

Math Review: Logic, Sets, Relations



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Background for Self-Study



- Topics of **sets** and **relations** were covered in EECS1019/1090.
- Slide 3 to Slide 23 contain what you should recall.

Propositional Logic (1)



- A **proposition** is a statement of claim that must be of either **true** or **false**, but not both.
- Basic logical operands are of type Boolean: **true** and **false**.
- We use logical operators to construct compound statements.
 - Unary logical operator: negation ($\neg$)

p	$\neg p$
true	false
false	true

- Binary logical operators: conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), equivalence ($\equiv$), and if-and-only-if ($\Leftrightarrow$).

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \equiv q$
true	true	true	true	true	true	true
true	false	false	true	false	false	false
false	true	false	true	true	false	false
false	false	false	false	true	true	true

Propositional Logic: Implication (1)



- Written as $p \Rightarrow q$ [pronounced as “p implies q”]
 - We call p the antecedent, assumption, or premise.
 - We call q the consequence or conclusion.
- Compare the **truth** of $p \Rightarrow q$ to whether a contract is **honoured**:
 - antecedent/assumption/premise $p \approx$ promised terms [e.g., salary]
 - consequence/conclusion $q \approx$ obligations [e.g., duties]
- When the promised terms are met, then the contract is:
 - **honoured** if the obligations fulfilled. [$(true \Rightarrow true) \Leftrightarrow true$]
 - **breached** if the obligations violated. [$(true \Rightarrow false) \Leftrightarrow false$]
- When the promised terms are not met, then:
 - Fulfilling the obligation (q) or not ($\neg q$) does **not breach** the contract.

p	q	$p \Rightarrow q$
false	true	true
false	false	true

Propositional Logic: Implication (2)



There are alternative, equivalent ways to expressing $p \Rightarrow q$:

- **q if p**
q is *true* if p is *true*
- **p only if q**
If p is *true*, then for $p \Rightarrow q$ to be *true*, it can only be that q is also *true*.
Otherwise, if p is *true* but q is *false*, then $(\text{true} \Rightarrow \text{false}) \equiv \text{false}$.
- Note.** To prove $p \equiv q$, prove $p \iff q$ (pronounced: "p if and only if q"):
 - **p if q** [$p \leftarrow q \equiv q \Rightarrow p$]
 - **p only if q** [$p \Rightarrow q$]
- p is **sufficient** for q [similar to q if p]
For q to be *true*, it is sufficient to have p being *true*.
- q is **necessary** for p [similar to p only if q]
If p is *true*, then it is necessarily the case that q is also *true*.
Otherwise, if p is *true* but q is *false*, then $(\text{true} \Rightarrow \text{false}) \equiv \text{false}$.
- q **unless** $\neg p$ [When is $p \Rightarrow q$ *true*?]
If q is *true*, then $p \Rightarrow q$ *true* regardless of p.
If q is *false*, then $p \Rightarrow q$ cannot be *true* unless p is *false*.

5 of 26

Propositional Logic (2)



- **Axiom:** Definition of $\Rightarrow$
$$p \Rightarrow q \equiv \neg p \vee q$$

- **Theorem:** Identity of $\Rightarrow$
$$\text{true} \Rightarrow p \equiv p$$

- **Theorem:** Zero of $\Rightarrow$
$$\text{false} \Rightarrow p \equiv \text{true}$$

- **Axiom:** De Morgan

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

- **Axiom:** Double Negation

$$p \equiv \neg(\neg p)$$

- **Theorem:** Contrapositive

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

7 of 26

Propositional Logic: Implication (3)



Given an implication $p \Rightarrow q$, we may construct its:

- **Inverse:** $\neg p \Rightarrow \neg q$ [negate antecedent and consequence]
- **Converse:** $q \Rightarrow p$ [swap antecedent and consequence]
- **Contrapositive:** $\neg q \Rightarrow \neg p$ [inverse of converse]

6 of 26

Predicate Logic (1)



- A **predicate** is a *universal* or *existential* statement about objects in some universe of discourse.
- Unlike propositions, predicates are typically specified using **variables**, each of which declared with some *range* of values.
- We use the following symbols for common numerical ranges:
 - $\mathbb{Z}$: the set of integers [$-\infty, \dots, -1, 0, 1, \dots, +\infty$]
 - $\mathbb{N}$: the set of natural numbers [$0, 1, \dots, +\infty$]
- Variable(s) in a predicate may be **quantified**:
 - **Universal quantification**:
All values that a variable may take satisfy certain property.
e.g., Given that i is a natural number, i is *always* non-negative.
 - **Existential quantification**:
Some value that a variable may take satisfies certain property.
e.g., Given that i is an integer, i *can be* negative.

8 of 26

Predicate Logic (2.1): Universal Q. ($\forall$)



- A **universal quantification** has the form $(\forall X \bullet R \Rightarrow P)$
 - X is a comma-separated list of variable names
 - R is a **constraint on types/ranges** of the listed variables
 - P is a **property** to be satisfied
- For all** (combinations of) values of variables listed in X that satisfies R , it is the case that P is satisfied.
 - $\forall i \bullet i \in \mathbb{N} \Rightarrow i \geq 0$ [true]
 - $\forall i \bullet i \in \mathbb{Z} \Rightarrow i \geq 0$ [false]
 - $\forall i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \Rightarrow i < j \vee i > j$ [false]
- Proof Strategies**
 - How to prove $(\forall X \bullet R \Rightarrow P)$ **true**?
 - Hint.** When is $R \Rightarrow P$ **true**? [true $\Rightarrow$ true, false $\Rightarrow$ -]
 - Show that for all instances of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
 - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.
 - How to prove $(\forall X \bullet R \Rightarrow P)$ **false**?
 - Hint.** When is $R \Rightarrow P$ **false**? [true $\Rightarrow$ false]
 - Give a **witness/counterexample** of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.

9 of 26

Predicate Logic (3): Exercises



- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 0$.
All 10 integers between 1 and 10 are greater than 0.
- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 1$.
Integer 1 (a witness/counterexample) in the range between 1 and 10 is **not** greater than 1.
- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 1$.
Integer 2 (a witness) in the range between 1 and 10 is greater than 1.
- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 10$?
All integers in the range between 1 and 10 are **not** greater than 10.

11 of 26

Predicate Logic (2.2): Existential Q. ($\exists$)



- An **existential quantification** has the form $(\exists X \bullet R \wedge P)$
 - X is a comma-separated list of variable names
 - R is a **constraint on types/ranges** of the listed variables
 - P is a **property** to be satisfied
- There exist** (a combination of) values of variables listed in X that satisfy both R and P .
 - $\exists i \bullet i \in \mathbb{N} \wedge i \geq 0$ [true]
 - $\exists i \bullet i \in \mathbb{Z} \wedge i \geq 0$ [true]
 - $\exists i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \wedge (i < j \vee i > j)$ [true]
- Proof Strategies**
 - How to prove $(\exists X \bullet R \wedge P)$ **true**?
 - Hint.** When is $R \wedge P$ **true**? [true $\wedge$ true]
 - Give a **witness** of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
 - How to prove $(\exists X \bullet R \wedge P)$ **false**?
 - Hint.** When is $R \wedge P$ **false**? [true $\wedge$ false, false $\wedge$ -]
 - Show that for all instances of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.
 - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.

10 of 26

Predicate Logic (4): Switching Quantifications



Conversions between $\forall$ and $\exists$:

$$(\forall X \bullet R \Rightarrow P) \iff \neg(\exists X \bullet R \wedge \neg P)$$

$$(\exists X \bullet R \wedge P) \iff \neg(\forall X \bullet R \Rightarrow \neg P)$$

12 of 26

Set of Tuples



Given n sets $S_1, S_2, \dots, S_n$, a **cross/Cartesian product** of these sets is a set of n -tuples.

Each **n -tuple** $(e_1, e_2, \dots, e_n)$ contains n elements, each of which is a member of the corresponding set.

$$S_1 \times S_2 \times \dots \times S_n = \{(e_1, e_2, \dots, e_n) \mid e_i \in S_i \wedge 1 \leq i \leq n\}$$

e.g., $\{a, b\} \times \{2, 4\} \times \{\$, \&\}$ is a set of triples:

$$\begin{aligned} & \{a, b\} \times \{2, 4\} \times \{\$, \&\} \\ = & \{(e_1, e_2, e_3) \mid e_1 \in \{a, b\} \wedge e_2 \in \{2, 4\} \wedge e_3 \in \{\$, \&\}\} \\ = & \left\{ \begin{array}{l} (a, 2, \$), (a, 2, \&), (a, 4, \$), (a, 4, \&), \\ (b, 2, \$), (b, 2, \&), (b, 4, \$), (b, 4, \&) \end{array} \right\} \end{aligned}$$

13 of 26

Relations (1): Constructing a Relation



A **relation** is a set of mappings, each being an **ordered pair** that maps a member of set S to a member of set T .

e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$

- $\emptyset$ is the **minimum** relation (i.e., an empty relation).
- $S \times T$ is the **maximum** relation (say r_1) between S and T , mapping from each member of S to each member in T :

$$\{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

- $\{(x, y) \mid (x, y) \in S \times T \wedge x \neq 1\}$ is a relation (say r_2) that maps only some members in S to every member in T :

$$\{(2, a), (2, b), (3, a), (3, b)\}$$

14 of 26

Relations (2.1): Set of Possible Relations



- We use the **power set** operator to express the set of **all possible relations** on S and T :

$$\mathbb{P}(S \times T)$$

Each member in $\mathbb{P}(S \times T)$ is a relation.

- To declare a relation variable r , we use the colon ($:$) symbol to mean **set membership**:

$$r : \mathbb{P}(S \times T)$$

- Or alternatively, we write:

$$r : S \leftrightarrow T$$

where the set $S \leftrightarrow T$ is synonymous to the set $\mathbb{P}(S \times T)$

15 of 26

Relations (2.2): Exercise



Enumerate $\{a, b\} \leftrightarrow \{1, 2, 3\}$.

- Hints:**
 - You may enumerate all relations in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$ via their **cardinalities**: $0, 1, \dots, |\{a, b\} \times \{1, 2, 3\}|$.
 - What's the **maximum** relation in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$?
 $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
- The answer is a set containing **all** of the following relations:
 - Relation with cardinality 0: $\emptyset$
 - How many relations with cardinality 1? $\left[\binom{|\{a, b\} \times \{1, 2, 3\}|}{1} = 6 \right]$
 - How many relations with cardinality 2? $\left[\binom{|\{a, b\} \times \{1, 2, 3\}|}{2} = \frac{6 \times 5}{2!} = 15 \right]$
 - ...
 - Relation with cardinality $|\{a, b\} \times \{1, 2, 3\}|$:
 $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

16 of 26

Relations (3.1): Domain, Range, Inverse



Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

- **domain** of r : set of first-elements from r
 - Definition: $\text{dom}(r) = \{d \mid (d, r') \in r\}$
 - e.g., $\text{dom}(r) = \{a, b, c, d, e, f\}$
- **range** of r : set of second-elements from r
 - Definition: $\text{ran}(r) = \{r' \mid (d, r') \in r\}$
 - e.g., $\text{ran}(r) = \{1, 2, 3, 4, 5, 6\}$
- **inverse** of r : a relation like r with elements swapped
 - Definition: $r^{-1} = \{(r', d) \mid (d, r') \in r\}$
 - e.g., $r^{-1} = \{(1, a), (2, b), (3, c), (4, a), (5, b), (6, c), (1, d), (2, e), (3, f)\}$

17 of 26

Relations (3.2): Image



Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

relational image of r over set s : sub-range of r mapped by s .

- Definition: $r[s] = \{r' \mid (d, r') \in r \wedge d \in s\}$
- e.g., $r[\{a, b\}] = \{1, 2, 4, 5\}$

18 of 26

Relations (3.3): Restrictions



Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

- **domain restriction** of r over set ds : sub-relation of r with domain ds .
 - Definition: $ds \triangleleft r = \{(d, r') \mid (d, r') \in r \wedge d \in ds\}$
 - e.g., $\{a, b\} \triangleleft r = \{(a, 1), (b, 2), (a, 4), (b, 5)\}$
- **range restriction** of r over set rs : sub-relation of r with range rs .
 - Definition: $r \triangleright rs = \{(d, r') \mid (d, r') \in r \wedge r' \in rs\}$
 - e.g., $r \triangleright \{1, 2\} = \{(a, 1), (b, 2), (d, 1), (e, 2)\}$

19 of 26

Relations (3.4): Subtractions



Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

- **domain subtraction** of r over set ds : sub-relation of r with domain not ds .
 - Definition: $ds \triangleleft r = \{(d, r') \mid (d, r') \in r \wedge d \notin ds\}$
 - e.g., $\{a, b\} \triangleleft r = \{(c, 3), (c, 6), (d, 1), (e, 2), (f, 3)\}$
- **range subtraction** of r over set rs : sub-relation of r with range not rs .
 - Definition: $r \triangleright rs = \{(d, r') \mid (d, r') \in r \wedge r' \notin rs\}$
 - e.g., $r \triangleright \{1, 2\} = \{(c, 3), (a, 4), (b, 5), (c, 6), (f, 3)\}$

20 of 26

Functions (1): Functional Property

- A **relation** r on sets S and T (i.e., $r \in S \leftrightarrow T$) is also a **function** if it satisfies the **functional property**:

$$\text{isFunction}(r)$$

$$\iff \forall s, t_1, t_2 \bullet (s \in S \wedge t_1 \in T \wedge t_2 \in T) \Rightarrow ((s, t_1) \in r \wedge (s, t_2) \in r \Rightarrow t_1 = t_2)$$
 - That is, in a **function**, it is forbidden for a member of S to map to more than one members of T .
 - Equivalently, in a **function**, two distinct members of T cannot be mapped by the same member of S .
- e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$, which of the following **relations** satisfy the above **functional property**?
 - $S \times T$ [No]
Witness 1: $(1, a), (1, b)$; **Witness 2:** $(2, a), (2, b)$; **Witness 3:** $(3, a), (3, b)$.
 - $(S \times T) \setminus \{(x, y) \mid (x, y) \in S \times T \wedge x = 1\}$ [No]
Witness 1: $(2, a), (2, b)$; **Witness 2:** $(3, a), (3, b)$
 - $\{(1, a), (2, b), (3, a)\}$ [Yes]
 - $\{(1, a), (2, b)\}$ [Yes]

21 of 26

Functions (2.1): Total vs. Partial

Given a **relation** $r \in S \leftrightarrow T$

- r is a **partial function** if it satisfies the **functional property**:

$$r \in S \nrightarrow T \iff (\text{isFunction}(r) \wedge \text{dom}(r) \subseteq S)$$
Remark. $r \in S \nrightarrow T$ means there may (or may not) be $s \in S$ s.t. $r(s)$ is **undefined** (i.e., $r[\{s\}] = \emptyset$).
 - e.g., $\{(2, a), (1, b)\}, \{(2, a), (3, a), (1, b)\} \subseteq \{1, 2, 3\} \nrightarrow \{a, b\}$
- r is a **total function** if there is a mapping for each $s \in S$:

$$r \in S \rightarrow T \iff (\text{isFunction}(r) \wedge \text{dom}(r) = S)$$
Remark. $r \in S \rightarrow T$ implies $r \in S \nrightarrow T$, but not vice versa. Why?
 - e.g., $\{(2, a), (3, a), (1, b)\} \in \{1, 2, 3\} \rightarrow \{a, b\}$
 - e.g., $\{(2, a), (1, b)\} \notin \{1, 2, 3\} \rightarrow \{a, b\}$

22 of 26

Functions (2.2): Relation Image vs. Function Application

- Recall: A **function** is a **relation**, but a **relation** is not necessarily a **function**.
- Say we have a **partial function** $f \in \{1, 2, 3\} \nrightarrow \{a, b\}$:

$$f = \{(3, a), (1, b)\}$$
 - With f wearing the **relation** hat, we can invoke **relational images**:

$$\begin{aligned} f[\{3\}] &= \{a\} \\ f[\{1\}] &= \{b\} \\ f[\{2\}] &= \emptyset \end{aligned}$$

Remark. $\Rightarrow |f[\{v\}]| \leq 1$ ∴

- each member in $\text{dom}(f)$ is mapped to at most one member in $\text{ran}(f)$
- each input set $\{v\}$ is a **singleton** set
- With f wearing the **function** hat, we can invoke **functional applications**:

$$\begin{aligned} f(3) &= a \\ f(1) &= b \\ f(2) &\text{ is } \text{undefined} \end{aligned}$$

23 of 26

Index (1)

Background for Self-Study

Propositional Logic (1)

Propositional Logic: Implication (1)

Propositional Logic: Implication (2)

Propositional Logic: Implication (3)

Propositional Logic (2)

Predicate Logic (1)

Predicate Logic (2.1): Universal Q. ($\forall$)

Predicate Logic (2.2): Existential Q. ($\exists$)

Predicate Logic (3): Exercises

Predicate Logic (4): Switching Quantifications

24 of 26

Index (2)



Set of Tuples

Relations (1): Constructing a Relation

Relations (2.1): Set of Possible Relations

Relations (2.2): Exercise

Relations (3.1): Domain, Range, Inverse

Relations (3.2): Image

Relations (3.3): Restrictions

Relations (3.4): Subtractions

Functions (1): Functional Property

Functions (2.1): Total vs. Partial

25 of 26

Index (3)



Functions (2.2):

Relation Image vs. Function Application

26 of 26