Asymptotic Analysis of Algorithms

EECS2101 X & Z:
Fundamentals of Data Structures
Winter 2025

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What You're Assumed to Know

- You will be required to ***implement*** Java classes and methods, and to ***test*** their correctness using JUnit.

  Review them if necessary:

  `https://www.eecs.yorku.ca/~jackie/teaching/lectures/index.html#EECS2030_F21`

  - Implementing classes and methods in Java [Weeks 1 – 2]
  - Testing methods in Java [Week 4]
- Also, make sure you know how to trace programs using a ***debugger***:

  `https://www.eecs.yorku.ca/~jackie/teaching/tutorials/index.html#java_from_scratch_w21`

  - Debugging actions (Step Over/Into/Return) [Parts C – E, Week 2]

Learning Outcomes

This module is designed to help you learn about:

- Notions of ***Algorithms*** and ***Data Structures***
- Measurement of the “goodness” of an algorithm
- Measurement of the ***efficiency*** of an algorithm
- Experimental measurement vs. ***Theoretical*** measurement
- Understand the purpose of *asymptotic* analysis.
- Understand what it means to say two algorithms are:
  - equally efficient, **asymptotically**
  - one is more efficient than the other, **asymptotically**
- Given an algorithm, determine its ***asymptotic upper bound*** .

Algorithm and Data Structure

- A *data structure* is:
  - A systematic way to store and organize data in order to facilitate ***access*** and ***modifications***
  - Never suitable for all purposes: it is important to know its ***strengths*** and ***limitations***
- A **well-specified** *computational problem* precisely describes the desired ***input/output relationship***.
  - **Input:** A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$
  - **Output:** A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$
  - An ***instance*** of the problem: $\langle 3, 1, 2, 5, 4 \rangle$
- An *algorithm* is:
  - A solution to a **well-specified** computational problem
  - A **sequence of computational steps** that takes value(s) as ***input*** and produces value(s) as ***output***
- An ***algorithm*** manipulates some chosen ***data structure(s)***.

Measuring “Goodness” of an Algorithm

1. ***Correctness*** :
   - Does the ***algorithm*** produce the **expected** output?
   - Use ***unit & regression testing*** (e.g., JUnit) to ensure this.
2. Efficiency:
   - ***Time Complexity***: processor time required to complete
   - ***Space Complexity***: memory space required to store data

Correctness is always the priority.

How about efficiency? Is time or space more of a concern?

Measuring Efficiency of an Algorithm

- ***Time*** is more of a concern than is ***storage***.
- Solutions (run on computers) should be ***as fast as possible***.
- Particularly, we are interested in how ***running time*** depends on two ***input factors*** :
  1. ***size***
     e.g., sorting an array of 10 elements vs. 1m elements
  2. ***structure***
     e.g., sorting an already-sorted array vs. a hardly-sorted array

Q. How does one determine the ***running time*** of an algorithm?

1. Measure time via ***experiments***
2. Characterize time as a ***mathematical function*** of the input size

Measure Running Time via Experiments

- Once the algorithm is implemented (e.g., in Java):
  - Execute program on ***test inputs*** of various ***sizes*** & ***structures***.
  - For each test, record the ***elapsed time*** of the execution.

```
long startTime = System.currentTimeMillis();
/* run the algorithm */
long endTime = System.currenctTimeMillis();
long elapsed = endTime - startTime;
```

  - ***Visualize*** the result of each test.
- To make ***sound statistical claims*** about the algorithm's ***running time***, the set of ***test inputs*** should be "***complete***".

  e.g., To experiment with the ***RT*** of a sorting algorithm:
  - **Unreasonable**: **only** consider small-sized and/or almost-sorted arrays
  - **Reasonable**: **also** consider large-sized, randomly-organized arrays

Example Experiment

- ***Computational Problem***:
  - **Input**: A character *c* and an integer *n*
  - **Output**: A string consisting of *n* repetitions of character *c*
    e.g., Given input `'*'` and `15`, output `***************`.
- ***Algorithm 1*** using `String` Concatenations:

```
public static String repeat1(char c, int n) {
  String answer = "";
  for (int i = 0; i < n; i ++) { answer += c; }
  return answer; }
```

- ***Algorithm 2*** using `append` from `StringBuilder`:

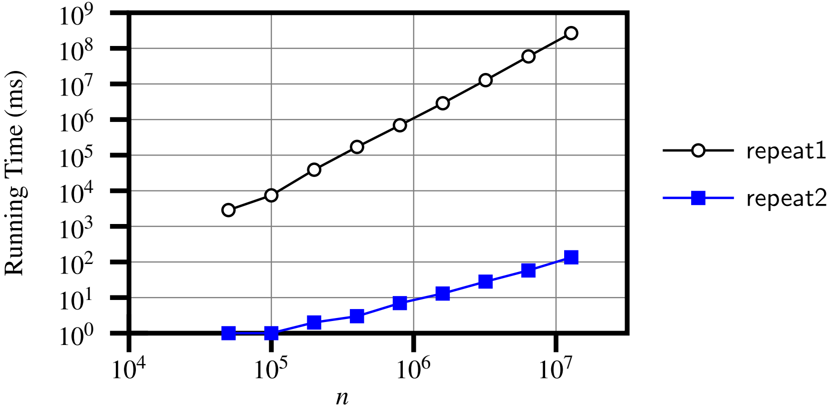
```
public static String repeat2(char c, int n) {
  StringBuilder sb = new StringBuilder();
  for (int i = 0; i < n; i ++) { sb.append(c); }
  return sb.toString(); }
```

Example Experiment: Detailed Statistics

n	`repeat1` (in ms)	`repeat2` (in ms)
50,000	2,884	1
100,000	7,437	1
200,000	39,158	2
400,000	170,173	3
800,000	690,836	7
1,600,000	2,847,968	13
3,200,000	12,809,631	28
6,400,000	59,594,275	58
12,800,000	265,696,421 ($\approx$ 3 days)	135

- As ***input size*** is doubled, ***rates of increase*** for both algorithms are ***linear***:
  - ***Running time*** of `repeat1` increases by $\approx$ 5 times.
  - ***Running time*** of `repeat2` increases by $\approx$ 2 times.

Example Experiment: Visualization



Experimental Analysis: Challenges

1. An algorithm must be ***fully implemented*** (e.g., in Java) in order study its runtime behaviour **experimentally**.
   - What if our purpose is to ***choose among alternative*** data structures or algorithms to implement?
   - Can there be a *higher-level analysis* to determine that one algorithm or data structure is more "**superior**" than others?
2. Comparison of multiple algorithms is only ***meaningful*** when experiments are conducted under the **same** working environment of:
   - ***Hardware***: CPU, running processes
   - ***Software***: OS, JVM version, Version of Compiler
3. Experiments can be done only on ***a limited set of test inputs***.
   - What if ***worst-case*** inputs were **not** included in the experiments?
   - What if "***important***" inputs were **not** included in the experiments?

Moving Beyond Experimental Analysis

- A better approach to analyzing the ***efficiency*** (e.g., ***running time***) of algorithms should be one that:
  - Can be applied using a *high-level description* of the algorithm (**without** fully implementing it).
    [e.g., Pseudo Code, Java Code (with "tolerances")]
  - Allows us to calculate the *relative efficiency* (rather than absolute elapsed time) of algorithms in a way that is ***independent of*** the hardware and software environment.
  - Considers *all* possible inputs (esp. the ***worst-case scenario***).
- We will learn a better approach that contains 3 ingredients:
  1. Counting ***primitive operations***
  2. Approximating running time as ***a function of input size***
  3. Focusing on the ***worst-case*** input (requiring most running time)

Counting Primitive Operations

- A ***primitive operation*** (***POs***) corresponds to a low-level instruction with a ***constant** execution time*.
  - (Variable) Assignment [e.g., `x = 5;`]
  - Indexing into an array [e.g., `a[i]`]
  - Arithmetic, relational, logical op. [e.g., `a + b`, `z > w`, `b1 && b2`]
  - Accessing an attribute of an object [e.g., `acc.balance`]
  - Returning from a method [e.g., `return result;`]

  Q: Is a ***method call*** a primitive operation?

  A: **Not** in general. It may be a call to:
  - a "***cheap***" method (e.g., printing `Hello World`), or
  - an "***expensive***" method (e.g., sorting an array of integers)
- ***RT*** of an ***algorithm*** is approximated as the number of ***POs*** involved (**despite** the execution environment).

Example: Counting Primitive Operations (1)

```
1 int findMax (int[] a, int n) {
2   currentMax = a[0];
3   for (int i = 1; i < n; ) {
4     if (a[i] > currentMax) {
5       currentMax = a[i]; }
6     i ++ }
7   return currentMax; }
```

# of times `i < n` in **Line 3** is executed? [n]

# of times the loop body (**Line 4** to **Line 6**) is executed? [$n - 1$]

- **Line 2**: 2 [1 indexing + 1 assignment]
- **Line 3**: $n + 1$ [1 assignment + n comparisons]
- **Line 4**: $(n - 1) \cdot 2$ [1 indexing + 1 comparison]
- **Line 5**: $(n - 1) \cdot 2$ [1 indexing + 1 assignment]
- **Line 6**: $(n - 1) \cdot 2$ [1 addition + 1 assignment]
- **Line 7**: 1 [1 return]
- **Total # of Primitive Operations**: 7n - 2

Example: Counting Primitive Operations (2)

Count the number of primitive operations for

```
boolean foundEmptyString = false;
int i = 0;
while (!foundEmptyString && i < names.length) {
  if (names[i].length() == 0) {
    /* set flag for early exit */
    foundEmptyString = true;
  }
  i = i + 1;
}
```

- # times the stay condition of the `while` loop is checked?

  [between 1 and `names.length` + 1]

  [***worst case***: `names.length` + 1 times]

- # times the body code of `while` loop is executed?

  [between 0 and `names.length`]

  [***worst case***: `names.length` times]

From Absolute RT to Relative RT

- Each ***primitive operation*** (***PO***) takes approximately the same, constant amount of time to execute. [say **t**]

  The absolute value of **t** depends on the ***execution environment***.

Q. How do you relate the ***number of POs*** required by an algorithm and its ***actual RT*** on a specific working environment?

A. ***Number of POs*** should be *proportional* to the actual ***RT***.

$$\boxed{\mathit{RT} = \mathbf{t} \cdot \textit{number of POs}}$$

- e.g., `findMax (int[] a, int n)` has ***7n - 2*** POs

  $\mathit{RT} = (7n - 2) \cdot \mathbf{t}$
- e.g., Say two algorithms with ***RT*** $(7n - 2) \cdot \mathbf{t}$ and ***RT*** $(10n + 3) \cdot \mathbf{t}$:
  It suffices to compare their *relative* running time:

  7n - 2 vs. ***10n + 3***.

∴ To determine the ***time efficiency*** of an algorithm, we only focus on their ***number of POs***.

Example: Approx. # of Primitive Operations

- Given # of primitive operations counted **precisely** as $7n - 2$, we view it as

$$7 \cdot n^1 - 2 \cdot n^0$$

- We say
  - n is the ***highest power***
  - 7 and 2 are the ***multiplicative constants***
  - 2 is the ***lower term***
- When **approximating** a ***function*** [e.g., RT $\approx$ f($\boldsymbol{n}$)] (considering that ***input size*** may be very large):
  - **Only** the ***highest power*** matters.
  - ***multiplicative constants*** and ***lower terms*** can be dropped.

  $\Rightarrow$ $7n - 2$ is approximately n

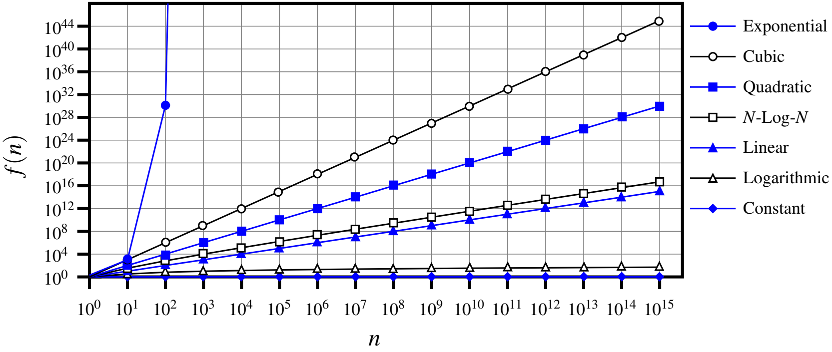
  Exercise: Consider $7n + 2n \cdot log\ n + 3n^2$:
  - ***highest power***? [n^2]
  - ***multiplicative constants***? [7, 2, 3]
  - ***lower terms***? [$7n$, $2n \cdot log\ n$]

Approximating Running Time as a Function of Input Size

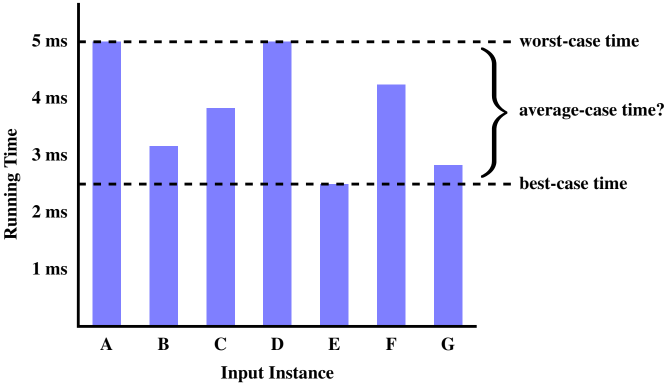
Given the ***high-level description*** of an algorithm, we associate it with a function f, such that $f(n)$ returns the ***number of primitive operations*** that are performed on an ***input of size*** n.

- $f(n) = 5$ [constant]
- $f(n) = log_2 n$ [logarithmic]
- $f(n) = 4 \cdot n$ [linear]
- $f(n) = n^2$ [quadratic]
- $f(n) = n^3$ [cubic]
- $f(n) = 2^n$ [exponential]

Rates of Growth: Comparison



Focusing on the Worst-Case Input



- ***Average-case*** analysis calculates the *expected running time* based on the probability distribution of input values.
- ***worst-case*** analysis or ***best-case*** analysis?

What is Asymptotic Analysis?

Asymptotic analysis

- Is a method of describing ***behaviour towards the limit***:
  - How the ***running time*** of the algorithm under analysis changes as the ***input size*** changes **without** bound
  - e.g., Contrast: $RT_1(n) = n$ vs. $RT_2(n) = n^2$
- Allows us to compare the ***relative performance*** of alternative algorithms:
  - For large enough inputs, the multiplicative constants and lower-order terms of an exact running time can be disregarded.
  - e.g., $RT_1(n) = 3n^2 + 7n + 18$ and $RT_1(n) = 100n^2 + 3n - 100$ are considered **equally efficient**, ***asymptotically***.
  - e.g., $RT_1(n) = n^3 + 7n + 18$ is considered **less efficient** than $RT_1(n) = 100n^2 + 100n + 2000$, ***asymptotically***.

Three Notions of Asymptotic Bounds

We may consider three kinds of ***asymptotic bounds*** for the ***running time*** of an algorithm:

- Asymptotic *upper* bound [O]
- Asymptotic lower bound [Ω]
- Asymptotic tight bound [Θ]

Asymptotic Upper Bound: Definition

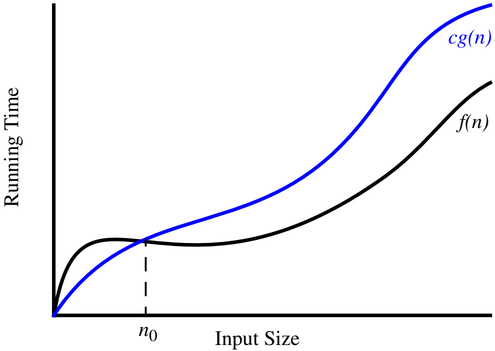
- Let ***f(n)*** and ***g(n)*** be functions mapping pos. integers (input size) to pos. real numbers (running time).
  - ***f(n)*** characterizes the running time of some algorithm.
  - ***O(g(n))***:
    - denotes a collection of functions
    - consists of all functions that can be ***upper bounded by g(n)***, starting at some point, using some constant factor
- ***f(n)*** $\in$ ***O(g(n))*** if there are:
  - A real ***constant*** $c > 0$
  - An integer ***constant*** $n_0 \geq 1$

  such that:

$$f(n) \leq c \cdot g(n) \quad \text{for } n \geq n_0$$

- For each member function ***f(n)*** in ***O(g(n))***, we say that:
  - $f(n) \in O(g(n))$ [f(n) is a member of "big-O of g(n)"]
  - $f(n)$ **is** $O(g(n))$ [f(n) is "big-O of g(n)"]
  - $f(n)$ **is order of** $g(n)$

Asymptotic Upper Bound: Visualization



From n_0, ***f(n)*** is ***upper bounded by*** $c \cdot$ ***g(n)***, so ***f(n)*** is $O(g(n))$.

Asymptotic Upper Bound: Example (1)

Prove: The function $8n + 5$ is $O(n)$.
Strategy: Choose a real constant $c > 0$ and an integer constant $n_0 \geq 1$, such that for every integer $n \geq n_0$:

$$8n + 5 \leq c \cdot n$$

Can we choose $c = 9$? What should the corresponding n_0 be?

n	8n + 5	9n
1	13	9
2	21	18
3	29	27
4	37	36
5	45	45
6	53	54

...

Therefore, we prove it by choosing $c = 9$ and $n_0 = 5$.
We may also prove it by choosing $c = 13$ and $n_0 = 1$. Why?

Asymptotic Upper Bound: Proposition

If $f(n)$ is a polynomial of degree d, i.e.,

$$f(n) = a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d$$

and $a_0, a_1, \ldots, a_d$ are integers, then $\boxed{f(n) \text{ is } O(n^d)}$.

- We prove by choosing

$$\begin{array}{rcl} c & = & |a_0| + |a_1| + \cdots + |a_d| \\ n_0 & = & 1 \end{array}$$

- We know that for $n \geq 1$: $n^0 \leq n^1 \leq n^2 \leq \cdots \leq n^d$
- Upper-bound effect: $n_0 = 1$? $[f(1) \leq (|a_0| + |a_1| + \cdots + |a_d|) \cdot 1^d]$

$$a_0 \cdot 1^0 + a_1 \cdot 1^1 + \cdots + a_d \cdot 1^d \leq |a_0| \cdot 1^d + |a_1| \cdot 1^d + \cdots + |a_d| \cdot 1^d$$

- Upper-bound effect holds? $[f(n) \leq (|a_0| + |a_1| + \cdots + |a_d|) \cdot n^d]$

$$a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d \leq |a_0| \cdot n^d + |a_1| \cdot n^d + \cdots + |a_d| \cdot n^d$$

Asymptotic Upper Bound: Example (2)

Prove: The function $f(n) = 5n^4 - 3n^3 + 2n^2 - 4n + 1$ is $O(n^4)$.

Strategy: Choose a real constant $c > 0$ and an integer constant $n_0 \geq 1$, such that for every integer $n \geq n_0$:

$$5n^4 + 3n^3 + 2n^2 + 4n + 1 \leq c \cdot n^4$$

Using the proven **proposition**, choose:

- $c = |5| + |-3| + |2| + |-4| + |1| = 15$
- $n_0 = 1$

Asymptotic Upper Bound: Families

- If a function $f(n)$ is ***upper bounded by*** another function $g(n)$ of degree d, $d \geq 0$, then $f(n)$ is also ***upper bounded by*** all other functions of a ***strictly higher degree*** (i.e., $d+1$, $d+2$, *etc.*).
  - e.g., Family of $O(n)$ contains all $f(n)$ that can be ***upper bounded by*** $g(n) = n^1$:

    $n, 2n, 3n, \ldots$ [functions with degree 1]
    $n^0, 2n^0, 3n^0, \ldots$ [functions with degree 0]
  - e.g., Family of $O(n^2)$ contains all $f(n)$ that can be ***upper bounded by*** $g(n) = n^2$:

    $n^2, 2n^2, 3n^2, \ldots$ [functions with degree 2]
    $n, 2n, 3n, \ldots$ [functions with degree 1]
    $n^0, 2n^0, 3n^0, \ldots$ [functions with degree 0]
- Consequently:

$$O(n^0) \subset O(n^1) \subset O(n^2) \subset \ldots$$

Using Asymptotic Upper Bound Accurately

- Use the big-O notation to characterize a function (of an algorithm's running time) ***as closely as possible***.

  For example, say $f(n) = 4n^3 + 3n^2 + 5$:
  - Recall: $O(n^3) \subset O(n^4) \subset O(n^5) \subset \ldots$
  - It is the *most accurate* to say that $f(n)$ is $O(n^3)$.
  - It is ***true***, but not very useful, to say that $f(n)$ is $O(n^4)$ and that $f(n)$ is $O(n^5)$.
  - It is ***false*** to say that $f(n)$ is $O(n^2)$, $O(n)$, or $O(1)$.
- Do **not** include ***constant factors*** and ***lower-order terms*** in the big-O notation.

  For example, say $f(n) = 2n^2$ is $O(n^2)$, do not say $f(n)$ is $O(4n^2 + 6n + 9)$.

Asymptotic Upper Bound: More Examples

- $5n^2 + 3n \cdot logn + 2n + 5$ is $O(n^2)$ $[c = 15, n_0 = 1]$
- $20n^3 + 10n \cdot logn + 5$ is $O(n^3)$ $[c = 35, n_0 = 1]$
- $3 \cdot logn + 2$ is $O(logn)$ $[c = 5, n_0 = 2]$
  - Why can't n_0 be 1?
  - Choosing $n_0 = 1$ means $\Rightarrow f(\boxed{1})$ ***is*** upper-bounded by $c \cdot log\boxed{1}$:
    - We have $f(\boxed{1}) = 3 \cdot log1 + 2$, which is 2.
    - We have $c \cdot log\boxed{1}$, which is 0.

  $\Rightarrow f(\boxed{1})$ ***is not*** upper-bounded by $c \cdot log\boxed{1}$ [Contradiction!]
- 2^{n+2} is $O(2^n)$ $[c = 4, n_0 = 1]$
- $2n + 100 \cdot logn$ is $O(n)$ $[c = 102, n_0 = 1]$

Classes of Functions

upper bound	class	cost
$O(1)$	constant	*cheapest*
$O(log(n))$	logarithmic	
$O(n)$	linear	
$O(n \cdot log(n))$	"n-log-n"	
$O(n^2)$	quadratic	
$O(n^3)$	cubic	
$O(n^k)$, $k \geq 1$	polynomial	
$O(a^n)$, $a > 1$	exponential	*most expensive*

Upper Bound of Algorithm: Example (1)

```
int maxOf (int x, int y) {
  int max = x;
  if (y > x) {
    max = y;
  }
  return max;
}
```

- # of primitive operations: 4

  2 assignments + 1 comparison + 1 return = 4
- Therefore, the running time is $\boxed{O(1)}$.
- That is, this is a *constant-time* algorithm.

Upper Bound of Algorithm: Example (2)

```
int findMax (int[] a, int n) {
  currentMax = a[0];
  for (int i = 1; i < n; ) {
    if (a[i] > currentMax) {
      currentMax = a[i]; }
    i ++ }
  return currentMax; }
```

- From last lecture, we calculated that the # of primitive operations is $7n - 2$.
- Therefore, the running time is $O(n)$.
- That is, this is a *linear-time* algorithm.

Upper Bound of Algorithm: Example (3)

```
1 boolean containsDuplicate (int[] a, int n) {
2   for (int i = 0; i < n; ) {
3     for (int j = 0; j < n; ) {
4       if (i != j && a[i] == a[j]) {
5         return true; }
6       j ++; }
7     i ++; }
8   return false; }
```

- Worst case is when we reach Line 8.
- # of primitive operations $\approx c_1 + n \cdot n \cdot c_2$, where c_1 and c_2 are some constants.
- Therefore, the running time is $O(n^2)$.
- That is, this is a *quadratic* algorithm.

Upper Bound of Algorithm: Example (4)

```
int sumMaxAndCrossProducts (int[] a, int n) {
  int max = a[0];
  for(int i = 1; i < n; i ++) {
    if (a[i] > max) { max = a[i]; }
  }
  int sum = max;
  for (int j = 0; j < n; j ++) {
    for (int k = 0; k < n; k ++) {
      sum += a[j] * a[k]; } }
  return sum; }
```

- # of primitive operations $\approx (c_1 \cdot n + c_2) + (c_3 \cdot n \cdot n + c_4)$, where c_1, c_2, c_3, and c_4 are some constants.
- Therefore, the running time is $O(n + n^2) = O(n^2)$.
- That is, this is a *quadratic* algorithm.

Upper Bound of Algorithm: Example (5)

```
int triangularSum (int[] a, int n) {
  int sum = 0;
  for (int i = 0; i < n; i ++) {
    for (int j = i; j < n; j ++) {
      sum += a[j]; } }
  return sum; }
```

- # of primitive operations $\approx n + (n-1) + \cdots + 2 + 1 = \frac{n \cdot (n+1)}{2}$
- Therefore, the running time is $O(\frac{n^2+n}{2}) = O(n^2)$.
- That is, this is a *quadratic* algorithm.

Beyond this lecture ...

- You will be required to ***implement*** Java classes and methods, and to ***test*** their correctness using JUnit.

  Review them if necessary:

  `https://www.eecs.yorku.ca/~jackie/teaching/lectures/index.html#EECS2030_F21`

  - Implementing classes and methods in Java [Weeks 1 – 2]
  - Testing methods in Java [Week 4]
- Also, make sure you know how to trace programs using a ***debugger***:

  `https://www.eecs.yorku.ca/~jackie/teaching/tutorials/index.html#java_from_scratch_w21`

  - Debugging actions (Step Over/Into/Return) [Parts C – E, Week 2]

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