Review of Math

EECS4315 Z:
Mission-Critical Systems
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Learning Outcomes of this Lecture

This module is designed to help you **review**:

- Propositional Logic
- Predicate Logic

Propositional Logic (1)

- A *proposition* is a statement of claim that must be of either *true* or *false*, but not both.
- Basic logical operands are of type Boolean: *true* and *false*.
- We use logical operators to construct compound statements.
  - Unary logical operator: negation ($\neg$)

p	$\neg p$
true	*false*
false	*true*

  - Binary logical operators: conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), equivalence ($\equiv$), and if-and-only-if ($\iff$).

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \iff q$	$p \equiv q$
true	*true*	*true*	*true*	*true*	*true*	*true*
true	*false*	*false*	*true*	*false*	*false*	*false*
false	*true*	*false*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*true*	*true*	*true*

Propositional Logic: Implication (1)

- Written as $p \Rightarrow q$ [pronounced as "p implies q"]
  - We call p the antecedent, assumption, or premise.
  - We call q the consequence or conclusion.
- Compare the *truth* of $p \Rightarrow q$ to whether a contract is *honoured*:
  - antecedent/assumption/premise $p \approx$ promised terms [e.g., salary]
  - consequence/conclusion $q \approx$ obligations [e.g., duties]
- When the promised terms are met, then the contract is:
  - *honoured* if the obligations fulfilled. [$(true \Rightarrow true) \iff true$]
  - *breached* if the obligations violated. [$(true \Rightarrow false) \iff false$]
- When the promised terms are not met, then:
  - Fulfilling the obligation (q) or not ($\neg q$) does *not breach* the contract.

p	q	$p \Rightarrow q$
false	*true*	*true*
false	*false*	*true*

Propositional Logic: Implication (2)

There are alternative, equivalent ways to expressing $p \Rightarrow q$:

- q **if** p

  q is *true* if p is *true*

- p **only if** q

  If p is *true*, then for $p \Rightarrow q$ to be *true*, it can only be that q is also *true*.
  Otherwise, if p is *true* but q is *false*, then $(true \Rightarrow false) \equiv false$.

  Note. To prove $p \equiv q$, prove $p \iff q$ (pronounced: "p if and only if q"):

  - p **if** q [$p \Leftarrow q \equiv q \Rightarrow p$]
  - p **only if** q [$p \Rightarrow q$]

- p is **sufficient** for q [similar to q **if** p]

  For q to be *true*, it is sufficient to have p being *true*.

- q is **necessary** for p [similar to p **only if** q]

  If p is *true*, then it is necessarily the case that q is also *true*.
  Otherwise, if p is *true* but q is *false*, then $(true \Rightarrow false) \equiv false$.

- q **unless** $\neg p$ [When is $p \Rightarrow q$ *true*?]

  If q is *true*, then $p \Rightarrow q$ *true* regardless of p.
  If q is *false*, then $p \Rightarrow q$ cannot be *true* unless p is *false*.

Propositional Logic: Implication (3)

Given an implication $p \Rightarrow q$, we may construct its:

- **Inverse**: $\neg p \Rightarrow \neg q$ [negate antecedent and consequence]
- **Converse**: $q \Rightarrow p$ [swap antecedent and consequence]
- **Contrapositive**: $\neg q \Rightarrow \neg p$ [inverse of converse]

Propositional Logic (2)

- **Axiom**: Definition of $\Rightarrow$

$$p \Rightarrow q \equiv \neg p \vee q$$

- **Theorem**: Identity of $\Rightarrow$

$$true \Rightarrow p \equiv p$$

- **Theorem**: Zero of $\Rightarrow$

$$false \Rightarrow p \equiv true$$

- **Axiom**: De Morgan

$$\begin{array}{rcl} \neg(p \wedge q) & \equiv & \neg p \vee \neg q \\ \neg(p \vee q) & \equiv & \neg p \wedge \neg q \end{array}$$

- **Axiom**: Double Negation

$$p \equiv \neg\,(\neg\, p)$$

- **Theorem**: Contrapositive

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

Predicate Logic (1)

- A *predicate* is a ***universal*** or ***existential*** statement about objects in some universe of disclosure.
- Unlike propositions, predicates are typically specified using ***variables***, each of which declared with some ***range*** of values.
- We use the following symbols for common numerical ranges:
  - $\mathbb{Z}$: the set of integers $[-\infty, \dots, -1, 0, 1, \dots, +\infty]$
  - $\mathbb{N}$: the set of natural numbers $[0, 1, \dots, +\infty]$
- Variable(s) in a predicate may be ***quantified***:
  - *Universal quantification* :
    All values that a variable may take satisfy certain property.
    e.g., Given that *i* is a natural number, *i* is *always* non-negative.
  - *Existential quantification* :
    Some value that a variable may take satisfies certain property.
    e.g., Given that *i* is an integer, *i* *can be* negative.

Predicate Logic (2.1): Universal Q. ($\forall$)

- A *universal quantification* has the form $(\forall X \bullet R \Rightarrow P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***For all*** (combinations of) values of variables listed in X that satisfies R, it is the case that P is satisfied.
  - $\forall i \bullet i \in \mathbb{N} \Rightarrow i \geq 0$ [*true*]
  - $\forall i \bullet i \in \mathbb{Z} \Rightarrow i \geq 0$ [*false*]
  - $\forall i, j \bullet i \in \mathbb{Z} \land j \in \mathbb{Z} \Rightarrow i < j \lor i > j$ [*false*]
- *Proof Strategies*
  1. How to prove $(\forall X \bullet R \Rightarrow P)$ ***true***?
     - **Hint.** When is $R \Rightarrow P$ ***true***? [*true* $\Rightarrow$ *true*, *false* $\Rightarrow$ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.
  2. How to prove $(\forall X \bullet R \Rightarrow P)$ ***false***?
     - **Hint.** When is $R \Rightarrow P$ ***false***? [*true* $\Rightarrow$ *false*]
     - Give a ***witness/counterexample*** of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.

Predicate Logic (2.2): Existential Q. ($\exists$)

- An *existential quantification* has the form $(\exists X \bullet R \land P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***There exist*** (a combination of) values of variables listed in X that satisfy both R and P.
  - $\exists i \bullet i \in \mathbb{N} \land i \geq 0$ [*true*]
  - $\exists i \bullet i \in \mathbb{Z} \land i \geq 0$ [*true*]
  - $\exists i, j \bullet i \in \mathbb{Z} \land j \in \mathbb{Z} \land (i < j \lor i > j)$ [*true*]
- *Proof Strategies*
  1. How to prove $(\exists X \bullet R \land P)$ ***true***?
     - **Hint.** When is $R \land P$ ***true***? [*true* $\land$ *true*]
     - Give a ***witness*** of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
  2. How to prove $(\exists X \bullet R \land P)$ ***false***?
     - **Hint.** When is $R \land P$ ***false***? [*true* $\land$ *false*, *false* $\land$ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.

Predicate Logic (3): Exercises

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 0$.

  All 10 integers between 1 and 10 are greater than 0.
- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 1$.

  Integer 1 (a witness/counterexample) in the range between 1 and 10 is *not* greater than 1.
- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 1$.

  Integer 2 (a witness) in the range between 1 and 10 is greater than 1.
- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 10$?

  All integers in the range between 1 and 10 are *not* greater than 10.

Predicate Logic (4): Switching Quantifications

Conversions between $\forall$ and $\exists$:

$$
\begin{array}{rcl}
(\forall X \bullet R \Rightarrow P) & \iff & \neg(\exists X \bullet R \land \neg P) \\
(\exists X \bullet R \land P) & \iff & \neg(\forall X \bullet R \Rightarrow \neg P)
\end{array}
$$

Index (1)