Priority Queues, Heaps, and Heap Sort

EECS2101 X & Z:
Fundamentals of Data Structures
Winter 2025

CHEN-WEI WANG

Learning Outcomes of this Lecture

This module is designed to help you understand:

- When the ***Worst-Case RT*** of a ***BST Search*** Occurs
- ***Height-Balance*** Property
- The ***Priority Queue*** (***PQ***) ADT
- Time Complexities of ***List***-Based ***PQ***
- The ***Heap*** Data Structure (Properties & Operations)
- ***Heap Sort***
- Time Complexities of ***Heap***-Based ***PQ***
- ***Heap*** Construction Methods: Top-Down vs. Bottom-Up
- ***Array***-Based Representation of a ***Heap***

Balanced Binary Search Trees: Motivation

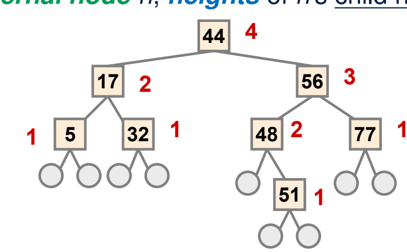
- After ***insertions*** into a BST, the ***worst-case RT*** of a ***search*** occurs when the ***height*** h is at its ***maximum***: ***O(n)*** :
  - e.g., Entries were inserted in an decreasing order of their keys
    $\langle 100, 75, 68, 60, 50, 1 \rangle$
    $\Rightarrow$ ***One-path***, ***left-slanted*** BST
  - e.g., Entries were inserted in an increasing order of their keys
    $\langle 1, 50, 60, 68, 75, 100 \rangle$
    $\Rightarrow$ ***One-path***, ***right-slanted*** BST
  - e.g., Last entry's key is in-between keys of the previous two entries
    $\langle 1, 100, 50, 75, 60, 68 \rangle$
    $\Rightarrow$ ***One-path***, ***side-alternating*** BST
- To avoid the worst-case RT ($\because$ a ***ill-balanced tree***), we need to take actions *as soon as* the tree becomes ***unbalanced***.

Balanced Binary Search Trees: Definition

- Given a node p, the ***height*** of the subtree rooted at p is:

$$\textit{height}(p) = \begin{cases} 0 & \text{if } p \text{ is } \textit{external} \\ 1 + \textbf{MAX}\ (\{\ \textit{height}(c) \mid \textit{parent}\ (c) = p\ \}) & \text{if } p \text{ is } \textit{internal} \end{cases}$$

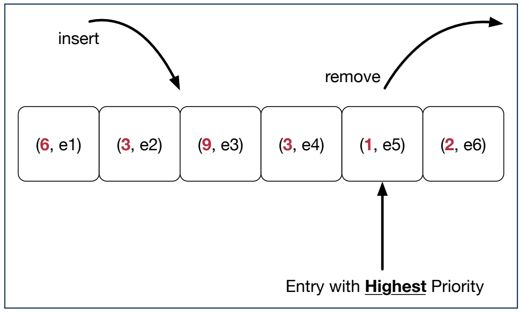
- A ***balanced*** BST T satisfies the ***height-balance property*** :
  For every ***internal node*** n, ***heights*** of n's child nodes differ ≤ 1.



Q: Is the above tree a ***balanced BST***? ✓
Q: Will the tree remain ***balanced*** after inserting 55? ×
Q: Will the tree remain ***balanced*** after inserting 63? ✓

What is a Priority Queue?

- A ***Priority Queue (PQ)*** stores a collection of ***entries***.



  - Each ***entry*** is a pair: an ***element*** and its ***key***.
  - The ***key*** of each ***entry*** denotes its ***element***'s "priority".
  - ***Keys*** in a Priority Queue (PQ) are **not** used for uniquely identifying an entry.
- In a PQ, the next entry to remove has the "***highest***" priority.
  - e.g., In the stand-by queue of a fully-booked flight, ***frequent flyers*** get the higher priority to replace any cancelled seats.
  - e.g., A network router, faced with insufficient bandwidth, may only handle ***real-time tasks*** (e.g., streaming) with highest priorities.

The Priority Queue (PQ) ADT

- ***min***
  [***precondition***: PQ is not empty]
  [***postcondition***: return entry with **highest priority** in PQ]
- ***size***
  [***precondition***: **none**]
  [***postcondition***: return number of entries inserted to PQ]
- ***isEmpty***
  [***precondition***: **none**]
  [***postcondition***: return whether there is no entry in PQ]
- ***insert(k, v)***
  [***precondition***: PQ is not full]
  [***postcondition***: insert the input entry into PQ]
- ***removeMin***
  [***precondition***: PQ is not empty]
  [***postcondition***: remove and return a **min** entry in PQ]

Two List-Based Implementations of a PQ

Consider two strategies for implementing a ***PQ***, where we maintain:

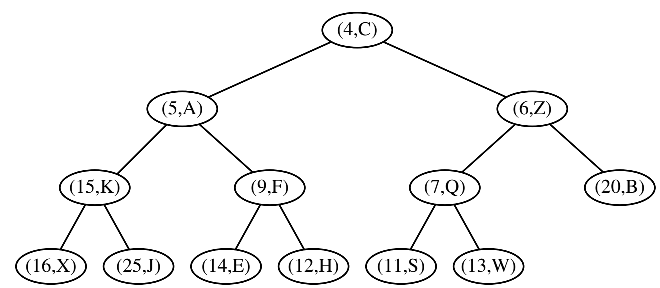
1. A list ***always sorted*** in a non-descending order [≈ **INSERTIONSORT**]
2. An ***unsorted*** list [≈ **SELECTIONSORT**]

PQ Method	**List Method**			
	SORTED LIST		UNSORTED LIST	
size	list.size ***O(1)***			
isEmpty	list.isEmpty ***O(1)***			
min	list.first	***O(1)***	search min	***O(n)***
insert	insert to "right" spot	***O(n)***	insert to front	***O(1)***
removeMin	list.removeFirst	***O(1)***	search min and remove	***O(n)***

Heaps

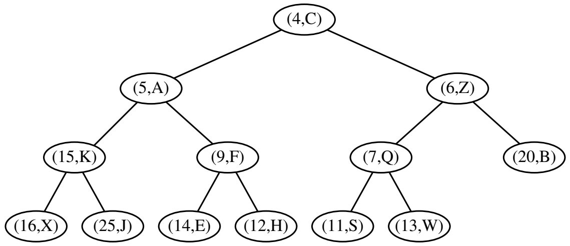
A ***heap*** is a ***binary tree*** which:

1. Stores in each node an ***entry*** (i.e., ***key*** and ***value***).



2. Satisfies a ***relational*** property of stored **keys**
3. Satisfies a ***structural*** property of tree **organization**

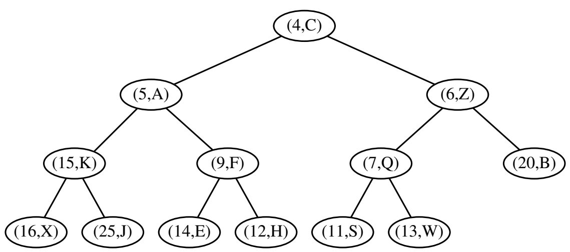
Heap Property 1: Relational



Keys in a ***heap*** satisfy the ***Heap-Order Property*** :

- Every node n (other than the root) is s.t. $key(n) \geq key(\textbf{parent}(n))$
  - ⇒ ***Keys*** in a ***root-to-leaf path*** are sorted in a non-descending order.
    e.g., Keys in entry path $\langle (4, C), (5, A), (9, F), (14, E) \rangle$ are sorted.
  - ⇒ The ***minimal key*** is stored in the ***root***.
    e.g., Root $(4, C)$ stores the minimal key 4.
- ***Keys*** of nodes from ***different subtrees*** are **not** constrained at all.
  e.g., For node $(5, A)$, key of its ***LST***'s root (15) is not minimal for its ***RST***.

Heap Property 2: Structural



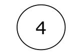
A ***heap*** with ***height h*** satisfies the ***Complete BT Property*** :

- Nodes with ***depth*** $\leq h - 2$ has two child nodes.
- Nodes with ***depth h - 1*** may have zero, one, or two child nodes.
- Nodes with ***depth h*** are filled from left to right.

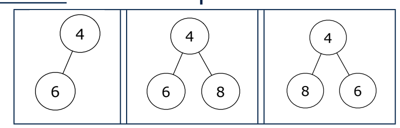
Q. When the # of nodes is n, what is h? $\lfloor \log_2 n \rfloor$
Q. # of nodes from Level 0 through Level $h - 1$? $2^h - 1$
Q. # of nodes at Level h? $n - (2^h - 1)$
Q. **Minimum** # of nodes of a complete BT? 2^h
Q. **Maximum** # of nodes of a complete BT? $2^{h+1} - 1$

Heaps: More Examples

- The ***smallest*** ***heap*** is just an empty binary tree.
- The ***smallest*** **non-empty** ***heap*** is a one-node heap.
  e.g.,



- Two-node and Three-node Heaps:



- These are **not** two-node heaps:

relational property violated | ***structural*** property violated

Heap Operations

- There are three main operations for a ***heap*** :
  1. **Extract the Entry with Minimal Key**:
     Return the stored entry of the ***root***. [$O(1)$]
  2. **Insert a New Entry**:
     A single ***root-to-leaf path*** is affected. [$O(h)$ or $O(\log n)$]
  3. **Delete the Entry with Minimal Key**:
     A single ***root-to-leaf path*** is affected. [$O(h)$ or $O(\log n)$]
- After performing each operation,
  both ***relational*** and ***structural*** properties must be maintained.

Updating a Heap: Insertion

To insert a new entry (k, v) into a heap with ***height h***:

1. Insert (k, v), possibly **temporarily** breaking the ***relational property***.

- **1.1** Create a new entry **e** = (k, v).
- **1.2** Create a new ***right-most*** node ***n*** at ***Level h***.
- **1.3** Store entry **e** in node ***n***.
  After steps **1.1** and **1.2**, the ***structural property*** is maintained.

2. Restore the **heap-order property** (**HOP**) using *Up-Heap Bubbling* :

- **2.1** Let ***c*** = ***n***.
- **2.2** While **HOP** is not restored and ***c*** is not the root:
  - **2.2.1** Let ***p*** be ***c***'s parent.
  - **2.2.2** **If** *key*(***p***) ≤ *key*(***c***), then **HOP** is restored.
    Else, swap nodes ***c*** and ***p***. ["upwards" along ***n***'s ***ancestor path***]

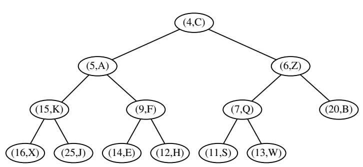
Running Time?

- All sub-steps in **1**, as well as steps **2.1**, **2.2.1**, and **2.2.2** take *O(1)*.
- Step **2.2** may be executed up to *O(h)* (or *O(log n)*) times.

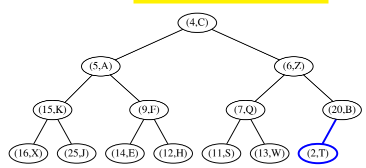
[*O(log n)*]

Updating a Heap: Insertion Example (1.1)

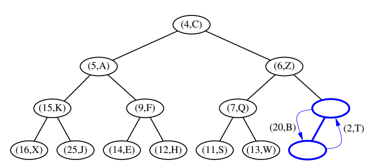
(0) A heap with height 3.



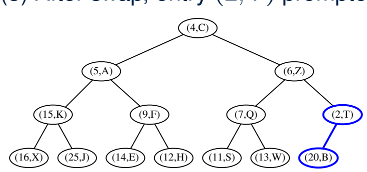
(1) Insert a new entry $(2, T)$ as the ***right-most*** node at Level 3. Perform *up-heap bubbling* from here.



(2) **HOP** violated ∵ 2 < 20 ∴ Swap.

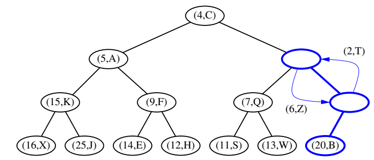


(3) After swap, entry $(2, T)$ prompted up.

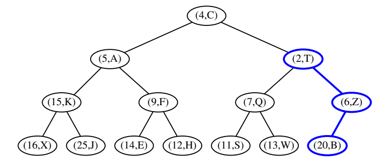


Updating a Heap: Insertion Example (1.2)

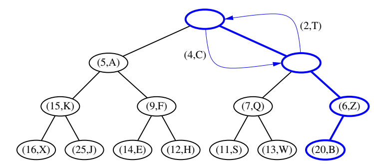
(4) **HOP** violated ∵ 2 < 6 ∴ Swap.



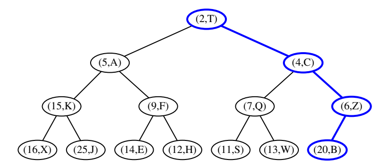
(5) After swap, entry $(2, T)$ prompted up.



(6) **HOP** violated ∵ 2 < 4 ∴ Swap.



(7) Entry $(2, T)$ becomes root ∴ Done.



Updating a Heap: Deletion

To delete the **root** (with the ***minimal*** key) from a heap with ***height h***:

1. Delete the **root**, possibly **temporarily** breaking **HOP**.

- **1.1** Let the ***right-most*** node at ***Level h*** be ***n***.
- **1.2** Replace the **root**'s entry by ***n***'s entry.
- **1.3** Delete ***n***.
  After steps **1.1** – **1.3**, the ***structural property*** is maintained.

2. Restore **HOP** using *Down-Heap Bubbling* :

- **2.1** Let ***p*** be the **root**.
- **2.2** While **HOP** is not restored and ***p*** is not external:
  - **2.2.1** **IF** ***p*** has no **right child**, let ***c*** be ***p***'s ***left child***.
    Else, let ***c*** be ***p***'s child with a ***smaller key value***.
  - **2.2.2** **If** *key*(***p***) ≤ *key*(***c***), then **HOP** is restored.
    Else, swap nodes ***p*** and ***c***. ["downwards" along a ***root-to-leaf path***]

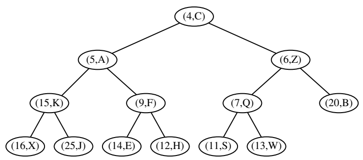
Running Time?

- All sub-steps in **1**, as well as steps **2.1**, **2.2.1**, and **2.2.2** take *O(1)*.
- Step **2.2** may be executed up to *O(h)* (or *O(log n)*) times.

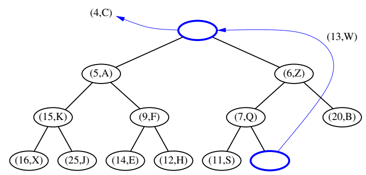
[*O(log n)*]

Updating a Heap: Deletion Example (1.1)

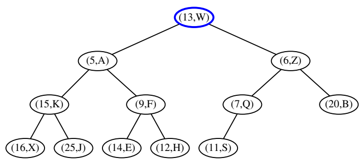
(0) Start with a heap with height 3.



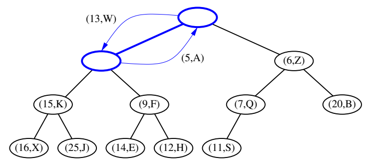
(1) Replace root with $(13, W)$ and delete ***right-most*** node from Level 3.



(2) $(13, W)$ becomes the root. Perform *down-heap bubbling* from here.

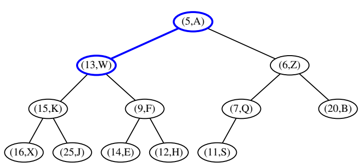


(3) Child with smaller key is $(5, A)$. **HOP** violated $\because$ 13 > 5 $\therefore$ Swap.

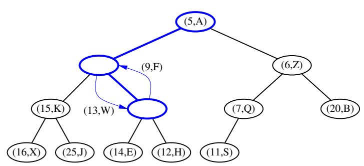


Updating a Heap: Deletion Example (1.2)

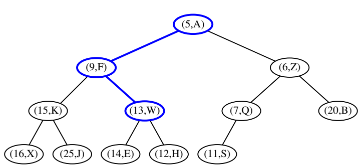
(4) After swap, entry $(13, W)$ demoted down.



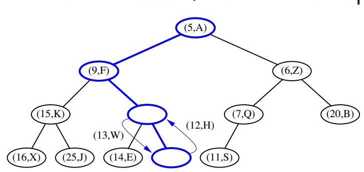
(5) Child with smaller key is $(9, F)$. **HOP** violated $\because$ 13 > 9 $\therefore$ Swap.



(6) After swap, entry $(13, W)$ demoted down.

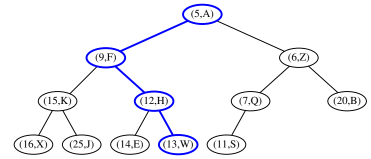


(7) Child with smaller key is $(12, H)$. **HOP** violated $\because$ 13 > 12 $\therefore$ Swap.



Updating a Heap: Deletion Example (1.3)

(8) After swap, entry $(13, W)$ becomes an external node $\therefore$ Done.



Heap-Based Implementation of a PQ

PQ Method	Heap Operation	RT
min	root	*O(1)*
insert	insert then up-heap bubbling	*O(log n)*
removeMin	delete then down-heap bubbling	*O(log n)*

Top-Down Heap Construction: List of Entries is Not Known in Advance

Problem: Build a heap out of N entires, supplied one at a time.

- Initialize an ***empty heap h***. [*O(1)*]
- As each new entry **e** = (k, v) is supplied, **insert** **e** into ***h***.
  - Each insertion triggers an ***up-heap bubbling*** step, which takes ***O(log n)*** time. [$n = 0, 1, 2, \ldots, N - 1$]
  - There are N insertions.

∴ Running time is $O(N \cdot \log N)$

Bottom-Up Heap Construction: List of Entries is Known in Advance

Problem: Build a heap out of N entires, supplied all at once.

- **Assume**: The resulting heap will be ***completely filled*** at **all** levels.
  $\Rightarrow N = 2^{h+1} - 1$ for some ***height*** $h \geq 1$ [$h = (\log(N + 1)) - 1$]
- Perform the following steps called *Bottom-Up Heap Construction*:

  Step 1: Treat the first $\frac{N+1}{2^1}$ list entries as heap roots.
  ∴ $\frac{N+1}{2^1}$ heaps with height 0 and size $2^1 - 1$ constructed.

  Step 2: Treat the next $\frac{N+1}{2^2}$ list entries as heap roots.
  - ◇ Each ***root*** sets two heaps from **Step 1** as its ***LST*** and ***RST***.
  - ◇ Perform ***down-heap bubbling*** to restore **HOP** if necessary.

  ∴ $\frac{N+1}{2^2}$ heaps, each with height 1 and size $2^2 - 1$, constructed.

  ...

  Step $h + 1$: Treat next $\frac{N+1}{2^{h+1}} = \frac{(2^{h+1}-1)+1}{2^{h+1}} = 1$ list entry as heap root.
  - ◇ Each ***root*** sets two heaps from **Step h** as its ***LST*** and ***RST***.
  - ◇ Perform ***down-heap bubbling*** to restore **HOP** if necessary.

  ∴ $\frac{N+1}{2^{h+1}} = 1$ heap, each with height h and size $2^{h+1} - 1$, constructed.

Bottom-Up Heap Construction: Example (1.1)

- Build a heap from the following list of 15 keys:

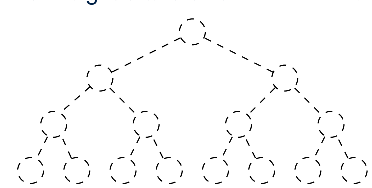
$$\langle 16, 15, 4, 12, 6, 7, 23, 20, 25, 9, 11, 17, 5, 8, 14 \rangle$$

- The resulting heap has:
  - ***Size N*** is 15
  - ***Height h*** is $(\log(15 + 1)) - 1 = 3$
- According to the ***bottom-up heap construction*** technique, we will need to perform $h + 1 = 4$ steps, utilizing 4 sublists:

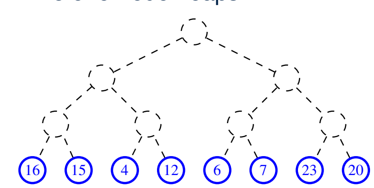
$$\langle \underbrace{16, 15, 4, 12, 6, 7, 23, 20}_{\frac{15+1}{2^1} = 8}, \underbrace{25, 9, 11, 17}_{\frac{15+1}{2^2} = 4}, \underbrace{5, 8}_{\frac{15+1}{2^3} = 2}, \underbrace{14}_{\frac{15+1}{2^4} = 1} \rangle$$

Bottom-Up Heap Construction: Example (1.2)

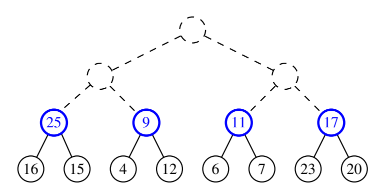
We know in advance to build a heap with height 3 and size $2^{3+1} - 1 = 15$



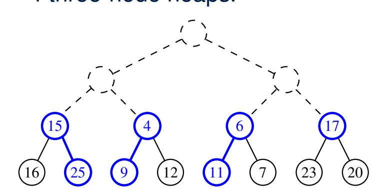
(**Step 1**) Treat first $\frac{15+1}{2^1}$ entries as roots.
∴ 8 one-node heaps.



(**Step 2**) Treat next $\frac{15+1}{2^2}$ entries as roots.
Set LST and RST of each root.

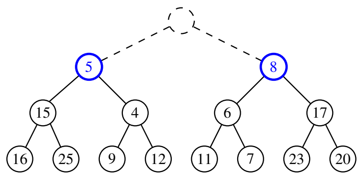


(**Step 2** cont.) Down-heap bubbling.
∴ 4 three-node heaps.

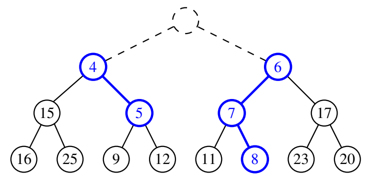


Bottom-Up Heap Construction: Example (1.3)

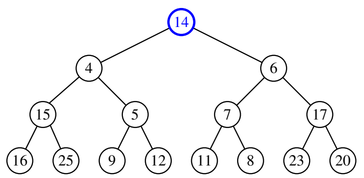
(**Step 3**) Treat next $\frac{15+1}{2^3}$ entries as roots.
Set LST and RST of each root.



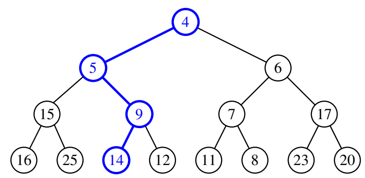
(**Step 3** cont.) Down-heap bubbling.
∴ 2 three-node heaps.



(**Step 4**) Treat next $\frac{15+1}{2^4}$ entry as roots.
Set LST and RST of each root.



(**Step 4** cont.) Down-heap bubbling.
∴ 1 fifteen-node heap.



RT of Bottom-Up Heap Construction

- Intuitively, the majority of the intermediate roots from which we perform ***down-heap bubbling*** are of very ***small height values***:
  - The first $\frac{n+1}{2}$ *1*-node heaps with ***height 0*** require **no** down-heap bubbling.
    [About 50% of the list entries processed]
  - Next $\frac{n+1}{4}$ *3*-node heaps with ***height 1*** require down-heap bubbling.
    [Another 25% of the list entries processed]
  - Next $\frac{n+1}{8}$ *7*-node heaps with ***height 2*** require down-heap bubbling.
    [Another 12.5% of the list entries processed]
  - . . .
  - Next two $\frac{N-1}{2}$-node heaps with ***height (h - 1)*** require down-heap
  - Final one ***N***-node heaps with ***height h*** requires down-heap bubbling.
- Running Time of the ***Bottom-Up Heap Construction*** takes only $O(n)$.

The Heap Sort Algorithm

Sorting Problem:

Given a list of **n** numbers $\langle a_1, a_2, \ldots, a_n \rangle$:
Precondition: NONE
Postcondition: A permutation of the input list $\langle a'_1, a'_2, \ldots, a'_n \rangle$ sorted in a non-descending order (i.e., $a'_1 \le a'_2 \le \ldots \le a'_n$)

The *Heap Sort* algorithm consists of two phases:

1. ***Construct*** a ***heap*** of size ***N*** out of the input array.
   - Approach 1: Top-Down "Continuous-Insertions" [$O(N \cdot \log N)$]
   - Approach 2: Bottom-Up Heap Construction [$O(N)$]
2. ***Delete N*** entries from the heap.
   - Each deletion takes $O(\log N)$ time.
   - 1st deletion extracts the ***minimum***, 2nd deletion the 2nd ***minimum***, . . .
     ⇒ Extracted ***minimums*** from ***N*** deletions form a *sorted* sequence.

∴ Running time of the *Heap Sort* algorithm is $O(N \cdot \log N)$.

The Heap Sort Algorithm: Exercise

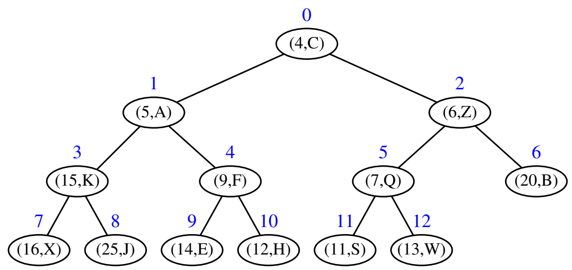
Sort the following array of integers

$$\langle 16, 15, 4, 12, 6, 7, 23, 20, 25, 9, 11, 17, 5, 8, 14 \rangle$$

into a ***non-descending*** order using the *Heap Sort Algorithm*.
Demonstrate:

1. Both top-down and bottom-up heap constructions in Phase 1
2. Extractions of minimums in Phase 2

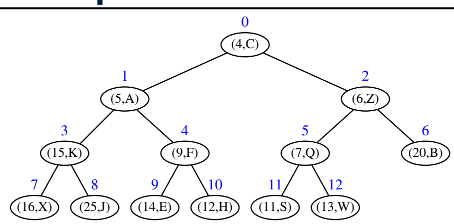
Array-Based Representation of a CBT (1)



$$index(x) = \begin{cases} 0 & \texttt{if } x \texttt{ is the root} \\ 2 \cdot index(\ parent(x)\ ) + 1 & \texttt{if } x \texttt{ is a left child} \\ 2 \cdot index(\ parent(x)\ ) + 2 & \texttt{if } x \texttt{ is a right child} \end{cases}$$

(4,C)	(5,A)	(6,Z)	(15,K)	(9,F)	(7,Q)	(20,B)	(16,X)	(25,J)	(14,E)	(12,H)	(11,S)	(13,W)
0	1	2	3	4	5	6	7	8	9	10	11	12

Array-Based Representation of a CBT (2)



(4,C)	(5,A)	(6,Z)	(15,K)	(9,F)	(7,Q)	(20,B)	(16,X)	(25,J)	(14,E)	(12,H)	(11,S)	(13,W)
0	1	2	3	4	5	6	7	8	9	10	11	12

- **Q1:** Where are nodes at ***Levels*** $0 .. h-1$ stored in the array?
  Indices $0 .. (2^h - 2) \equiv 0 .. (2^{\lfloor log_2 N \rfloor} - 2)$ [e.g., Indices $0 .. 2^3 - 2$]
- **Q2:** Where are nodes at ***Level*** h stored in the array?
  Indices $2^h - 1 .. (N-1) \equiv 2^{\lfloor log_2 N \rfloor} - 1 .. (N-1)$ [e.g., Indices 7 .. 12]
- **Q3:** How do we determine if a non-root node x is a ***left or right child***?
  IF $index(x)\ \%\ 2 == 1$ **THEN** *left* **ELSE** *right*
- **Q4:** Given a non-root node x, how do we determine the ***index of x's parent***?
  IF $index(x)\ \%\ 2 == 1$ **THEN** $\frac{index(x)-1}{2}$ **ELSE** $\frac{index(x)-2}{2}$

Index (1)

Learning Outcomes of this Lecture
Balanced Binary Search Trees: Motivation
Balanced Binary Search Trees: Definition
What is a Priority Queue?
The Priority Queue (PQ) ADT
Two List-Based Implementations of a PQ
Heaps
Heap Property 1: Relational
Heap Property 2: Structural
Heaps: More Examples
Heap Operations

Index (2)

Updating a Heap: Insertion
Updating a Heap: Insertion Example (1.1)
Updating a Heap: Insertion Example (1.2)
Updating a Heap: Deletion
Updating a Heap: Deletion Example (1.1)
Updating a Heap: Deletion Example (1.2)
Updating a Heap: Deletion Example (1.3)
Heap-Based Implementation of a PQ
Top-Down Heap Construction:
List of Entries is Not Known in Advance
Bottom-Up Heap Construction:
List of Entries is Known in Advance

Index (3)

Bottom-up Heap Construction: Example (1.1)
Bottom-up Heap Construction: Example (1.2)
Bottom-up Heap Construction: Example (1.3)
RT of Bottom-up Heap Construction
The Heap Sort Algorithm
The Heap Sort Algorithm: Exercise
Array-Based Representation of a CBT (1)
Array-Based Representation of a CBT (2)