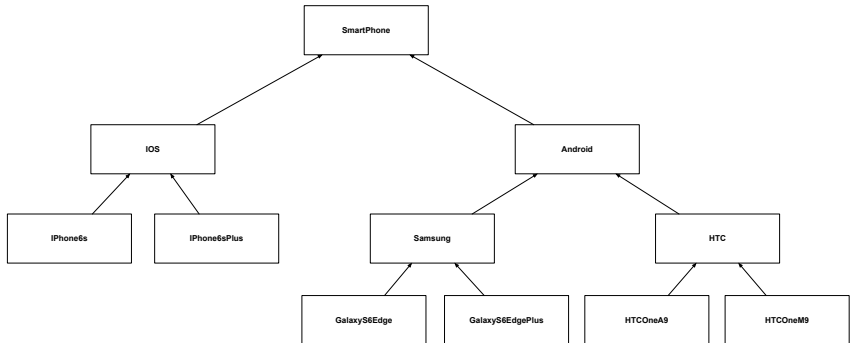
Directed vs. Undirected Graphs

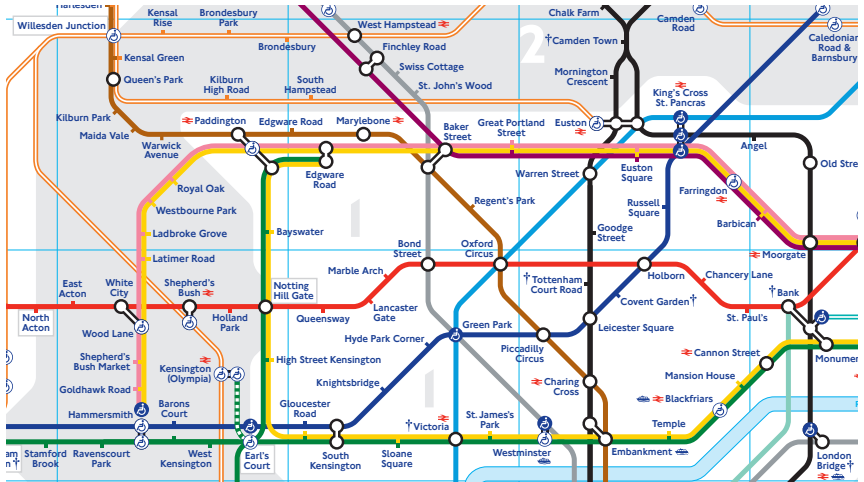
- In a **directed graph**, **all** edges are directed.
e.g., dependency graphs (inheritance relationships, method calls, *etc.*)
- In an **undirected graph**, all edges are undirected.
e.g., Subway map of Young-University Line
- In a **mixed graph**, **some** edges directed; **some** undirected.
e.g., A city map has street intersections as vertices and streets as edges: each street may be one-way (a directed edge) or both-way (an undirected edge).

Directed Graph Example (1): A Flight Network

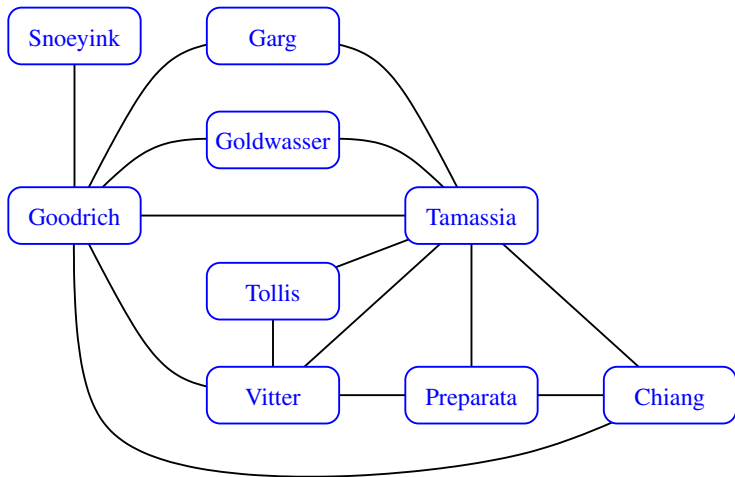


Directed Graph Example (2): Class Inheritance





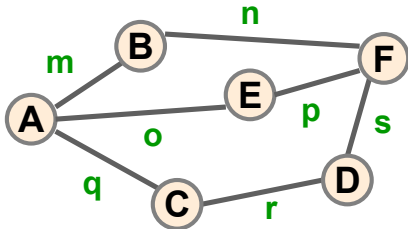
Undirected Graph Example (2): Co-authorship



Basic Properties of Graphs (1)

- Given a simple, undirected graph $G = (V, E)$ with $|E| = m$:

$$\sum_{v \in V} \text{degree}(v) = 2 \cdot m$$

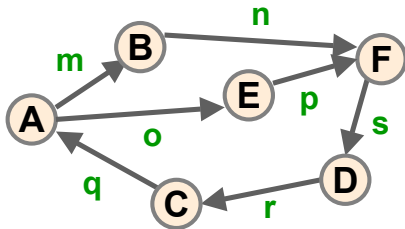


- Intuition:** Each edge (u, v) contributes to degrees of both u and v .
 - Formal Proof:** *Mathematical induction* on $|V|$.
- Prove that the claim still holds on graphs that are not simple.

Basic Properties of Graphs (2)

- Given a **simple, directed** graph $G = (V, E)$ with $|E| = m$:

$$\sum_{v \in V} \text{in-degree}(v) = \sum_{v \in V} \text{out-degree}(v)$$

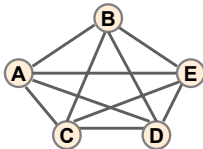


- Intuition:** Each directed edge (u, v) contributes to the out-degree of origin u and the in-degree of destination v .
 - Formal Proof:** *Mathematical induction* on $|V|$.
- Prove that the claim still holds on graphs that are **not simple**.

Basic Properties of Graphs (3)

- Given a simple, undirected graph $G = (V, E)$, $|V| = n$, $|E| = m$:

$$m \leq \frac{n \cdot (n - 1)}{2}$$



- Intuition:** Say $V = \{v_1, v_2, \dots, v_n\}$
 - Maximum** value of m is obtained when each vertex is connected to all other $n - 1$ vertices: $n \cdot (n - 1)$
 - Since G is undirected, for each pair of vertices v_i and v_j , we have double-counted (v_i, v_j) and (v_j, v_i) : $\frac{n \cdot (n - 1)}{2}$
- G is a **complete graph** when $m = \frac{n \cdot (n - 1)}{2}$

Paths and Cycles (1)

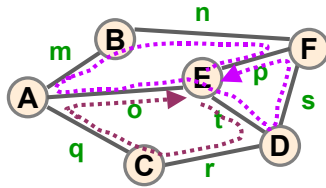
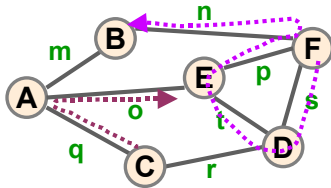
Given a graph $G = (V, E)$:

- A **path** of G is a sequence of **alternating** vertices and edges, which **starts** and **ends** at vertices:

$$\langle v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n \rangle \quad v_i \in V, 1 \leq i \leq n, e_j \in E, 1 \leq j < n$$

- A **cycle** of G is a **path** of G with the **same** vertex appearing at its **both** ends (and the rest of it **may** also contain duplicates).
- A **simple path** of G is a **path** of G with **distinct** vertices.
- A **simple cycle** of G is a **cycle** of G with **distinct** vertices (except the **beginning** and **end** vertices that form the cycle).
- Given two vertices u and v in G , vertex v is **reachable** from vertex u if there exists a **path** of G such that its **start vertex** is u and **end vertex** is v .
 - Vertex v may be reachable from vertex u via more than one paths.
 - Any of the **reachable paths** from u to v contains a cycle $\Rightarrow$ An **infinite** number of reachable paths from u to v .

Paths and Cycles (2)



Path = (F, s, D, t, E, p, F, n, B) Cycle = (E, p, F, n, B, m, A, o, E, t, D, s, F, p, E)

Simple Path = (C, q, A, o, E) Simple Cycle = (E, t, D, r, C, q, A, o, E)

Vertex *F* is **reachable** from vertex *A* via:

- (A, m, B, n, F)
- (A, o, E, p, F)
- (A, o, E, t, D, s, F)

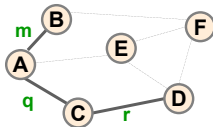
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Subgraphs vs. Spanning Subgraphs

Given a graph $G = (V, E)$:

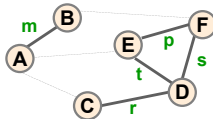
- A **subgraph** of G is another graph $G' = (V', E')$ such that $V' \subseteq V$ and that $E' \subseteq E$.

e.g., $G_1 = (\{A, B, C, D, E, F\}, \{m, q, r\})$



- A **spanning subgraph** of G is another graph $G' = (V', E')$ s.t. $V' = V$ and that $E' \subseteq E$.

e.g., $G_2 = (\{A, B, C, D, E, F\}, \{m, p, s, t, r\})$

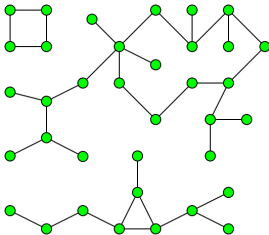


Connected Graph vs. Connected Components

Given a graph $G = (V, E)$:

- G is **connected**: there is a **path** between any two vertices of G .
e.g., Spanning subgraph G_2 extended with the edge n , o , or q
- G 's **connected components**: G 's maximal **connected subgraphs**.

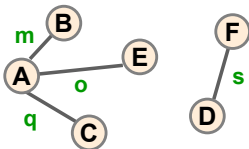
A **CC** is maximal in that it **cannot** be expanded any further.
e.g., How many **connected components** does the following graph have?



Answer: 3

Forests vs. Trees

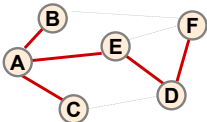
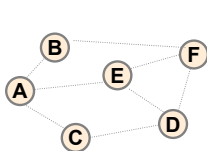
- A **forest** is an **undirected** graph without **cycles**.



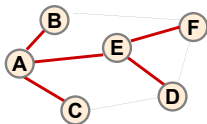
- Acyclic**:
Any two **vertices** are connected via at most one **path**.
- A **forest** may or may not be **connected**.
 $(\exists v_1, v_2 \bullet \{v_1, v_2\} \subseteq V \wedge \neg \text{connected}(v_1, v_2)) \Rightarrow \neg \text{connected}(\text{Forest } G)$
- A **tree** is a connected **forest**.
 - Acyclic & Connected**:
Any two **vertices** are connected via exactly one path.
 - e.g., Add either edge (E, F) or (E, D) to the above forest.

Spanning Trees

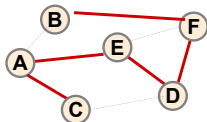
- A **spanning tree** of graph G : a **spanning subgraph** that is also a **tree**
 - $\Rightarrow$ A **spanning tree** of G is a connected **spanning subgraph** of G that contains no cycles.
 - $\Rightarrow \neg \text{connected}(G) \Rightarrow \neg(\exists G' \bullet G' \text{ is a spanning tree of } G)$



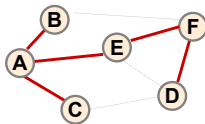
Spanning Tree 3



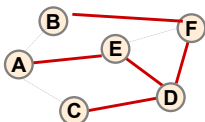
Spanning Tree 1



Spanning Tree 4



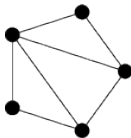
Spanning Tree 2



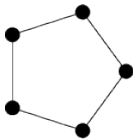
Spanning Tree 5

Exercise (2)

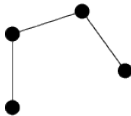
Given a graph



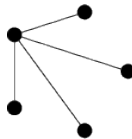
Which one of the following is a *spanning tree*?



(a)



(b)



(c)

- (a): *spanning subgraph* containing a *cycle* ($\therefore$ not a *tree*).
- (b): *tree* but not *spanning*.

Basic Properties of Graphs (4)

Given $G = (V, E)$ an **undirected** graph with $|V| = n$, $|E| = m$:

$$\begin{cases} m = n - 1 & \text{if } G \text{ is a } \textit{spanning tree} \\ m \leq n - 1 & \text{if } G \text{ is a } \textit{forest} \\ m \geq n - 1 & \text{if } G \text{ is } \textit{connected} \\ m \geq n & \text{if } G \text{ contains a } \textit{cycle} \end{cases}$$

- Prove the **spanning tree** case via **mathematical induction on n** :
 - **Base Cases**: $n = 1 \Rightarrow m = 0$, $n = 2 \Rightarrow m = 1$, $n = 3 \Rightarrow m = 2$
 - **Inductive Cases**: Assume a spanning tree with $|V| = n$ and $|E| = n - 1$.
 - When adding a new vertex v' into the existing graph, we may only expand the **existing spanning tree** by connecting v' to **exactly one** of the existing vertices; otherwise there will be a **cycle**.
 - This makes the new spanning tree contains $n + 1$ vertices and n edges.
- When G is a **forest**, it may be:
 - **unconnected** $\Rightarrow m < n - 1$
 - **connected & acyclic** $\Rightarrow m = n - 1$
- When G is **connected**, it may be:
 - **acyclic** $\Rightarrow m = n - 1$
 - **cyclic** $\Rightarrow m > n - 1$

Graph Traversals: Definition

Given a graph $G = (V, E)$:

- A **traversal** of G is a systematic procedure for examining all its vertices V and edges E .
- A **traversal** of G is considered **efficient** if its **running time** is **linear** on $|V|$ and/or $|E|$.
[e.g., $O(|V| + |E|)$]

Graph Traversals: Applications

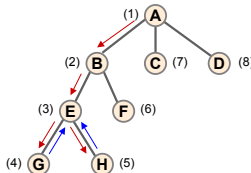
Fundamental questions about graphs involve *reachability*.

Given a graph $G = (V, E)$ (directed or undirected):

- Given a vertex u , find all other vertices in G *reachable* from u .
- Given a vertex u and a vertex v :
 - compute a *path* from u to v , or report that there is no such a path.
 - compute a *path* from u to v that involves the *minimum* number of edges, or report that there is no such a path.
- Determine whether or not G is *connected*.
- Given that G is *connected*, compute a *spanning tree* of G .
- Compute the *connected components* of G .
- Identify a *cycle* in G , or report that G is *acyclic*.

Depth-First Search (DFS)

- A **Depth-First Search (DFS)** of graph $G = (V, E)$, starting from some vertex $v \in V$, proceeds along a **path** from v .
 - The **path** is constructed by following an incident edge.
 - The **path** is extended **as far as possible**, until all incident edges lead to vertices that have already been **visited**.
 - Once the **path** originated from v cannot be extended further, **backtrack** to the **latest** vertex whose **incident edges** lead to some **unvisited** vertices.



- DFS resembles the **preorder traversal** in trees.
- Use a **LIFO stack** to keep track of the nodes to be visited.
- **Contrast**: DFS on a undirected vs. directed graph.

DFS on Undirected Graph: Marking Vertices and Edges

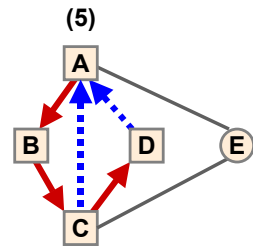
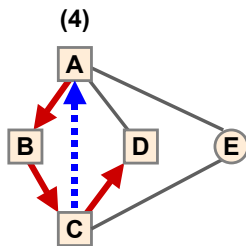
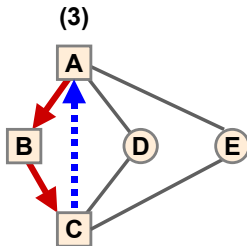
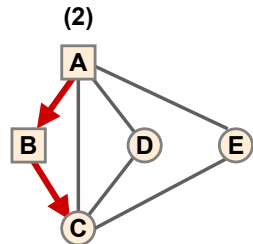
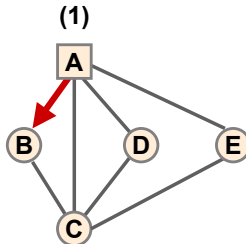
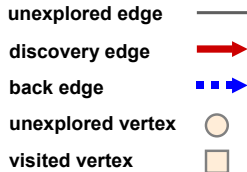
Before the **DFS** on a undirected graph starts:

- All vertices are **unvisited**.
- All edges are **unexplored/unmarked**.

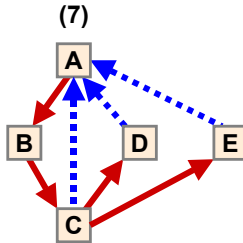
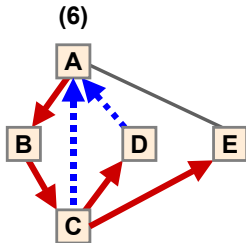
Over the course of a **DFS**, we mark vertices and edges:

- A vertex v is marked **visited** when it is first encountered.
- Then, we iterate through each of v 's **incident edges**, say e :
 - If edge e is already marked, then skip it.
 - Otherwise, mark edge e as:
 - A **discovery** edge if it leads to an **unvisited** vertex
 - A **back** edge if it leads to a vertex on the current DFS path (i.e., an **ancestor** that's currently on the DFS **stack**)

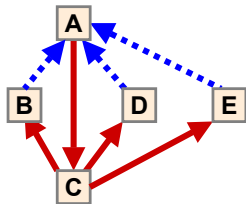
DFS on Undirected Graph: Illustration (1.1)



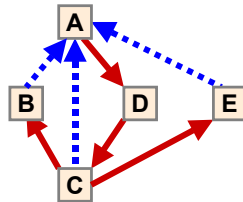
DFS on Undirected Graph: Illustration (1.2)



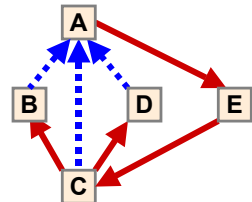
Other solutions (different *incident edges* on vertex *A* to get started):



edge AC visited first

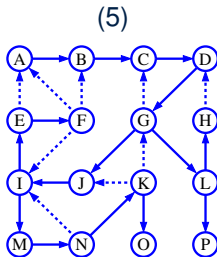
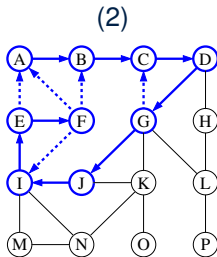


edge AD visited first



edge AE visited first

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DFS on a Directed Graph: Notations (1)

- A **DFS run** refers to a single execution/application of **DFS**.
 - ≥ 1 **DFS runs** (starting from different vertices) are needed to visit all vertices.
- Let u be a vertex discovered in the current DFS run:
 - $T(u)$: **DFS tree** formed by the identified DISCOVERY/TREE edges
 - $r(u)$: the **start vertex** of the current DFS run, and the **root** of $T(u)$
 - v is u 's **ancestor** if v is on u 's **ancestor path**, the unique path from u to $r(u)$.
 - v is a **proper ancestor** of u if $v \neq u$.
 - w is a **descendant** of u if u is w 's **ancestor**: w lies in the **subtree** rooted at u .
 - w is a **proper descendant** of u if $w \neq u$.
- 3 possible **colours** for each **vertex**:
 -  WHITE [unvisited/undiscovered]
 - A vertex is white until it is first **pushed** to the **DFS stack**.
 -  GREY [visited/discovered, unfished]
 - A vertex becomes grey when it is first **pushed** to the **DFS stack**.
 - A vertex remains grey while it is still in the **DFS stack** and has some of its **outgoing** edges not yet explored/processed.
 -  BLACK [visited/discovered, finished]
 - A vertex becomes black when it is **popped** from the **DFS stack** and has all its **outgoing** edges explored/processed.

DFS on a Directed Graph: Notations (2)

- 2 **timestamps** for each **vertex** u :
 - DISCOVERY time: $d(u)$ [when u becomes grey]
 - FINISH time: $f(u)$ [when u becomes black]
 - **Requirement**: $d(u) < f(u)$
- 4 possible **categories** for each **edge** (u, v) : [u is grey when examined]
 - DISCOVERY/TREE edge: v is white.
 - BACK edge: v is grey, and v is u 's **proper ancestor** in $T(u)$.
 - FORWARD edge: v is black, and v is u 's **proper descendant** in $T(u)$.
 - CROSS edge: v is black, and one of the following holds:
 - $v \in T(u)$: v is neither ancestor nor descendant of u in $T(u)$.
 - $T(v) \neq T(u)$: v is in a **different**, previously **finished** **DFS tree**.

DFS on a Directed Graph: Exercises

- Can a vertex v 's **colour** decide whether $d(v)$ or $f(v)$ is assigned yet?
 - WHITE $\Rightarrow$ both $d(v)$ and $f(v)$ undefined
 - GREY $\Rightarrow$ $d(v)$ defined but $f(v)$ undefined
 - BLACK $\Rightarrow$ both $d(v)$ and $f(v)$ defined
- For every vertex v that's **black**, what must be true for $d(v)$ and $f(v)$?

$$d(v) < f(v)$$
- When **categorizing** an edge (u, v) , **colours** of u and v ? Are they related?
 - In all cases, u is GREY.
 - TREE/DISCOVERY edge: v is WHITE
 - BACK edge: v is GREY [$\equiv v$ is an ancestor of u]
 - FORWARD edge: v is BLACK $\wedge v$ is a proper descendant of u
 - CROSS edge: v is BLACK $\wedge v$ is neither an ancestor nor a descendant of u
- Decide if u is an ancestor, a descendant, or neither, of v via d and f .
 - v is a descendant of $u \iff (d(u) < d(v) \wedge f(v) < f(u))$
 - v is an ancestor of $u \iff (d(v) < d(u) \wedge f(u) < f(v))$
 - v is neither an ancestor nor a descendant of $u \iff (f(u) < d(v) \vee f(v) < d(u))$

DFS: Properties

1. Running Time?

- Every **vertex** is set as **visited** at most once.
- Each **edge** is set as either **DISCOVERY** or **BACK** at most once.

$$\Rightarrow O(m + n)$$

2. For a **DFS** starting from vertex u in a graph $G = (V, E)$:

- 2.1 $|visited\ nodes| = |V| \Rightarrow G$ is **connected**
- 2.2 $|visited\ nodes| < |V| \Rightarrow G$ has > 1 **connected components**
- 2.3 There are no **back edges** $\Rightarrow G$ is **acyclic**

3. For a **DFS** starting from vertex u in an undirected graph G :

- 3.1 The traversal visits all nodes in the **connected component** containing u .
- 3.2 **Discovery edges** form a **spanning tree** (with $|V| - 1$ edges) of the **connected component** containing u .

4. If a graph G is not **connected**, then it takes multiple runs of **DFS** to identify all G 's **connected components**.

Graph Questions: Adapting DFS

- Given a (directed or undirected) graph $G = (V, E)$:
 - Find a **path** between vertex u and vertex v .
Start a DFS from u and stop as soon as v is encountered.
 - Is vertex v **reachable** from vertex u ?
No if a DFS starting from u never encounters v .
 - Find all **connected components** of G .
 - Continuously apply **DFS**'s until the entire set V is visited.
 - Each **DFS** produces a **subgraph** representing a new **CC**.
 - Given that G is **connected**, find a **spanning tree** of it.
 G is **connected**. $\Rightarrow G$'s only **CC** is its **spanning tree**.
- Given an undirected graph $G = (V, E)$:
 - Is G **connected**?
 - Start a **DFS** from an arbitrary vertex, and count # of visited nodes.
 - When the traversal completes, compare the counter value against $|V|$.
 - Is G **acyclic**?
 - Start a **DFS** from an arbitrary vertex.
 - Return **no** (i.e., a **cycle** exists) as soon as a **back edge** is found.

Graphs in Java: DL Node and List

For each graph, maintain two *doubly-linked lists* for *vertices* and *edges*.

```
public class DLNode<E> { /* Doubly-Linked Node */
    private E element;
    private DLNode<E> prev; private DLNode<E> next;
    public DLNode(E e, DLNode<E> p, DLNode<E> n) { ... }
    /* setters and getters for prev and next */
}
```

```
public class DoublyLinkedList<E> {
    private int size;
    private DLNode<E> header; private DLNode<E> trailer;
    public void remove (DLNode<E> node) {
        DLNode<E> pred = node.getPrev();
        DLNode<E> succ = node.getSucc();
        pred.setNext(succ); succ.setPrev(pred);
        node.setNext(null); node.setPrev(null);
        size --;
    }
}
```

Graphs in Java: Vertex and Edge

```
public abstract class Vertex<V> {  
    private V element;  
    public Vertex(V element) { this.element = element; }  
    /* setter and getter for element */  
}
```

```
public abstract class Edge<E, V> {  
    private E element;  
    private Vertex<V> origin;  
    private Vertex<V> destination;  
    public Edge(E element) { this.element = element; }  
    /* setters and getters for element, origin, and destination */  
}
```

Graphs in Java: Interface (1)

```
public interface Graph<V,E> {  
    /* Number of vertices of the graph */  
    public int getNumberOfVertices();  
  
    /* Number of edges of the graph */  
    public int getNumberOfEdges();  
  
    /* Vertices of the graph */  
    public Iterable<EdgeListVertex<V>> getVertices();  
  
    /** Edges of the graph */  
    public Iterable<EdgeListEdge<E, V>> getEdges();  
  
    /* Number of edges leaving vertex v. */  
    public int getOutDegreeOf(EdgeListVertex<V> v);  
  
    /* Number of edges for which vertex v is the destination. */  
    public int getInDegreeOf(EdgeListVertex<V> v);  
  
    public int getDegreeOf(EdgeListVertex<V> v);  
}
```

Graphs in Java: Interface (2)

```
/* Edges for which vertex v is the origin. */  
public Iterable<Edge<E, V>> getOutgoingEdgesOf(Vertex<V> v);  
  
/* Edges for which vertex v is the destination. */  
public Iterable<Edge<E, V>> getIncomingEdgesOf(Vertex<V> v);  
  
/* The edge from u to v, or null if they are not adjacent. */  
public Edge<E, V> getEdgeBetween(Vertex<V> u, Vertex<V> v);
```

Graphs in Java: Interface (3)

```
/* Inserts a new vertex, storing given element. */  
public Vertex<V> addVertex(V element);  
  
/* Inserts a new edge between vertices u and v,  
 * storing given element.  
 */  
public Edge<E, V> addEdge(Vertex<V> u, Vertex<V> v, E element);  
  
/* Removes a vertex and all its incident edges from the graph. */  
public void removeVertex(Vertex<V> v);  
  
/* Removes an edge from the graph. */  
public void removeEdge(Edge<E, V> e);  
} /* end Graph */
```

Graphs in Java: Edge List (1)

Each **vertex** or **edge** stores a **reference** to its **position** in the respective vertex or edge list.

⇒ **$O(1)$ deletion** of the vertex or edge from the list.

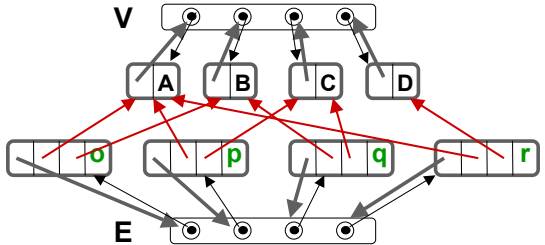
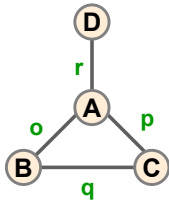
```
public class EdgeListVertex<V> extends Vertex<V> {  
    private DLNode<Vertex<V>> vertexListPosition;  
    /* setter and getter for vertexListPosition */  
}
```

```
public class EdgeListEdge<E, V> extends Edge<E, V> {  
    private DLNode<Edge<E, V>> edgeListPosition;  
    /* setter and getter for edgeListPosition */  
}
```

Graphs in Java: Edge List (2)

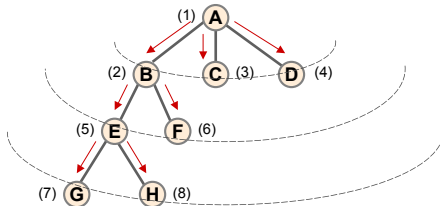
```
public class EdgeListGraph<V, E> implements Graph<V, E> {  
    private DoublyLinkedList<EdgeListVertex<V>> vertices;  
    private DoublyLinkedList<EdgeListEdge<E, V>> edges;  
    private boolean isDirected;  
  
    /* initialize an empty graph */  
    public EdgeListGraph(boolean isDirected) {  
        this.vertices = new DoublyLinkedList<>();  
        this.edges = new DoublyLinkedList<>();  
        this.isDirected = isDirected;  
    }  
    ...  
}
```

Graphs in Java: Edge List (3)



Breadth-First Search (BFS)

- A **breadth-first search (BFS)** of graph $G = (V, E)$, starting from some vertex $v \in V$:
 - Visits every vertex **adjacent** to v before visiting any other (**more distant**) vertices



- BFS** attempts to stay as **close** as possible, whereas **DFS** attempts to move as **far** as possible
- BFS** proceeds in rounds and divides the vertices into **levels**
 - No backtracking** in **BFS**: it is completed **as soon as** the **most distant level** of vertices from the start vertex v are visited.
- Use a **FIFO queue** to keep track of the nodes to be visited.

BFS in Java: Marking Vertices and Edges

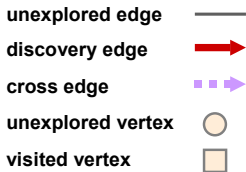
Before the **BFS** starts:

- All vertices are **unvisited**.
- All edges are **unexplored/unmarked**.

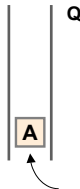
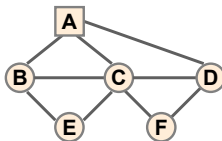
Over the course of a **BFS**, we **mark** vertices and edges:

- A vertex is marked **visited** when it is **first** encountered.
- Then, we iterate through each of v's **incident edges**, say *e*:
 - If edge *e* is already **marked**, then skip it.
 - Otherwise, for an **undirected** graph, an edge is marked as:
 - A **discovery** edge if it leads to an **unvisited** vertex
 - A **cross** edge if it leads to a **visited** vertex
(i.e., from a different **branch** at the same **level**).
- A **cross** edge:
 - Always connects to vertices at the same level
 - Can not connect to vertices at an upper or a lower level
∴ It would've been or will be marked as a **discovery edge**.

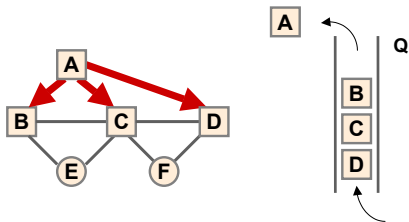
Iterative BFS: Illustration (1.1)



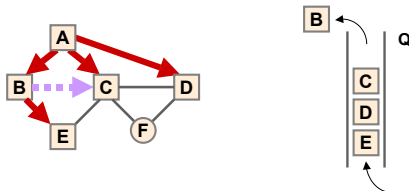
(0)



(1)

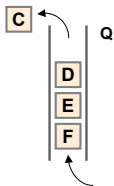
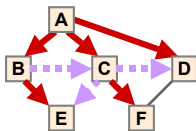


(2)

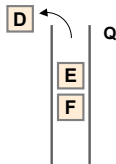
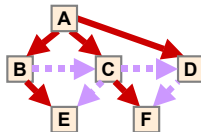


Iterative BFS: Illustration (1.2)

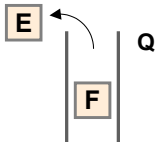
(3)



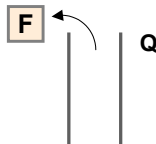
(4)



(5)

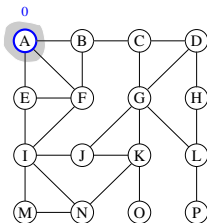


(6)

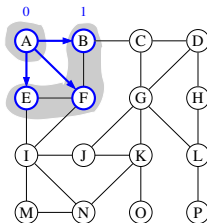


BFS: Illustration (2)

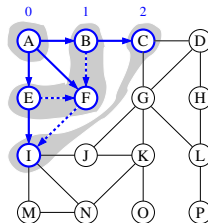
(0)



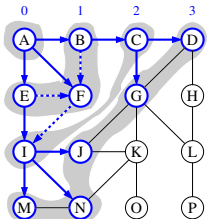
(1)



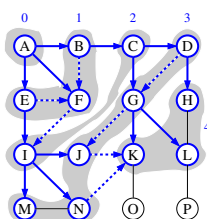
(2)



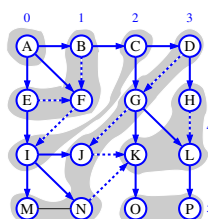
(3)



(4)



(5)



BFS: Properties

1. Running Time?

- Every **vertex** is set as **visited at most once**.
- Each **edge** is set as either **DISCOVERY** or **CROSS at most once**.

$$\Rightarrow O(m + n)$$

2. For a **BFS** starting from vertex u in a graph $G = (V, E)$:

- 2.1 $|visited\ nodes| = |V| \Rightarrow G$ is **connected**
- 2.2 $|visited\ nodes| < |V| \Rightarrow G$ has > 1 **connected components**
- 2.3 A **cross edge** connects vertices that in different **branches**.

In a directed graph, this does **not** necessarily creates a **cycle**.

3. For a **BFS** starting from vertex u in an undirected graph G :

- 3.1 The traversal visits all nodes in the **connected component** containing u .
- 3.2 **Discovery edges** form a **spanning tree** or **level tree** (with $|V| - 1$ edges) of the **connected component** containing u .

4. If a graph G is **not connected**, then it takes **multiple** runs of **BFS** to identify all G 's **connected components**.

Graph Questions: Adapting BFS

- Given a (directed or undirected) graph $G = (V, E)$:
 - Find a **shortest path** (by edges) between vertex u and vertex v .
Start a BFS from u and stop as soon as v is encountered.
 - Is vertex v **reachable** from vertex u ?
No if a BFS starting from u never encounters v .
 - Find all **connected components** of G .
 - Continuously apply **BFS**'s until the entire set V is visited.
 - Each **BFS** produces a **subgraph** representing a new **CC**.
 - Given that G is **connected**, find a **spanning tree** of it.
 G is **connected**. $\Rightarrow G$'s only **CC** is its **spanning tree**.
- Given an undirected graph $G = (V, E)$:
 - Is G **connected**?
 - Start a **BFS** from an arbitrary vertex, and count # of visited nodes.
 - When the traversal completes, compare the counter value against $|V|$.
 - Is an undirected G **acyclic**?
 - Start a **BFS** from an arbitrary vertex.
 - Return **no** (i.e., a **cycle** exists) as soon as a **cross edge** is found.

Graphs in Java: Adjacency List (1)

- Extends the *edge list* structure
- Each *vertex* v also stores a list of *incident edges*.
 ⇒ vertex-based methods such as `outgoingEdges` and `removeVertex` takes $O(d_v)$ rather than $O(|E|)$
- Each *edge* also stores references to its *positions* in both *lists of incident edges* of its two end vertices.
 ⇒ $O(1)$ deletion of the *edge* from the *incident edges list*.

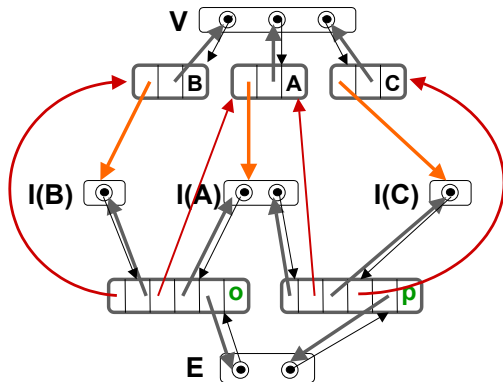
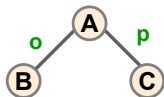
```
class AdjacencyListVertex<V> extends EdgeListVertex<V> {
    private DoublyLinkedList<AdjacencyListEdge<E, V>> incidentEdges;
    /* getter for incidentEdges */
}
```

```
class AdjacencyListEdge<V> extends EdgeListEdge<V> {
    DLNode<Edge<E, V>> originIncidentListPos;
    DLNode<Edge<E, V>> destIncidentListPos;
}
```

Graphs in Java: Adjacency List (2)

```
class AdjacencyListGraph<V, E> implements Graph<V, E> {  
    private DoublyLinkedList<AdjacencyListVertex<V>> vertices;  
    private DoublyLinkedList<AdjacencyListEdge<E, V>> edges;  
    private boolean isDirected;  
  
    /* initialize an empty graph */  
    AdjacencyListGraph(boolean isDirected) {  
        this.vertices = new DoublyLinkedList<>();  
        this.edges = new DoublyLinkedList<>();  
        this.isDirected = isDirected;  
    }  
}
```

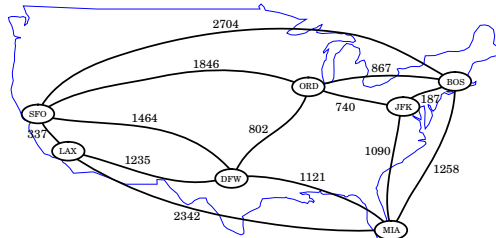
Graphs in Java: Adjacency List (3)



Weighted Graphs

A graph may usefully carry **weights** on its **edges**:

- e.g., Edges of a graph of cities denote distances.



- In a **weighted graph**, for each **edge** $e = (u, v)$, we write $w(u, v)$ to denote the numerical value of **edge e 's weight**.
e.g., $w(JFK, ORD) = 740$, $w(ORD, DFW) = 802$, $w(DFW, LAX) = 1235$
- **Weights** on edges may be considered as “**cost**”.
⇒ When there are **more than one paths** existing between two **vertices**, choose one whose “**total cost**” is the **minimum**.
- We assume that all edge weights are **non-negative** (i.e., ≥ 0).

Shortest Paths in Weighted Graphs

- Given a **path** $P = (v_0, v_1, \dots, v_k)$, with $k + 1$ **vertices** and k **edges**, we define $w(P)$ as the **length** (or **weight**) of P :

$$w(P) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$$

e.g., $w(JFK, ORD, DFW, LAX) = 2777$, $w(JFK, MIA, DFW, LAX) = 3446$

- $d(u, v)$ denotes the **distance** or **shortest path** between u and v : **minimum** weight sum of a path between u and v .

e.g.

$$\begin{aligned} d(JFK, LAX) &= w(JFK, ORD, DFW, LAX) \\ &> w(JFK, MIA, DFW, LAX) \end{aligned}$$

- If there is no path existing between u and v , then $d(u, v) = \infty$

Dijkstra's Shortest Path Algorithm

Starting from a **source vertex s** , perform a **BFS**-like procedure:

1. Initially:

1.1 Set $D(s) = 0$, and every other vertex $t \neq s$, $D(t) = \infty$. [distance]

1.2 Set $a(v) = \text{nil}$ for every vertex v . [ancestor in shortest path]

1.3 Insert all vertices into a **priority queue Q** [keyed by D]

2. While Q is not empty, repeat the following:

2.1 Find vertex u in Q s.t. $D(u)$ is the **minimum**.

2.2 For every vertex v **adjacent to** u , if:

$v \in Q \wedge D(u) + w(u, v) < D(v)$, then:

- Set $D(v) = D(u) + w(u, v)$
- Set $a(v) = u$

2.3 Remove vertex u from Q .

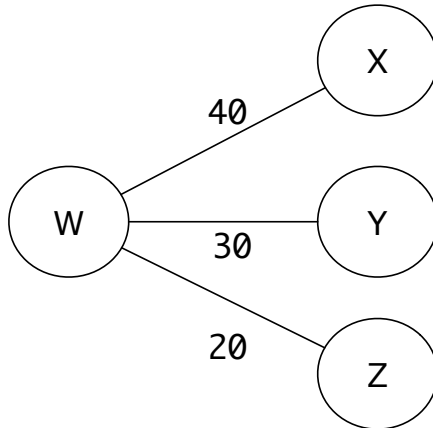
Upon completion, for every vertex t ($t \neq s$):

• $D(t) = d(s, t)$ (i.e., weight of **shortest path** from s to t).

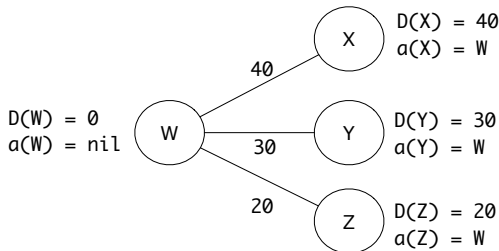
• **Reversing** t 's **ancestor path** $\rightarrow$ **shortest path**: $\langle s, \dots, a(t), t \rangle$

Dijkstra's Algorithm: Example (1) Input

Perform Dijkstra's algorithm on the following graph, starting with a **source vertex W**:



Dijkstra's Algorithm: Example (1) Output



History of Q's contents:

W	X	Y	Z
---	---	---	---

Shortest paths and distances:

- W to X: $\langle a(X), X \rangle = \langle W, X \rangle$;
- W to Y: $\langle a(Y), Y \rangle = \langle W, Y \rangle$;
- W to Z: $\langle a(Z), Z \rangle = \langle W, Z \rangle$;

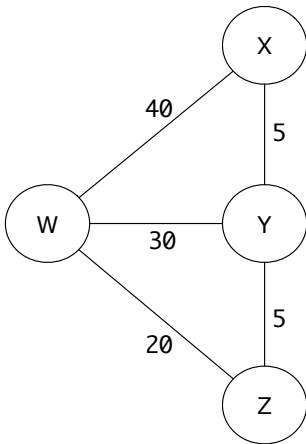
$$d(W, X) = D(X) = 40$$

$$d(W, Y) = D(Y) = 30$$

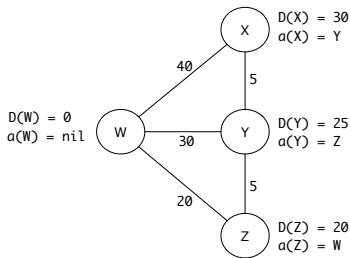
$$d(W, Z) = D(Z) = 20$$

Dijkstra's Algorithm: Example (2) Input

Perform Dijkstra's algorithm on the following graph, starting with a **source vertex W**:



Dijkstra's Algorithm: Example (2) Output



History of Q's contents:

W	X	Y	Z
---	---	---	---

Shortest paths and distances:

- W to X: $\langle a(Z), a(Y), a(X), X \rangle = \langle W, Z, Y, X \rangle$;
- W to Y: $\langle a(Z), a(Y), Y \rangle = \langle W, Z, Y \rangle$;
- W to Z: $\langle a(Z), Z \rangle = \langle W, Z \rangle$;

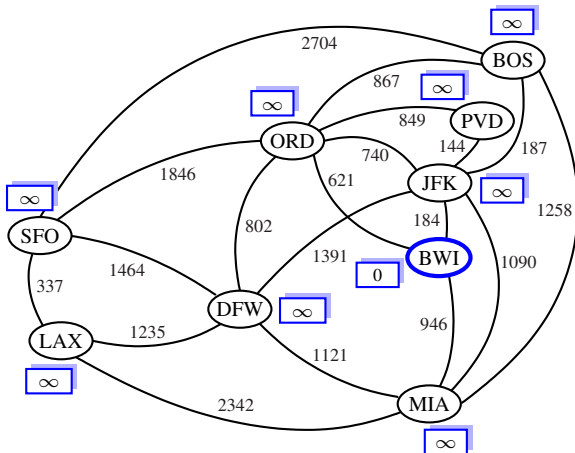
$$d(W, X) = D(X) = 30$$

$$d(W, Y) = D(Y) = 25$$

$$d(W, Z) = D(Z) = 20$$

Dijkstra's Algorithm: Exercise

Perform Dijkstra's algorithm on the following initial configuration, starting with a **source vertex BWI**:



Correctness of Loops

How do we prove that the following loop is correct?

```

{  $Q$  }
 $S_{init}$ 
while ( $B$ ) {
     $S_{body}$ 
}
{  $R$  }
  
```

In case of C/Java, B denotes the **stay condition**.

- In C/Java, there is **not** native, syntactic support for checking the **correctness** of loops.
- Instead, we have to **manually** add assertions to encode:
 - **LOOP INVARIANT** [for establishing **partial correctness**]
 - **LOOP VARIANT** [for ensuring **termination**]

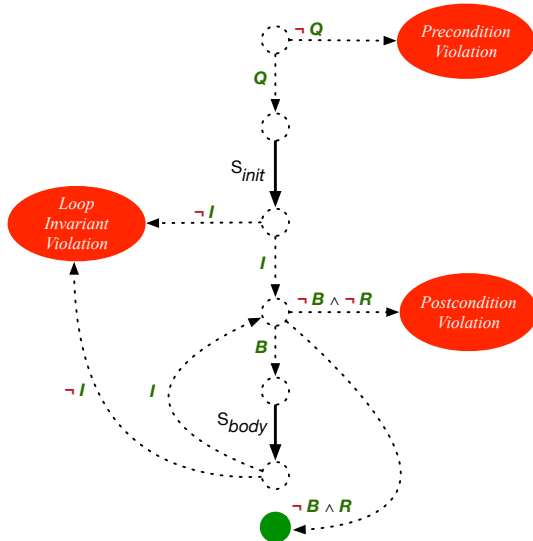
Specifying Loops

- **Loop Invariant** (**LI**): Boolean expression for measuring/proving *partial correctness*
 - **Established** before the very first iteration.
 - **Maintained** TRUE after each iteration.

Specifying Loops: Syntax

```
void myAlgorithm() {  
    assert  $Q$ ; /* Precondition */  
     $S_{init}$   
    assert  $I$ ; /* Is LI established? */  
    while(  $B$  ) {  
         $S_{body}$   
        assert  $I$ ; /* Is LI preserved? */  
    }  
    assert  $R$ ; /* Postcondition */  
}
```

Specifying Loops: Runtime Checks (1)

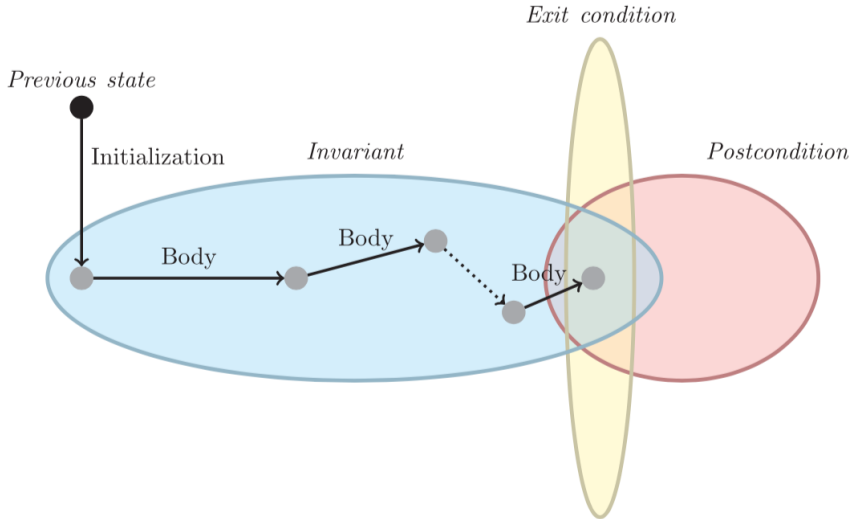


Specifying Loops: Runtime Checks (2)

```
1 void testLI() { /* Assume: integer attribute i */  
2   assert i == 1; /* Precondition */  
3   assert (1 <= i) && (i <= 6); /* Is LI established? */  
4   while (i <= 5) {  
5     i = i + 1;  
6     assert (1 <= i) && (i <= 6); /* Is LI maintained? */  
7   }  
8   assert i == 6; /* Postcondition */  
9 }
```

L1: Change to `1 <= i && i <= 5` for a **Loop Invariant Violation**.

Specifying Loops: Visualization



Digram Source: page 5 in *Loop Invariants: Analysis, Classification, and Examples*

Dijkstra's SP Algorithm: Loop Invariant

- Recall: A **loop invariant** (**LI**) is a Boolean condition.
 - **LI** is established before the 1st iteration.
 - **LI** is preserved **at the end of** each subsequent iteration.
- The (iterative) Dijkstra's algorithm has **LI**:

For every vertex u that has already been removed from the priority queue Q (i.e., u is considered visited), $D(u)$ equals the **true** shortest-path distance from source s to u .

- Formally:

$$\forall u \bullet u \in V \wedge u \notin Q \Rightarrow D(u) = d(s, u)$$

- Important assumption: weights are **non-negative**.
- To relax this assumption, update **visited** nodes.

[ref: Bellman-Ford]

$\Rightarrow$ Worse running time: $O(|V|^3)$ (rather than $O(|V|^2 \cdot \log |V|)$)

Running Time of Dijkstra's Algorithm (1)

```
1  ALGORITHM: Dijkstra-Shortest-Path
2  INPUT: Graph  $G = (V, E)$ ; Source Vertex  $s \in V$ 
3  OUTPUT: For  $t \in V$  ( $t \neq s$ ),
4      •  $D(t) := d(s, t)$ 
5      • Shortest Path:  $\langle s, \dots, a(a(t)), a(t), t \rangle$ 
6  PROCEDURE:
7       $D(s) = 0$ 
8      for  $(t \in (V \setminus \{s\}))$ :  $D(t) := \infty$ 
9      for  $(v \in V)$ :  $a(v) := \text{nil}$ 
10     for  $(v \in V)$ :  $Q.\text{insert}(v)$  --  $Q$  is a PQ keyed by  $D$ 
11     while ( $\neg Q.\text{isEmpty}()$ ):
12          $u := Q.\text{min}()$ 
13         for  $(v \text{ adjacent to } u)$ :
14             if  $(v \in Q \wedge D(u) + w(u, v) < D(v))$ :
15                  $D(v) := D(u) + w(u, v)$ 
16                  $a(v) := u$ 
17             else:
18                 skip
19      $Q.\text{removeMin}()$ 
```

Running Time of Dijkstra's Algorithm (2)

- When implemented using a *heap*, the *priority queue* Q can perform each insertion and deletion in $O(\log n)$ time.
- Given $|V| = n$ and $|E| = m$, *time complexity* breaks down to:
 - L7 – L9**: initializing D and a for all vertices $[O(n)]$
 - L10**: n insertions to Q $[O(n \cdot \log n)]$
 - L11**: while loop has n iterations (**L12 – L19**).
 - L12**: retrieving the root of heap n times $[O(n)]$
 - Q**. How many iterations for **L14 – L18**?
A. # adjacency edges across all vertices: $\sum_{u \in V} \text{degree}(u) = m$
 - L15**: *upward bubbling* to restore relational property of Q $[O(m \cdot \log n)]$
 - L14, L16-18**: constant operations $[O(m)]$
 - L19**: removing min-root of heap n times $[O(n \cdot \log n)]$
- Efficient implementation of Dijkstra Algorithm: $O((n + m) \cdot \log n)$
- G almost *complete* (i.e., $m = O(n^2)$) $\Rightarrow$ RT is $O(n^2 \cdot \log n)$

Graphs in Java: Adjacency Matrix (1)

- Extends the *edge list* structure
- Each *vertex* v also stores an *integer index* that is used to index into a 2-dimensional *adjacency matrix*.
⇒ locating an edge between two vertices takes **$O(1)$**

```
class AdjacencyMatrixVertex<V> extends EdgeListVertex<V> {  
    private int index;  
    /* getter and setter for index */  
}
```

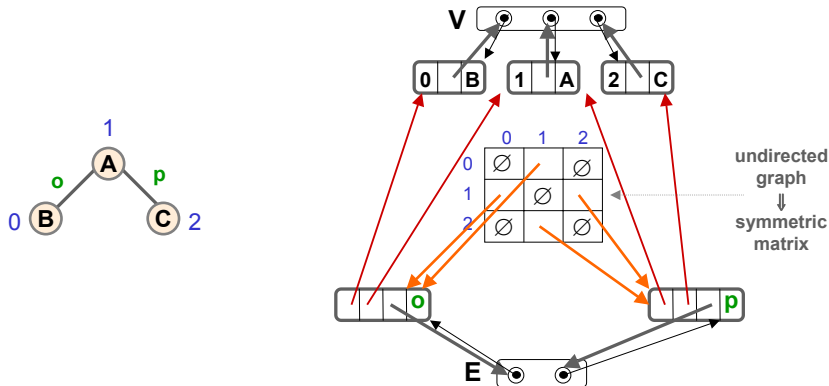
Graphs in Java: Adjacency Matrix (2)

```
class AdjacencyMatrixGraph<V, E> implements Graph<V, E> {  
    private DoublyLinkedList<AdjacencyMatrixVertex<V>> vertices;  
    private DoublyLinkedList<EdgeListEdge<E, V>> edges;  
    private boolean isDirected;  
  
    private EdgeListEdge<E, V>[][] matrix;  
  
    /* initialize an empty graph */  
    AdjacencyMatrixGraph(boolean isDirected) {  
        this.vertices = new DoublyLinkedList<>();  
        this.edges = new DoublyLinkedList<>();  
        this.isDirected = isDirected;  
    }  
}
```

Space Requirements?

$O(|V|^2)$

Graphs in Java: Adjacency Matrix (3)



- Each **row index** in matrix represents an **origin vertex**.
- Each **column index** in matrix represents a **destination vertex**.
e.g., edge (B, A): `matrix[0][1]`
- For an undirected graph, cells are **symmetric**.
e.g., `matrix[0][1] == matrix[1][0]`

Graphs in Java: Comparing Strategies (1)

	VERTEX	EDGE	GRAPH
ADJACENCY LIST	incidentEdges	originIncidentListPos	isDirected vertices edges
		destIncidentListPos	
EDGE LIST	vertexListPosition	edgeListPosition	
ADJACENCY MATRIX	index		
			matrix

Graphs in Java: Comparing Strategies (2)

	EDGE LIST	ADJACENCY LIST	ADJACENCY MATRIX
numVertices() numEdges()	$O(1)$		
vertices()	$O(n)$		
edges()	$O(m)$		
getEdge(u, v)	$O(m)$	$O(\min(d_u, d_v))$	$O(1)$
outDegree(v) inDegree(v)	$O(m)$	$O(d_v)$	$O(n)$
outgoingEdges(v) incomingEdges(v)	$O(m)$	$O(d_v)$	$O(n)$
insertVertex(x)	$O(1)$		$O(n^2)$
removeVertex(v)	$O(m)$	$O(d_v)$	$O(n^2)$
insertEdge(u, v, x) removeEdge(e)	$O(1)$		

Directed Acyclic Graph (DAG)

- **Directed Acyclic Graph (DAG)**: directed graph with no cycle.
- A **DAG** has many applications where **dependency** exists between **vertices**:
 - e.g., Prerequisites between courses of the undergrad program
 - e.g., Inheritance hierarchy among Java classes
 - e.g., Scheduling constraints between tasks
 - e.g., Dependency between variables in transactional updates
- In a **DAG**, an **edge** (v_i, v_j) means v_i “occurs before” v_j
 - e.g., $(eecs2101, eecs3101)$
 - e.g., $(\text{int } x = 0, \text{println}(x))$
- Given a **DAG** $G = (V, E)$, where $|V| = n$, a **topological ordering** of G is a sequence of n vertices

$$v_1, v_2, \dots, v_n$$

such that

$$\forall i, j \bullet 1 \leq i, j \leq n \wedge (v_i, v_j) \in E \Rightarrow i < j$$

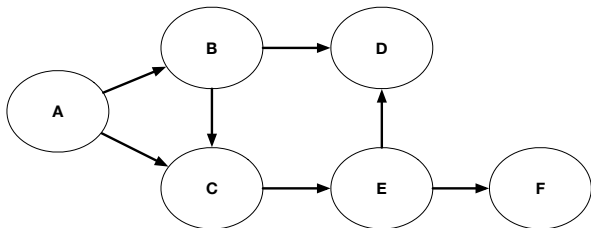
Using DFS for Topological Sort

- Given a **DAG**, the process of computing a **topological ordering** of G is called performing a **topological sort**.
- Initialize an empty **sorted list** of vertices.
- Repetitively perform an extended version of **DFS**, say **DFS_{topo}**, until all vertices in the **DAG** are visited.
- Each **DFS_{topo}**:
 - Starts with an arbitrary unvisited vertex v
 - Returns a **sorted list** that corresponds to the reverse order in which vertices **backtracked**.
 - When visiting v , only **push** v 's **adjacent** vertices that are unvisited.
 - When **popping** a vertex v , add v to the front of the **sorted list**.

⇒ The **sorted list** contains all vertices **reachable** from v , within the current DFS run (i.e., all vertices that must “occur after” v).

 - Add the produced **sorted list** to the front of the list accumulated from the previous **DFS_{topo}**.

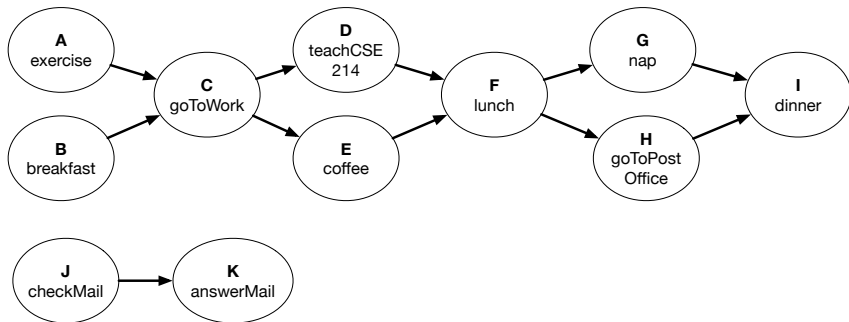
DAG: Illustration (1)



Topologically Sorted lists produced by extended DFS:

- List from $DFS_{topo}(F)$ $\langle F \rangle$
- List from $DFS_{topo}(D)$ $\langle D \rangle$
- List from $DFS_{topo}(E)$ $\langle E, D, F \rangle$ or $\langle E, F, D \rangle$
- List from $DFS_{topo}(C)$ $\langle C, E, D, F \rangle$ or $\langle C, E, F, D \rangle$
- List from $DFS_{topo}(B)$ $\langle B, C, E, D, F \rangle$ or $\langle B, C, E, F, D \rangle$
- List from $DFS_{topo}(A)$ $\langle A, B, C, E, D, F \rangle$ or $\langle A, B, C, E, F, D \rangle$

DAG: Illustration (2)



Possible **topological orderings** after a **topological sort**:

- $\langle A, B, C, D, E, F, G, H, I, J, K \rangle$
- $\langle B, A, C, E, D, F, H, G, I, J, K \rangle$
- $\langle J, B, A, C, E, D, F, H, G, I, K \rangle$

DAG: Topological Sort in Java (1)

```
Iterable<Vertex<V>> topologicalSort(Graph<V, E> g) {  
    ArrayList<Vertex<V>> order = new ArrayList<>();  
    for(Vertex<V> v: g.vertices()) {  
        if(!v.isVisited()) {  
            DFStopo(g, v, order)  
        }  
    }  
    return order;  
}
```

DAG: Topological Sort in Java (2)

```

1  DFSTopo(Graph<V, E> g, Vertex<V> v, ArrayList<Vertex<V>> order){
2      Stack s = new LinkedStack(); v.setVisited(); s.push(v);
3      while(!s.isEmpty()) {
4          Vertex<V> top = s.peek();
5          Iterator<Edge<E, V>> it = g.outGoingEdges(top);
6          boolean foundUnexploredEdge = false;
7          while(it.hasNext() && !foundUnexploredEdge) {
8              Edge<E, V> e = it.next();
9              Vertex<V> opposite = e.getDestination();
10             if(!opposite.isVisited()) { /* discovery edge */
11                 foundUnexploredEdge = true;
12                 opposite.setVisited(); s.push(opposite);
13             }
14         }
15         if(!foundUnexploredEdge) { order.addFirst(top); s.pop(); }
16     }
17 }
  
```

Minimum Spanning Trees (MSTs): Problem

- **Minimum Spanning Tree (MST) problem:**

Given a simple, and undirected, weighted graph $G = (V, E)$, find a **spanning tree** T with the **minimum** total weight (over all spanning trees). More precisely:

$$w(T) = \sum_{(u,v) \in T} w(u,v)$$

is the **minimum** among all **spanning trees**.

- Solving the **MST problem** in practice:
 - e.g., **Telecom network design**: Build a network connecting all cell towers via fiber links with the minimum installation cost or distance.
 - e.g., **Chip design**: Connect all ground pins through wiring backbone with minimal wire length or delay.

MST Problem: The Greedy Method

- Intuitively, to design using the **Greedy Method**:
 - Build an **iterative** solution.
 - At each iteration:
 - Multiple **feasible** choices exist to keep the partial solution valid.
 - Pick the one that looks best **right now** w.r.t. a simple **cost function**.
 - The choice made is **not** necessarily the **globally** best-looking.
 - After a choice is made, **never** undo it (i.e., **no backtracking**).
- A **cost/score function** assigns a number:
 - to **rank** possible choices; and
 - to **measure** the **partial solution** constructed so far.

We may either minimize the cost (e.g., smallest edge weight, smallest distance), or maximize the score (e.g., largest profit).

- A **greedy algorithm** builds the solution **incrementally**, always picking the **locally-optimal choice** (i.e., a **feasible** option with the **best** score at the moment by the **cost function**).

MST Problem: Kruskal's Algorithm

```
1  ALGORITHM: Find-MST-Kruskal
2  INPUT: Simple, Undirected, Weighted, Connected  $G = (V, E)$ 
3  OUTPUT: A minimum spanning tree  $T$  of  $G$ 
4  PROCEDURE:
5    for  $v \in V$ :  $C(v) := \{v\}$  -- build  $|V|$  elementary clusters
6    Initialize a priority queue  $Q$  containing  $E$  -- keyed by weights
7     $T := \emptyset$ 
8    while  $|T| \neq n - 1$ :
9       $(u, v) := Q.removeMin()$ 
10     let  $C(u)$  be the cluster containing  $u$ 
11     let  $C(v)$  be the cluster containing  $v$ 
12     if  $C(u) \neq C(v)$  then
13        $T := T \cup \{(u, v)\}$ 
14       Merge  $C(u)$  and  $C(v)$  into one cluster
```

MST Problem: Tracing Kruskal's (1)

Apply Kruskal's algorithm on this graph:

$$V = \{A, B, C, D, E, F, G, H, G\}$$

edge	weight
(A, B)	1
(A, D)	3
(B, C)	2
(B, H)	10
(C, D)	3
(C, F)	6
(D, E)	5
(E, F)	4
(E, H)	8
(F, G)	7
(G, H)	9

MST Problem: Tracing Kruskal's (2)

ITERATION	MIN EDGE	PROCESSING	RESULTING PARTITION	T: MST UNDER CONSTRUCTION
Init.	—	—	$\left\{ \begin{array}{l} \{A\}, \{B\}, \{C\}, \{D\}, \\ \{E\}, \{F\}, \{G\}, \{H\} \end{array} \right\}$	$\emptyset$
1	$w(A, B) = 1$	$\therefore C(A) \neq C(B) \therefore$ Tree Edge	$\left\{ \begin{array}{l} \{A, B\}, \{C\}, \{D\}, \\ \{E\}, \{F\}, \{G\}, \{H\} \end{array} \right\}$	$\{ (A, B) \}$
2	$w(B, C) = 2$	$\therefore C(B) \neq C(C) \therefore$ Tree Edge	$\left\{ \begin{array}{l} \{A, B, C\}, \{D\}, \\ \{E\}, \{F\}, \{G\}, \{H\} \end{array} \right\}$	$\{ (A, B), (B, C) \}$
3	$w(A, D) = 3$	$\therefore C(A) \neq C(D) \therefore$ Tree Edge	$\left\{ \begin{array}{l} \{A, B, C, D\}, \\ \{E\}, \{F\}, \{G\}, \{H\} \end{array} \right\}$	$\{ (A, B), (B, C), (A, D) \}$
4	$w(C, D) = 3$	$\therefore C(C) = C(D) \therefore$ Internal Edge	No Change	
5	$w(E, F) = 4$	$\therefore C(E) \neq C(F) \therefore$ Tree Edge	$\left\{ \begin{array}{l} \{A, B, C, D\}, \\ \{E, F\}, \{G\}, \{H\} \end{array} \right\}$	$\{ (A, B), (B, C), (A, D), (E, F) \}$
6	$w(D, E) = 5$	$\therefore C(D) \neq C(E) \therefore$ Tree Edge	$\left\{ \begin{array}{l} \{A, B, C, D, E, F\}, \\ \{G\}, \{H\} \end{array} \right\}$	$\left\{ \begin{array}{l} (A, B), (B, C), (A, D), (E, F), \\ (D, E) \end{array} \right\}$
7	$w(C, F) = 6$	$\therefore C(C) = C(F) \therefore$ Internal Edge	No Change	
8	$w(F, G) = 7$	$\therefore C(F) \neq C(G) \therefore$ Tree Edge	$\left\{ \begin{array}{l} \{A, B, C, D, E, F, G\}, \\ \{H\} \end{array} \right\}$	$\left\{ \begin{array}{l} (A, B), (B, C), (A, D), (E, F), \\ (D, E), (F, G) \end{array} \right\}$
9	$w(E, H) = 8$	$\therefore C(E) \neq C(H) \therefore$ Tree Edge	$\{ \{A, B, C, D, E, F, G, H\} \}$	$\left\{ \begin{array}{l} (A, B), (B, C), (A, D), (E, F), \\ (D, E), (F, G), (E, H) \end{array} \right\}$

MST Problem: From Clusters to Cuts

- **Partition** (of V) [V broken into one or more pieces]
 - A set P of non-empty, **disjoint** vertex sets whose **union** equals V .

$$\bigcup_{x \in P} x = V$$

$$\wedge (\forall x_1, x_2 \bullet x_1 \in P \wedge x_2 \in P \wedge x_1 \neq \emptyset \wedge x_2 \neq \emptyset \wedge x_1 \neq x_2 \Rightarrow x_1 \cap x_2 = \emptyset)$$

- **Cluster** [one piece of the current partition]
 - In each iteration of executing Kruskal's algorithm (say $x, y \in V$):
 - A **cluster** is a set $C(x)$ that's a member of **partition** P (i.e., $C(x) \in P$)
e.g., **L10** and **L11** in Kruskal's algorithm
 - $C(x) = C(y) \Rightarrow x$ and y in the same **connected component**.
 - $C(x) \neq C(y) \Rightarrow x$ and y in different **connected components**.
- **Cut** (of V) [V broken into two pieces]
 - A **partition** of V into **two** non-empty, **disjoint** sets:

$$P = \{S, V \setminus S\}$$

- An edge **crosses the cut** with its endpoints in S and $V \setminus S$.

MST Problem: Cut Property of Safe Edges

Cut Property. Given:

- $G = (V, E)$ a weighted, **connected** graph
 - Any **cut** $\{S, V \setminus S\}$ of the vertices
 - e a **minimum-weight edge** among all edges that **cross this cut**
- ⇒ There exists **an MST** that contains e .

We say: e is a **safe edge** for some **MST**.

In Kruskal's algorithm:

- **L10:** **Cluster** $C(u)$ helps form a **cut**: $\{C(u), V \setminus C(u)\}$
 - $C(u) \neq \emptyset$
 - $V \setminus C(u) \neq \emptyset$ [i.e., not in the **last** iteration]
- **L12:** $C(u) \neq C(v)$ means $C(u) \cap C(v) = \emptyset$ and $C(v) \subseteq V \setminus C(u)$.
 - Recall: Edges extracted from Q in a non-decreasing order on weights
 - **Clusters** only merge; never split.
 - Any edge (x, y) with $x \in C(u)$ and $w(x, y) < w(u, v)$ would've been processed (as a **tree** edge or as an **internal** edge), so now $x, y \in C(u)$.
 ⇒ $\nexists$ edge "cheaper" than (u, v) **crossing the cut** $\{C(u), V \setminus C(u)\}$
- ∴ (u, v) from **L9** is a **min-weight edge** **crossing the cut**
- By the **Cut Property**, **L13** is justified:
 - (u, v) is a **safe edge**: $\exists$ some **MST** containing $T \cup \{(u, v)\}$
 - Adding (u, v) keeps us on the right track to reach some **MST**.

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