Parser: Syntactic Analysis

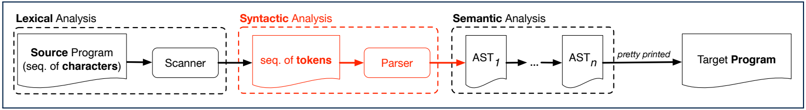
Readings: EAC2 Chapter 3

EECS4302 A:
Compilers and Interpreters
Summer 2025

CHEN-WEI WANG

Parser in Context

- Recall:



- Treats the input programas as a ***a sequence of classified tokens/words***
- Applies rules ***parsing*** token sequences as ***abstract syntax trees (ASTs)*** [***syntactic*** analysis]
- Upon termination:
  - Reports token sequences not derivable as ASTs
  - Produces an ***AST***
- No longer considers ***every character*** in input program.
- *Derivable* token sequences constitute a *context-free language (CFL)*.

Context-Free Languages: Introduction

- We have seen ***regular languages***:
  - Can be described using ***finite automata*** or ***regular expressions***.
  - Satisfy the ***pumping lemma***.
- Language with ***recursive*** structures are provably ***non-regular***.
  e.g., $\{0^n1^n \mid n \geq 0\}$
- *Context-Free Grammars (CFG's)* are used to describe strings that can be generated in a ***recursive*** fashion.
- *Context-Free Languages (CFL's)* are:
  - Languages that can be described using CFG's.
  - A proper superset of the set of regular languages.

CFG: Example (1.1)

- The following language that is ***non-regular***

$$\{0^n \# 1^n \mid n \geq 0\}$$

can be described using a ***context-free grammar (CFG)***:

$$\begin{array}{lcl} A & \rightarrow & 0A1 \\ A & \rightarrow & B \\ B & \rightarrow & \# \end{array}$$

- A grammar contains a collection of ***substitution*** or ***production*** rules, where:
  - A **terminal** is a word $w \in \Sigma^*$ (e.g., 0, 1, *etc.*).
  - A ***variable*** or ***non-terminal*** is a word $w \notin \Sigma^*$ (e.g., A, B, *etc.*).
  - A ***start variable*** occurs on the LHS of the topmost rule (e.g., A).

CFG: Example (1.2)

- Given a grammar, generate a string by:
  1. Write down the ***start variable***.
  2. Choose a production rule where the ***start variable*** appears on the LHS of the arrow, and ***substitute*** it by the RHS.
  3. There are two cases of the re-written string:
     - **3.1** It contains **no** variables, then you are done.
     - **3.2** It contains **some** variables, then ***substitute*** each variable using the relevant ***production rules***.
  4. Repeat Step 3.
- e.g., We can generate an infinite number of strings from

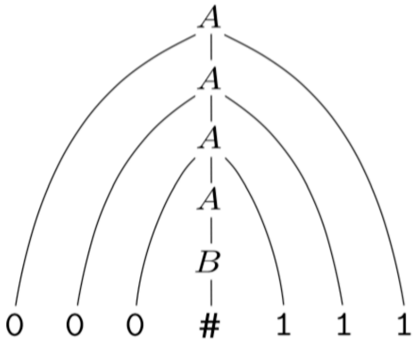
$$\begin{array}{lcl} A & \rightarrow & 0A1 \\ A & \rightarrow & B \\ B & \rightarrow & \# \end{array}$$

  - $A \Rightarrow B \Rightarrow \#$
  - $A \Rightarrow 0A1 \Rightarrow 0B1 \Rightarrow 0\#1$
  - $A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$
  - ...

CFG: Example (1.2)

Given a CFG, a string's ***derivation*** can be shown as a ***parse tree***.

e.g., The derivation of 000#111 has the parse tree



CFG: Example (2)

Design a CFG for the following language:

$$\{w \mid w \in \{0,1\}^* \wedge w \texttt{ is a palidrome}\}$$

e.g., 00, 11, 0110, 1001, *etc.*

$$\begin{array}{lcl} P & \rightarrow & \epsilon \\ P & \rightarrow & 0 \\ P & \rightarrow & 1 \\ P & \rightarrow & 0P0 \\ P & \rightarrow & 1P1 \end{array}$$

CFG: Example (3)

Design a CFG for the following language:

$$\{ww^R \mid w \in \{0,1\}^*\}$$

e.g., 00, 11, 0110, *etc.*

$$\begin{array}{lcl} P & \rightarrow & \epsilon \\ P & \rightarrow & 0P0 \\ P & \rightarrow & 1P1 \end{array}$$

CFG: Example (4)

Design a CFG for the set of binary strings, where each block of 0's followed by at least as many 1's.

e.g., 000111, 0001111, *etc.*

- We use S to represent one such string, and A to represent each such block in S.

$$
\begin{array}{lcll}
S & \rightarrow & \epsilon & \{\texttt{BC of } S\} \\
S & \rightarrow & AS & \{\texttt{RC of } S\} \\
A & \rightarrow & \epsilon & \{\texttt{BC of } A\} \\
A & \rightarrow & 01 & \{\texttt{BC of } A\} \\
A & \rightarrow & 0A1 & \{\texttt{RC of } A\texttt{: equal 0's and 1's}\} \\
A & \rightarrow & A1 & \{\texttt{RC of } A\texttt{: more 1's}\}
\end{array}
$$

CFG: Example (5.1) Version 1

Design the grammar for the following small expression language, which supports:

- Arithmetic operations: `+`, `−`, `*`, `/`
- Relational operations: `>`, `<`, `>=`, `<=`, `==`, `/=`
- Logical operations: `true`, `false`, `!`, `&&`, `||`, `=>`

  Start with the variable ***Expression***.
- There are two possible versions:
  1. All operations are mixed together.
  2. Relevant operations are grouped together.
     Try both!

CFG: Example (5.2) Version 1

$$
\begin{array}{lcl}
\textit{Expression} & \rightarrow & \textit{IntegerConstant} \\
& | & -\textit{IntegerConstant} \\
& | & \textit{BooleanConstant} \\
& | & \textit{BinaryOp} \\
& | & \textit{UnaryOp} \\
& | & (\ \textit{Expression}\ ) \\
\\
\textit{IntegerConstant} & \rightarrow & \textit{Digit} \\
& | & \textit{Digit IntegerConstant} \\
\\
\textit{Digit} & \rightarrow & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \\
\\
\textit{BooleanConstant} & \rightarrow & \texttt{TRUE} \\
& | & \texttt{FALSE}
\end{array}
$$

CFG: Example (5.3) Version 1

$$
\begin{array}{lcl}
\textit{BinaryOp} & \rightarrow & \textit{Expression} + \textit{Expression} \\
& | & \textit{Expression} - \textit{Expression} \\
& | & \textit{Expression} * \textit{Expression} \\
& | & \textit{Expression} \; / \; \textit{Expression} \\
& | & \textit{Expression} \;\&\& \; \textit{Expression} \\
& | & \textit{Expression} \; || \; \textit{Expression} \\
& | & \textit{Expression} => \textit{Expression} \\
& | & \textit{Expression} == \textit{Expression} \\
& | & \textit{Expression} \; /= \textit{Expression} \\
& | & \textit{Expression} > \textit{Expression} \\
& | & \textit{Expression} < \textit{Expression} \\
\\
\textit{UnaryOp} & \rightarrow & ! \; \textit{Expression}
\end{array}
$$

CFG: Example (5.4) Version 1

However, Version 1 of CFG:

- ***Parses*** string that requires further ***semantic analysis*** (e.g., type checking):

  e.g., `3 => 6`
- Is ***ambiguous***, meaning?
  - Some string may have more than one ways to interpreting it.
  - An interpretation is either visualized as a ***parse tree***, or written as a sequence of ***derivations***.

  e.g., Draw the parse tree(s) for `3 * 5 + 4`

CFG: Example (5.5) Version 2

$$
\begin{array}{lcl}
\textit{Expression} & \rightarrow & \textit{ArithmeticOp} \\
& | & \textit{RelationalOp} \\
& | & \textit{LogicalOp} \\
& | & (\ \textit{Expression}\ ) \\
\\
\textit{IntegerConstant} & \rightarrow & \textit{Digit} \\
& | & \textit{Digit IntegerConstant} \\
\\
\textit{Digit} & \rightarrow & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \\
\\
\textit{BooleanConstant} & \rightarrow & \texttt{TRUE} \\
& | & \texttt{FALSE}
\end{array}
$$

CFG: Example (5.6) Version 2

$$
\begin{array}{lcl}
\textit{ArithmeticOp} & \rightarrow & \textit{ArithmeticOp} + \textit{ArithmeticOp} \\
& | & \textit{ArithmeticOp} - \textit{ArithmeticOp} \\
& | & \textit{ArithmeticOp} \star \textit{ArithmeticOp} \\
& | & \textit{ArithmeticOp} \; / \; \textit{ArithmeticOp} \\
& | & (\textit{ArithmeticOp}) \\
& | & \textit{IntegerConstant} \\
& | & -\textit{IntegerConstant} \\
\textit{RelationalOp} & \rightarrow & \textit{ArithmeticOp} == \textit{ArithmeticOp} \\
& | & \textit{ArithmeticOp} \; /= \; \textit{ArithmeticOp} \\
& | & \textit{ArithmeticOp} > \textit{ArithmeticOp} \\
& | & \textit{ArithmeticOp} < \textit{ArithmeticOp} \\
\textit{LogicalOp} & \rightarrow & \textit{LogicalOp} \; \&\& \; \textit{LogicalOp} \\
& | & \textit{LogicalOp} \; || \; \textit{LogicalOp} \\
& | & \textit{LogicalOp} \Rightarrow \textit{LogicalOp} \\
& | & ! \; \textit{LogicalOp} \\
& | & (\textit{LogicalOp}) \\
& | & \textit{RelationalOp} \\
& | & \textit{BooleanConstant}
\end{array}
$$

CFG: Example (5.7) Version 2

However, Version 2 of CFG:

- Eliminates some cases for further semantic analysis:
  e.g., `(1 + 2) => (5 / 4)` [no parse tree]
- Still ***parses*** strings that might require further ***semantic analysis***:
  e.g., `(1 + 2) / (5 - (2 + 3))`
- Still is ***ambiguous***.
  e.g., Draw the parse tree(s) for `3 * 5 + 4`

CFG: Formal Definition (1)

- A ***context-free grammar (CFG)*** is a 4-tuple (V, Σ, R, S):
  - V is a finite set of **variables**.
  - Σ is a finite set of ***terminals***. $[V \cap \Sigma = \varnothing]$
  - R is a finite set of ***rules*** s.t.

$$R \subseteq \{v \rightarrow s \mid v \in V \land s \in (V \cup \Sigma)^*\}$$

  - $S \in V$ is is the **start variable**.
- Given strings $u, v, w \in (V \cup \Sigma)^*$, variable $A \in V$, a rule $A \rightarrow w$:
  - $\boxed{uAv \Rightarrow uwv}$ menas that uAv ***yields*** uwv.
  - $\boxed{u \stackrel{*}{\Rightarrow} v}$ means that u ***derives*** v, if:
    - $u = v$; or
    - $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k \Rightarrow v$ [a ***yield sequence***]
- Given a CFG $G = (V, \Sigma, R, S)$, the language of G

$$L(G) = \{w \in \Sigma^* \mid S \stackrel{*}{\Rightarrow} w\}$$

CFG: Formal Definition (2): Example

- Design the ***CFG*** for strings of properly-nested parentheses.
  e.g., ( ) , ( ) ( ) , ((( ) ( ))) ( ) , *etc.*
  Present your answer in a ***formal*** manner.
- $G = (\{S\}, \{(,)\}, R, S)$, where R is

$$S \rightarrow (\ S \ ) \mid SS \mid \epsilon$$

- Draw ***parse trees*** for the above three strings that G generates.

CFG: Formal Definition (3): Example

- Consider the grammar $G = (V, \Sigma, R, S)$:
  - R is

$$
\begin{array}{lcl}
Expr & \rightarrow & Expr + Term \\
 & | & Term \\
Term & \rightarrow & Term * Factor \\
 & | & Factor \\
Factor & \rightarrow & (Expr) \\
 & | & \texttt{a}
\end{array}
$$

  - $V = \{Expr, Term, Factor\}$
  - $\Sigma = \{\texttt{a}, +, *, (,)\}$
  - $S = Expr$
- ***Precedence*** of operators $+$, $*$ is embedded in the grammar.
  - "Plus" is specified at a **higher** level (*Expr*) than is "times" (*Term*).
  - Both operands of a multiplication (*Factor*) may be **parenthesized**.

Regular Expressions to CFG's

- Recall the semantics of regular expressions (assuming that we do not consider $\varnothing$):

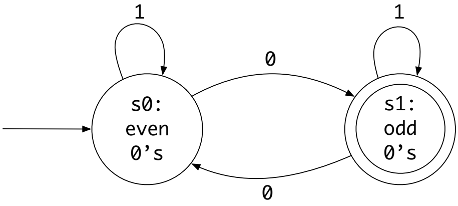
$$
\begin{array}{lcl}
L(\ \epsilon\ ) & = & \{\epsilon\} \\
L(\ a\ ) & = & \{a\} \\
L(\ E + F\ ) & = & L(E) \cup L(F) \\
L(\ EF\ ) & = & L(E)L(F) \\
L(\ E^*\ ) & = & (L(E))^* \\
L(\ (E)\ ) & = & L(E)
\end{array}
$$

- e.g., Grammar for $(00 + 1)^* + (11 + 0)^*$

$$
\begin{array}{lcl}
S & \rightarrow & A \mid B \\
A & \rightarrow & \epsilon \mid AC \\
C & \rightarrow & 00 \mid 1 \\
B & \rightarrow & \epsilon \mid BD \\
D & \rightarrow & 11 \mid 0
\end{array}
$$

DFA to CFG's

- Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$:
  - Make a ***variable*** R_i for each ***state*** $q_i \in Q$.
  - Make R_0 the ***start variable***, where q_0 is the ***start state*** of M.
  - Add a rule $R_i \rightarrow aR_j$ to the grammar if $\delta(q_i, a) = q_j$.
  - Add a rule $R_i \rightarrow \epsilon$ if $q_i \in F$.
- e.g., Grammar for



$$
\begin{aligned}
R_0 &\rightarrow 1R_0 \mid 0R_1 \\
R_1 &\rightarrow 0R_0 \mid 1R_1 \mid \epsilon
\end{aligned}
$$

CFG: Leftmost Derivations (1)

$$\begin{array}{lcl} Expr & \rightarrow & Expr + Term \mid Term \\ Term & \rightarrow & Term \star Factor \mid Factor \\ Factor & \rightarrow & (\, Expr \,) \mid a \end{array}$$

- Given a string ($\in (V \cup \Sigma)^*$), a ***left-most derivation (LMD)*** keeps substituting the leftmost non-terminal ($\in V$).
- ***Unique LMD*** for the string `a + a * a`:

$$\begin{array}{lcl} Expr & \Rightarrow & Expr + Term \\ & \Rightarrow & Term + Term \\ & \Rightarrow & Factor + Term \\ & \Rightarrow & a + Term \\ & \Rightarrow & a + Term \star Factor \\ & \Rightarrow & a + Factor \star Factor \\ & \Rightarrow & a + a \star Factor \\ & \Rightarrow & a + a \star a \end{array}$$

- This ***LMD*** suggests that `a * a` is the right operand of $+$.

CFG: Rightmost Derivations (1)

$$\begin{array}{lcl} Expr & \rightarrow & Expr + Term \mid Term \\ Term & \rightarrow & Term \star Factor \mid Factor \\ Factor & \rightarrow & (Expr) \mid a \end{array}$$

- Given a string ($\in (V \cup \Sigma)^*$), a ***right-most derivation (RMD)*** keeps substituting the rightmost non-terminal ($\in V$).
- ***Unique RMD*** for the string `a + a * a`:

$$\begin{array}{lcl} Expr & \Rightarrow & Expr + Term \\ & \Rightarrow & Expr + Term \star Factor \\ & \Rightarrow & Expr + Term \star a \\ & \Rightarrow & Expr + Factor \star a \\ & \Rightarrow & Expr + a \star a \\ & \Rightarrow & Term + a \star a \\ & \Rightarrow & Factor + a \star a \\ & \Rightarrow & a + a \star a \end{array}$$

- This ***RMD*** suggests that `a * a` is the right operand of `+`.

CFG: Leftmost Derivations (2)

$$\begin{array}{lcl} Expr & \rightarrow & Expr + Term \mid Term \\ Term & \rightarrow & Term * Factor \mid Factor \\ Factor & \rightarrow & (Expr) \mid a \end{array}$$

- ***Unique LMD*** for the string `(a + a) * a`:

$$\begin{array}{lcl} Expr & \Rightarrow & Term \\ & \Rightarrow & Term * Factor \\ & \Rightarrow & Factor * Factor \\ & \Rightarrow & (Expr) * Factor \\ & \Rightarrow & (Expr + Term) * Factor \\ & \Rightarrow & (Term + Term) * Factor \\ & \Rightarrow & (Factor + Term) * Factor \\ & \Rightarrow & (a + Term) * Factor \\ & \Rightarrow & (a + Factor) * Factor \\ & \Rightarrow & (a + a) * Factor \\ & \Rightarrow & (a + a) * a \end{array}$$

- This ***LMD*** suggests that `(a + a)` is the left operand of $*$.

CFG: Rightmost Derivations (2)

$$\begin{array}{lcl} Expr & \rightarrow & Expr + Term \mid Term \\ Term & \rightarrow & Term \star Factor \mid Factor \\ Factor & \rightarrow & (\, Expr\,) \mid a \end{array}$$

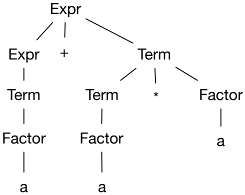
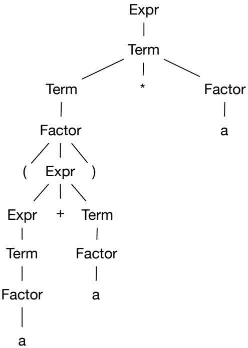
- ***Unique RMD*** for the string `(a + a) * a`:

$$\begin{array}{lcl} Expr & \Rightarrow & Term \\ & \Rightarrow & Term \star Factor \\ & \Rightarrow & Term \star a \\ & \Rightarrow & Factor \star a \\ & \Rightarrow & (\ Expr\ ) \star a \\ & \Rightarrow & (\ Expr + Term\ ) \star a \\ & \Rightarrow & (\ Expr + Factor\ ) \star a \\ & \Rightarrow & (\ Expr + a\ ) \star a \\ & \Rightarrow & (\ Term + a\ ) \star a \\ & \Rightarrow & (\ Factor + a\ ) \star a \\ & \Rightarrow & (\ a + a\ ) \star a \end{array}$$

- This ***RMD*** suggests that `(a + a)` is the left operand of $\star$.

CFG: Parse Trees vs. Derivations (1)

- *Parse trees* for (leftmost & rightmost) *derivations* of expressions:

a + a * a	(a + a) * a
	

- Orders in which ***derivations*** are performed are ***not*** reflected on parse trees.

CFG: Parse Trees vs. Derivations (2)

- A string $w \in \Sigma^*$ may have more than one ***derivations***.

  Q: distinct ***derivations*** for $w \in \Sigma^*$ $\Rightarrow$ distinct ***parse trees*** for w?

  A: Not in general $\because$ Derivations with ***distinct orders*** of variable substitutions may still result in the ***same parse tree***.
- For example:

$$
\begin{array}{lcl}
Expr & \rightarrow & Expr \; + \; Term \mid Term \\
Term & \rightarrow & Term \; \star \; Factor \mid Factor \\
Factor & \rightarrow & (\, Expr \,) \mid a
\end{array}
$$

For string `a + a * a`, the ***LMD*** and ***RMD*** have ***distinct orders*** of variable substitutions, but their corresponding ***parse trees are the same***.

CFG: Ambiguity: Definition

Given a grammar $G = (V, \Sigma, R, S)$:

- A string $w \in \Sigma^*$ is derived *ambiguously* in G if there exist two or more ***distinct*** ***parse trees*** or, equally, two or more ***distinct*** ***LMDs*** or, equally, two or more ***distinct*** ***RMDs***.

  We require that all such derivations are completed by following a consisten order (**leftmost** or **rightmost**) to avoid ***false positive***.

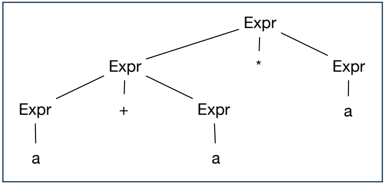
- G is *ambiguous* if it generates some string ambiguously.

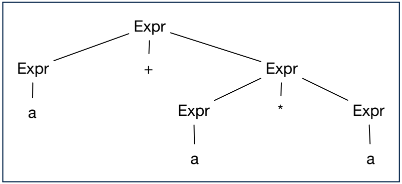
CFG: Ambiguity: Exercise (1)

- Is the following grammar *ambiguous*?

$$Expr \rightarrow Expr + Expr \mid Expr * Expr \mid (Expr) \mid a$$

- Yes ∵ it generates the string `a + a * a` *ambiguously*:





- ***Distinct ASTs*** (for the ***same input***) imply ***distinct semantic interpretations***: e.g., a pre-order traversal for evaluation
- **Exercise**: Show ***LMDs*** for the two parse trees.

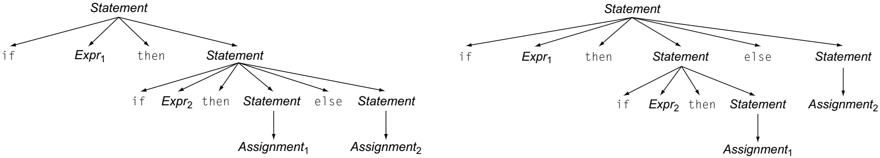
CFG: Ambiguity: Exercise (2.1)

- Is the following grammar *ambiguous*?

$$
\begin{array}{lcl}
\textit{Statement} & \rightarrow & \texttt{if}\ \textit{Expr}\ \texttt{then}\ \textit{Statement} \\
 & | & \texttt{if}\ \textit{Expr}\ \texttt{then}\ \textit{Statement}\ \texttt{else}\ \textit{Statement} \\
 & | & \textit{Assignment} \\
 & \ldots &
\end{array}
$$

- Yes ∵ it derives the following string *ambiguously*:

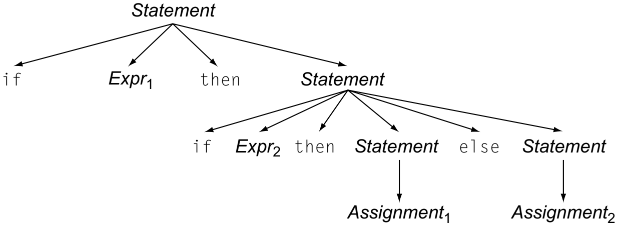
$\texttt{if}\ \textit{Expr}_1\ \texttt{then}\ \texttt{if}\ \textit{Expr}_2\ \texttt{then}\ \textit{Assignment}_1\ \texttt{else}\ \textit{Assignment}_2$



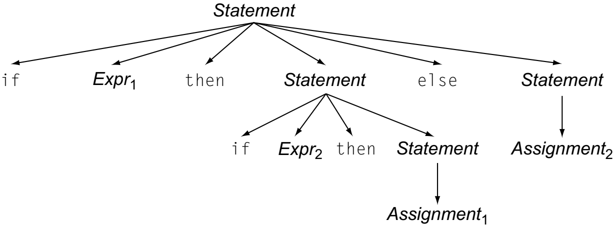
- This is called the ***dangling else*** problem.
- **Exercise:** Show ***LMDs*** for the two parse trees.

CFG: Ambiguity: Exercise (2.2)

(***Meaning 1***) *Assignment*$_2$ may be associated with the <u>inner</u> `if`:



(***Meaning 2***) *Assignment*$_2$ may be associated with the <u>outer</u> `if`:



CFG: Ambiguity: Exercise (2.3)

- We may remove the ***ambiguity*** by specifying that the *dangling* `else` is associated with the **nearest** **`if`**:

$$
\begin{array}{lcl}
\textit{Statement} & \rightarrow & \texttt{if}\ \textit{Expr}\ \texttt{then}\ \textit{Statement} \\
 & | & \texttt{if}\ \textit{Expr}\ \texttt{then}\ \textit{WithElse}\ \texttt{else}\ \textit{Statement} \\
 & | & \textit{Assignment} \\
\textit{WithElse} & \rightarrow & \texttt{if}\ \textit{Expr}\ \texttt{then}\ \textit{WithElse}\ \texttt{else}\ \textit{WithElse} \\
 & | & \textit{Assignment}
\end{array}
$$

- When applying `if ... then` *WithElse* `else` *Statement*:
  - The ***true*** branch will be produced via *WithElse*.
  - The ***false*** branch will be produced via *Statement*.

  There is ***no circularity*** between the two non-terminals.

Discovering Derivations

- Given a CFG $G = (V, \Sigma, R, S)$ and an input program $p \in \Sigma^*$:
  - So far we **manually** come up a valid ***derivation*** s.t. $S \stackrel{*}{\Rightarrow} p$.
  - A *parser* is supposed to **automate** this ***derivation*** process.
    - Input: **A sequence of (t, c) pairs**, where each ***token*** t (e.g., `r241`) belongs to a ***syntactic category*** c (e.g., `register`); and a ***CFG*** G.
    - Output: A ***valid derivation*** (as an ***AST***); or A ***parse error***.
- In the process of constructing an ***AST*** for the input program:
  - ***Root*** of AST: The ***start symbol*** S of G
  - ***Internal nodes***: A ***subset of variables*** V of G
  - ***Leaves*** of AST: A ***token/terminal*** sequence

    $\Rightarrow$ Discovering the *grammatical connections* (w.r.t. R of G) between the ***root***, ***internal nodes***, and ***leaves*** is the hard part!
- Approaches to Parsing: $[\ w \in (V \cup \Sigma)^*, A \in V, \boxed{A \rightarrow w} \in R\ ]$
  - **Top-down** parsing

    For a node representing ***A***, extend it with a subtree representing ***w***.
  - **Bottom-up** parsing

    For a substring matching ***w***, build a node representing ***A*** accordingly.

TDP: Discovering Leftmost Derivation

```
ALGORITHM: TDParse
  INPUT: CFG G = (V, Σ, R, S)
  OUTPUT: Root of a Parse Tree or Syntax Error
PROCEDURE:
  root := a new node for the start symbol S
  focus := root
  initialize an empty stack trace
  trace.push(null)
  word := NextWord()
  while (true):
    if focus ∈ V then
      if ∃ unvisited rule focus → β1 β2 ... βn ∈ R then
        create β1, β2 ... βn as children of focus
        trace.push(βn βn−1 ... β2)
        focus := β1
      else
        if focus = S then report syntax error
        else backtrack
    elseif word matches focus then
      word := NextWord()
      focus := trace.pop()
    elseif word = EOF ∧ focus = null then return root
    else backtrack
```

backtrack ≙ pop *focus*.siblings; *focus* := *focus*.parent; *focus*.resetChildren

TDP: Exercise (1)

- Given the following CFG **G**:

$$
\begin{array}{lcl}
Expr & \rightarrow & Expr \; \texttt{+} \; Term \\
 & | & Term \\
Term & \rightarrow & Term \; \star \; Factor \\
 & | & Factor \\
Factor & \rightarrow & \texttt{(} \, Expr \, \texttt{)} \\
 & | & \texttt{a}
\end{array}
$$

  Trace *TDParse* on how to build an AST for input `a + a * a`.

- Running *TDParse* with **G** results an ***infinite loop*** !!!
  - *TDParse* focuses on the ***leftmost*** non-terminal.
  - The grammar **G** contains ***left-recursions***.
- We must first convert left-recursions in **G** to ***right-recursions***.

TDP: Exercise (2)

- Given the following CFG **G**:

$$\begin{array}{lcl} \textit{Expr} & \rightarrow & \textit{Term} \quad \textit{Expr}' \\ \textit{Expr}' & \rightarrow & + \ \textit{Term} \quad \textit{Expr}' \\ & | & \epsilon \\ \textit{Term} & \rightarrow & \textit{Factor} \quad \textit{Term}' \\ \textit{Term}' & \rightarrow & \star \ \textit{Factor} \quad \textit{Term}' \\ & | & \epsilon \\ \textit{Factor} & \rightarrow & (\ \textit{Expr}\ ) \\ & | & \mathtt{a} \end{array}$$

  Exercise. Trace *TDParse* on building AST for `a + a * a`.

  Exercise. Trace *TDParse* on building AST for `(a + a) * a`.

  Q: How to handle ϵ-productions (e.g., $\textit{Expr} \rightarrow \epsilon$)?

  A: Execute $\textit{focus} := \textit{trace}.\texttt{pop()}$ to advance to next node.

- Running *TDParse* will ***terminate*** $\because$ **G** is ***right-recursive***.
- We will learn about a systematic approach to converting left-recursions in a given grammar to ***right-recursions***.

Left-Recursions (LR): Direct vs. Indirect

Given CFG $G = (V, \Sigma, R, S)$, $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$, G contains:

- A *cycle* if $\exists A \in V \bullet A \overset{*}{\Rightarrow} A$
- A ***direct*** LR if $\mathbf{A} \rightarrow \mathbf{A}\alpha \in R$ for non-terminal $A \in V$

e.g.,

$$
\begin{array}{lcl}
Expr & \rightarrow & Expr + Term \\
 & | & Term \\
Term & \rightarrow & Term * Factor \\
 & | & Factor \\
Factor & \rightarrow & (\, Expr\,) \\
 & | & \mathtt{a}
\end{array}
$$

e.g.,

$$
\begin{array}{lcl}
Expr & \rightarrow & Expr + Term \\
 & | & Expr - Term \\
 & | & Term \\
Term & \rightarrow & Term * Factor \\
 & | & Term \,/\, Factor \\
 & | & Factor
\end{array}
$$

- An ***indirect*** LR if $\mathbf{A} \rightarrow \mathbf{B}\beta \in R$ for non-terminals $A, B \in V$, $\mathbf{B} \overset{*}{\Rightarrow} \mathbf{A}\gamma$

$$
\begin{array}{lcl}
A & \rightarrow & Br \\
B & \rightarrow & Cd \\
C & \rightarrow & At
\end{array}
$$

$A \rightarrow Br, B \overset{*}{\Rightarrow} Atd$

$$
\begin{array}{lcllll}
A & \rightarrow & Ba & | & b & \\
B & \rightarrow & Cd & | & e & \\
C & \rightarrow & Df & | & g & \\
D & \rightarrow & f & | & Aa & | \;\; Cg
\end{array}
$$

$A \rightarrow Ba, B \overset{*}{\Rightarrow} Aafd$

TDP: (Preventively) Eliminating LRs

```
1  ALGORITHM: RemoveLR
2    INPUT: CFG G = (V, Σ, R, S)
3    ASSUME: G has no ϵ-productions
4    OUTPUT: G′ s.t. G′ ≡ G, G′ has no
5            indirect & direct left-recursions
6  PROCEDURE:
7    impose an order on V: ⟨⟨A1, A2, ..., An⟩⟩
8    for i: 1 .. n:
9      for j: 1 .. i−1:
10       if ∃ Ai → Ajγ ∈ R ∧ Aj → δ1 | δ2 | ... | δm ∈ R then
11         replace Ai → Ajγ with Ai → δ1γ | δ2γ | ... | δmγ
12       end
13     for Ai → Aiα | β ∈ R:
14       replace it with: Ai → βAi′, Ai′ → αAi′ | ϵ
```

- **L9** to **L12**: Remove ***indirect*** left-recursions from A_1 to A_{i-1}.
- **L13** to **L14**: Remove ***direct*** left-recursions from A_1 to A_{i-1}.
- ***Loop Invariant*** **(outer for-loop)**? At the start of i^{th} iteration:
  - No ***direct*** or ***indirect*** left-recursions for $A_1, A_2, \ldots, A_{i-1}$.
  - More precisely: $\forall j : j < i \bullet \neg(\exists k \bullet k \leq j \land A_j \rightarrow A_k \cdots \in R)$

CFG: Eliminating ϵ-Productions (1)

- Motivations:
  - ***TDParse*** handles each ϵ-production as a special case.
  - ***RemoveLR*** produces CFG which may contain ϵ-productions.
- $\epsilon \notin L \Rightarrow \exists$ CFG $G = (V, \Sigma, R, S)$ s.t. G has no ϵ-productions.

  An *ϵ-production* has the form $A \rightarrow \epsilon$.
- A variable A is ***nullable*** if $A \overset{*}{\Rightarrow} \epsilon$.
  - Each terminal symbol is ***not nullable***.
  - Variable A is ***nullable*** if either:
    - $A \rightarrow \epsilon \in R$; or
    - $A \rightarrow B_1 B_2 \dots B_k \in R$, where each variable B_i $(1 \leq i \leq k)$ is a ***nullable***.
- Given a production $B \rightarrow CAD$, if only variable A is ***nullable***, then there are 2 versions of B: $B \rightarrow CAD \mid CD$
- In general, given a production $A \rightarrow X_1 X_2 \dots X_k$ with k symbols, if m of the k symbols are ***nullable***:
  - $m < k$: There are 2^m versions of A.
  - $m = k$: There are $2^m - 1$ versions of A. [excluding $A \rightarrow \epsilon$]

CFG: Eliminating ϵ-Productions (2)

- Eliminate ϵ-productions from the following grammar:

$$\begin{array}{lcl} S & \rightarrow & AB \\ A & \rightarrow & aAA \mid \epsilon \\ B & \rightarrow & bBB \mid \epsilon \end{array}$$

- Which are the ***nullable*** variables? [S, A, B]

$$\begin{array}{lcll} S & \rightarrow & A \mid B \mid AB & \{S \rightarrow \epsilon \texttt{ not included}\} \\ A & \rightarrow & aAA \mid aA \mid a & \{A \rightarrow aA \texttt{ duplicated}\} \\ B & \rightarrow & bBB \mid bB \mid b & \{B \rightarrow bB \texttt{ duplicated}\} \end{array}$$

Backtrack-Free Parsing (1)

- `TDParse` automates the ***top-down***, ***leftmost*** derivation process by consistently choosing production rules (e.g., in order of their appearance in CFG).
  - This ***inflexibility*** may lead to ***inefficient*** runtime performance due to the need to *backtrack*.
  - e.g., It may take the ***construction of a giant subtree*** to find out a ***mismatch*** with the input tokens, which end up requiring it to *backtrack* all the way back to the ***root*** (start symbol).
- We may avoid backtracking with a modification to the parser:
  - When deciding which production rule to choose, consider:
    (1) the ***current*** input symbol
    (2) the consequential ***first*** symbol if a rule was applied for `focus`
    [*lookahead* symbol]
  - Using a ***one symbol lookhead***, w.r.t. a ***right-recursive*** CFG, each alternative for the ***leftmost nonterminal*** leads to a ***unique terminal***, allowing the parser to decide on a choice that prevents *backtracking*.
  - Such CFG is ***backtrack free*** with the ***lookhead*** of one symbol.
  - We also call such backtrack-free CFG a ***predictive grammar***.

The FIRST Set: Definition

- Say we write $T \subset \mathbb{P}(\Sigma^*)$ to denote the set of valid tokens recognizable by the scanner.
- **FIRST** $(\alpha) \triangleq$ set of symbols that can appear as the ***first word*** in some string derived from α.
- More precisely:

$$\textbf{FIRST}(\alpha) = \begin{cases} \{\alpha\} & \text{if } \alpha \in T \\ \{w \mid w \in \Sigma^* \wedge \alpha \overset{*}{\Rightarrow} w\beta \wedge \beta \in (V \cup \Sigma)^*\} & \text{if } \alpha \in V \end{cases}$$

The FIRST Set: Examples

- Consider this ***right***-recursive CFG:

0	$Goal \rightarrow Expr$			6	$Term' \rightarrow$	x	$Factor\ Term'$
1	$Expr \rightarrow Term\ Expr'$			7	$\mid$	÷	$Factor\ Term'$
2	$Expr' \rightarrow$	+	$Term\ Expr'$	8	$\mid$	ϵ	
3	$\mid$	-	$Term\ Expr'$	9	$Factor \rightarrow$	(	$Expr$)
4	$\mid$	ϵ		10	$\mid$	num	
5	$Term \rightarrow Factor\ Term'$			11	$\mid$	name	

- Compute **FIRST** for each terminal (e.g., num, +, ( ):

	num	name	+	-	×	÷	(	)	eof	ϵ
FIRST	num	name	+	-	x	÷	(	)	eof	ϵ

- Compute **FIRST** for each non-terminal (e.g., *Expr*, *Term'*):

	Expr	***Expr'***	***Term***	***Term'***	***Factor***
FIRST	(, name, num	+, -, ϵ	(, name, num	x, ÷, ϵ	(, name, num

Computing the FIRST Set

$$\text{FIRST}(\alpha) = \begin{cases} \{\alpha\} & \text{if } \alpha \in T \\ \{w \mid w \in \Sigma^* \land \alpha \overset{*}{\Rightarrow} w\beta \land \beta \in (V \cup \Sigma)^*\} & \text{if } \alpha \in V \end{cases}$$

ALGORITHM: *GetFirst*
  INPUT: *CFG* $G = (V, \Sigma, R, S)$
  $T \subset \Sigma^*$ *denotes valid terminals*
  OUTPUT: $\text{FIRST} : V \cup T \cup \{\epsilon, eof\} \longrightarrow \mathbb{P}(T \cup \{\epsilon, eof\})$
PROCEDURE:
  for $\alpha \in (T \cup \{eof, \epsilon\})$: $\text{FIRST}(\alpha) := \{\alpha\}$
  for $A \in V$: $\text{FIRST}(A) := \emptyset$
  $lastFirst := \emptyset$
  while ($lastFirst \neq \text{FIRST}$):
    $lastFirst := \text{FIRST}$
    for $A \to \beta_1 \beta_2 \dots \beta_k \in R$ *s.t.* $\forall \beta_j : \beta_j \in (T \cup V)$:
      rhs $:= \text{FIRST}(\beta_1) - \{\epsilon\}$
      for ($i := 1$; $\epsilon \in \text{FIRST}(\beta_i) \land i < k$; i++):
        rhs $:=$ ***rhs*** $\cup (\text{FIRST}(\beta_{i+1}) - \{\epsilon\})$
      if $i = k \land \epsilon \in \text{FIRST}(\beta_k)$ **then**
        rhs $:=$ ***rhs*** $\cup \{\epsilon\}$
      end
      $\text{FIRST}(A) := \text{FIRST}(A) \cup$ ***rhs***

Computing the FIRST Set: Extension

- Recall: **FIRST** takes as input a token or a variable.

$$\textbf{FIRST} : V \cup T \cup \{\epsilon, eof\} \longrightarrow \mathbb{P}(T \cup \{\epsilon, eof\})$$

- The computation of variable ***rhs*** in algoritm `GetFirst` actually suggests an extended, overloaded version:

$$\textbf{FIRST} : (V \cup T \cup \{\epsilon, eof\})^* \longrightarrow \mathbb{P}(T \cup \{\epsilon, eof\})$$

FIRST may also take as input a string $\beta_1 \beta_2 \dots \beta_n$ (RHS of rules).

- More precisely:

$$\textbf{FIRST}(\beta_1 \beta_2 \dots \beta_n) = \left\{ \textbf{FIRST}(\beta_1) \cup \textbf{FIRST}(\beta_2) \cup \dots \cup \textbf{FIRST}(\beta_{k-1}) \cup \textbf{FIRST}(\beta_k) \;\middle|\; \begin{array}{l} \forall i : 1 \leq i < k \bullet \epsilon \in \textbf{FIRST}(\beta_i) \\ \wedge \\ \epsilon \notin \textbf{FIRST}(\beta_k) \end{array} \right.$$

Note. β_k is the first symbol whose **FIRST** set does not contain ϵ.

Extended FIRST Set: Examples

Consider this ***right***-recursive CFG:

0	*Goal*	$\rightarrow$	*Expr*		6	*Term′*	$\rightarrow$	x *Factor Term′*
1	*Expr*	$\rightarrow$	*Term Expr′*		7		|	÷ *Factor Term′*
2	*Expr′*	$\rightarrow$	+ *Term Expr′*		8		|	ϵ
3		|	- *Term Expr′*		9	*Factor*	$\rightarrow$	(*Expr*)
4		|	ϵ		10		|	num
5	*Term*	$\rightarrow$	*Factor Term′*		11		|	name

e.g., FIRST(*Term Expr′*) = FIRST(*Term*) = {(, name, num}

e.g., FIRST(+ *Term Expr′*) = FIRST(+) = {+}

e.g., FIRST(− *Term Expr′*) = FIRST(−) = {−}

e.g., FIRST(ϵ) = {ϵ}

Is the FIRST Set Sufficient

- Consider the following three productions:

$$\begin{array}{llllll} Expr' & \rightarrow & + & Term & Term' & \texttt{(1)} \\ & | & - & Term & Term' & \texttt{(2)} \\ & | & \epsilon & & & \texttt{(3)} \end{array}$$

  In TDP, when the parser attempts to expand an *Expr′* node, it ***looks ahead with one symbol*** to decide on the choice of rule: **FIRST**$(+) = \{+\}$, **FIRST**$(-) = \{-\}$, and **FIRST**$(\epsilon) = \{\epsilon\}$.

  Q. When to choose rule `(3)` (causing ***focus := trace.pop()***)?

  A?. Choose rule `(3)` when *focus* $\neq$ **FIRST**$(+) \wedge$ *focus* $\neq$ **FIRST**$(-)$?

  - ***Correct*** but ***inefficient*** in case of illegal input string: syntax error is only reported after possibly a long series of ***backtrack***.
  - Useful if parser knows which words can appear, after an application of the ϵ-production (rule `(3)`), as leadling symbols.

- **FOLLOW** $(v : V) \triangleq$ set of symbols that can appear to the immediate right of a string derived from v.

$$\textbf{FOLLOW}(v) = \{\boldsymbol{w} \mid w, x, y \in \Sigma^* \wedge v \overset{*}{\Rightarrow} \boldsymbol{x} \wedge S \overset{*}{\Rightarrow} \boldsymbol{xwy}\}$$

The FOLLOW Set: Examples

- Consider this ***right***-recursive CFG:

0	$Goal$	$\rightarrow$	$Expr$	6	$Term'$	$\rightarrow$	x $Factor\ Term'$
1	$Expr$	$\rightarrow$	$Term\ Expr'$	7		|	÷ $Factor\ Term'$
2	$Expr'$	$\rightarrow$	+ $Term\ Expr'$	8		|	ϵ
3		|	- $Term\ Expr'$	9	$Factor$	$\rightarrow$	($Expr$)
4		|	ϵ	10		|	num
5	$Term$	$\rightarrow$	$Factor\ Term'$	11		|	name

- Compute **FOLLOW** for each non-terminal (e.g., *Expr*, *Term'*):

	Expr	***Expr'***	***Term***	***Term'***	***Factor***
FOLLOW	eof,)	eof,)	eof, +, -,)	eof, +, -,)	eof, +, -, x, ÷,)

Computing the FOLLOW Set

$$\text{FOLLOW}(v) = \{ w \mid w, x, y \in \Sigma^* \wedge v \overset{*}{\Rightarrow} x \wedge S \overset{*}{\Rightarrow} xwy \}$$

```
ALGORITHM: GetFollow
  INPUT: CFG G = (V, Σ, R, S)
  OUTPUT: FOLLOW : V ⟶ ℙ(T ∪ {eof})
PROCEDURE:
  for A ∈ V: FOLLOW(A) := ∅
  FOLLOW(S) := {eof}
  lastFollow := ∅
  while(lastFollow ≠ FOLLOW):
    lastFollow := FOLLOW
    for A → β1β2 ... βk ∈ R:
      trailer := FOLLOW(A)
      for i: k .. 1:
        if βi ∈ V then
          FOLLOW(βi) := FOLLOW(βi) ∪ trailer
          if ϵ ∈ FIRST(βi)
            then trailer := trailer ∪ (FIRST(βi) − ϵ)
            else trailer := FIRST(βi)
        else
          trailer := FIRST(βi)
```

Backtrack-Free Grammar

- A ***backtrack-free grammar*** (for a **top-down parser**), when expanding the `focus` ***internal node***, is always able to choose a unique rule with the ***one-symbol lookahead*** (or report a ***syntax error*** when no rule applies).
- To formulate this, we first define:

$$\textbf{START}(A \rightarrow \beta) = \begin{cases} \textbf{FIRST}(\beta) & \text{if } \epsilon \notin \textbf{FIRST}(\beta) \\ \textbf{FIRST}(\beta) \cup \textbf{FOLLOW}(A) & \text{otherwise} \end{cases}$$

  FIRST(β) is the extended version where β may be $\beta_1 \beta_2 \dots \beta_n$
- A ***backtrack-free grammar*** has each of its productions $A \rightarrow \gamma_1 \mid \gamma_2 \mid \dots \mid \gamma_n$ satisfying:

$$\forall i, j : 1 \leq i, j \leq n \wedge i \neq j \bullet \textbf{START}(\gamma_i) \cap \textbf{START}(\gamma_j) = \varnothing$$

TDP: Lookahead with One Symbol

```
ALGORITHM: TDParse
  INPUT: CFG G = (V, Σ, R, S)
  OUTPUT: Root of a Parse Tree or Syntax Error
PROCEDURE:
  root := a new node for the start symbol S
  focus := root
  initialize an empty stack trace
  trace.push(null)
  word := NextWord()
  while (true):
    if focus ∈ V then
      if ∃ unvisited rule focus → β1 β2 ... βn ∈ R ∧ word ∈ START(β) then
        create β1, β2 ... βn as children of focus
        trace.push(βn βn−1 ... β2)
        focus := β1
      else
        if focus = S then report syntax error
        else backtrack
    elseif word matches focus then
      word := NextWord()
      focus := trace.pop()
    elseif word = EOF ∧ focus = null then return root
    else backtrack
```

backtrack ≜ pop *focus*.siblings; *focus* := *focus*.parent; *focus*.resetChildren

Backtrack-Free Grammar: Exercise

Is the following CFG ***backtrack free***?

11	*Factor*	$\rightarrow$	`name`
12		|	`name` [*ArgList*]
13		|	`name` (*ArgList*)
15	*ArgList*	$\rightarrow$	*Expr MoreArgs*
16	*MoreArgs*	$\rightarrow$	, *Expr MoreArgs*
17		|	ϵ

- $\epsilon \notin$ **FIRST**(*Factor*) $\Rightarrow$ **START**(*Factor*) = **FIRST**(*Factor*)
- **FIRST**(*Factor* → `name`) = {`name`}
- **FIRST**(*Factor* → `name` [*ArgList*]) = {`name`}
- **FIRST**(*Factor* → `name` (*ArgList*)) = {`name`}

$\therefore$ The above grammar is ***not*** backtrack free.
$\Rightarrow$ To expand an AST node of *Factor*, with a ***lookahead*** of `name`, the parser has no basis to choose among rules 11, 12, and 13.

Backtrack-Free Grammar: Left-Factoring

- A CFG is not backtrack free if there exists a ***common prefix*** (`name`) among the RHS of ***multiple*** production rules.
- To make such a CFG ***backtrack-free***, we may transform it using *left factoring*: a process of extracting and isolating ***common prefixes*** in a set of production rules.
  - Identify a common prefix α:

$$A \rightarrow \alpha\beta_1 \mid \alpha\beta_2 \mid \ldots \mid \alpha\beta_n \mid \gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_j$$

[each of $\gamma_1, \gamma_2, \ldots, \gamma_j$ does not begin with α]

  - Rewrite that production rule as:

$$\begin{array}{lcl} A & \rightarrow & \alpha B \mid \gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_j \\ B & \rightarrow & \beta_1 \mid \beta_2 \mid \ldots \mid \beta_n \end{array}$$

  - New rule $B \rightarrow \beta_1 \mid \beta_2 \mid \ldots \mid \beta_n$ may also contain ***common prefixes***.
  - Rewriting continues until no common prefixes are identified.

Left-Factoring: Exercise

- Use *left-factoring* to remove all ***common prefixes*** from the following grammar.

11	*Factor*	$\rightarrow$	`name`
12		|	`name` [*ArgList*]
13		|	`name` (*ArgList*)
15	*ArgList*	$\rightarrow$	*Expr MoreArgs*
16	*MoreArgs*	$\rightarrow$	, *Expr MoreArgs*
17		|	ϵ

- Identify common prefix `name` and rewrite rules 11, 12, and 13:

Factor	$\rightarrow$	`name` *Arguments*
Arguments	$\rightarrow$	[*ArgList*]
	|	(*ArgList*)
	|	ϵ

Any more ***common prefixes***? [No]

TDP: Terminating and Backtrack-Free

- Given an arbitrary CFG as input to a ***top-down parser***:
  - **Q.** How do we avoid a ***non-terminating*** parsing process?
    A. Convert left-recursions to right-recursion.
  - **Q.** How do we minimize the need of ***backtracking***?
    A. left-factoring & one-symbol lookahead using **START**
- ***Not*** every context-free language has a corresponding ***backtrack-free*** context-free grammar.

  Given a CFL l, the following is ***undecidable***:

  $$\exists cfg \mid L(cfg) = l \wedge isBacktrackFree(cfg)$$

- Given a CFG $g = (V, \Sigma, R, S)$, whether or not g is ***backtrack-free*** is ***decidable***:

  For each $A \rightarrow \gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_n \in R$:

  $$\forall i, j : 1 \leq i, j \leq n \wedge i \neq j \bullet \textbf{START}(\gamma_i) \cap \textbf{START}(\gamma_j) = \varnothing$$

Backtrack-Free Parsing (2.1)

- A *recursive-descent* parser is:
  - A top-down parser
  - Structured as a set of *mutually recursive* procedures

    Each procedure corresponds to a ***non-terminal*** in the grammar.

    See an example.
- Given a ***backtrack-free*** grammar, a tool (a.k.a. *parser generator*) can automatically generate:
  - **FIRST**, **FOLLOW**, and **START** sets
  - An efficient ***recursive-descent*** parser

    This generated parser is called an *LL(1) parser*, which:
    - Processes input from **L**eft to right
    - Constructs a **L**eftmost derivation
    - Uses a lookahead of **1** symbol
- *LL(1) grammars* are those working in an *LL(1)* scheme.

  LL(1) grammars are ***backtrack-free*** by definition.

Backtrack-Free Parsing (2.2)

- Consider this CFG with **START** sets of the RHSs:

	Production			FIRST$^+$
2	*Expr′*	$\rightarrow$	+ *Term Expr′*	{ + }
3		|	- *Term Expr′*	{ - }
4		|	ϵ	{ ϵ, eof,) }

- The corresponding ***recursive-descent*** parser is structured as:

```
ExprPrim()
  if word = + ∨ word = - then /* Rules 2, 3 */
    word := NextWord()
    if(Term())
      then return ExprPrim()
      else return false
  elseif word = ) ∨ word = eof then /* Rule 4 */
    return true
  else
    report a syntax error
    return false
  end

Term()
  ...
```

See: parser generator

LL(1) Parser: Exercise

Consider the following grammar:

$$
\begin{array}{lcl}
L & \rightarrow & R\ \mathtt{a} \\
  & | & Q\ \mathtt{ba}
\end{array}
\qquad
\begin{array}{lcl}
R & \rightarrow & \mathtt{aba} \\
  & | & \mathtt{caba} \\
  & | & R\ \mathtt{bc}
\end{array}
\qquad
\begin{array}{lcl}
Q & \rightarrow & \mathtt{bbc} \\
  & | & \mathtt{bc}
\end{array}
$$

Q. Is it suitable for a ***top-down predictive*** parser?

- If so, show that it satisfies the *LL(1)* condition.
- If not, identify the problem(s) and correct it (them). Also show that the revised grammar satisfies the *LL(1)* condition.

BUP: Discovering Rightmost Derivation

- In TDP, we build the start variable as the ***root node***, and then work towards the ***leaves***. [**leftmost** derivation]
- In Bottom-Up Parsing (BUP):
  - Words (terminals) are still returned from **left** to **right** by the scanner.
  - As terminals, or a mix of terminals and variables, are identified as *reducible* to some variable *A* (i.e., matching the RHS of some production rule for *A*), then a layer is added.
  - Eventually:
    - ***accept***:
      The ***start variable*** is reduced and **all** words have been consumed.
    - ***reject***:
      The next word is not `eof`, but no further *reduction* can be identified.

Q. Why can BUP find the ***rightmost*** derivation (RMD), if any?

A. BUP discovers steps in a ***RMD*** in its *reverse* order.

BUP: Discovering Rightmost Derivation (1)

- ***table***-driven *LR(1)* parser: an implementation for BUP, which
  - Processes input from **L**eft to right
  - Constructs a **R**ightmost derivation
  - Uses a lookahead of **1** symbol
- A language has the *LR(1)* property if it:
  - Can be parsed in a single **L**eft to right scan,
  - To build a *reversed* **R**ightmost derivation,
  - Using a lookahead of **1** symbol to determine parsing actions.
- Critical step in a ***bottom-up parser*** is to find the ***next*** *handle*.

BUP: Discovering Rightmost Derivation (2)

```
ALGORITHM: BUParse
  INPUT: CFG G = (V, Σ, R, S), Action & Goto Tables
  OUTPUT: Report Parse Success or Syntax Error
PROCEDURE:
  initialize an empty stack trace
  trace.push(0) /* start state */
  word := NextWord()
  while(true)
    state := trace.top()
    act := Action[state, word]
    if act = ``accept'' then
      succeed()
    elseif act = ``reduce based on A → β'' then
      trace.pop() 2 × |β| times /* word + state */
      state := trace.top()
      trace.push(A)
      next := Goto[state, A]
      trace.push(next)
    elseif act = ``shift to s_i'' then
      trace.push(word)
      trace.push(i)
      word := NextWord()
    else
      fail()
```

BUP: Example Tracing (1)

- Consider the following grammar for parentheses:

1	*Goal*	$\rightarrow$ *List*
2	*List*	$\rightarrow$ *List Pair*
3		| *Pair*
4	*Pair*	$\rightarrow$ (*Pair*)
5		| ( )

- Assume: tables ***Action*** and ***Goto*** constructed accordingly:

State	Action Table			Goto Table	
	eof	(	)	*List*	*Pair*
0		s 3		1	2
1	acc	s 3			4
2	r 3	r 3			
3		s 6	s 7		5
4	r 2	r 2			
5			s 8		
6		s 6	s 10		9
7	r 5	r 5			
8	r 4	r 4			
9			s 11		
10			r 5		
11			r 4		

In ***Action*** table:

- s_i: shift to state i
- r_j: reduce to the LHS of production #j

BUP: Example Tracing (2.1)

Consider the steps of performing BUP on input `()`:

Iteration	State	word	Stack	Handle	Action
initial	—	(	$ 0	— *none* —	—
1	0	(	$ 0	— *none* —	*shift 3*
2	3	)	$ 0 (3	— *none* —	*shift 7*
3	7	eof	$ 0 (3) 7	( )	*reduce 5*
4	2	eof	$ 0 *Pair* 2	*Pair*	*reduce 3*
5	1	eof	$ 0 *List* 1	*List*	*accept*

BUP: Example Tracing (2.2)

Consider the steps of performing BUP on input `( ( ) ) ( )`:

Iteration	State	word	Stack	Handle	Action
initial	—	(	$ 0	— *none* —	—
1	0	(	$ 0	— *none* —	*shift 3*
2	3	(	$ 0 (3	— *none* —	*shift 6*
3	6	)	$ 0 (3 (6	— *none* —	*shift 10*
4	10	)	$ 0 (3 (6) 10	( )	*reduce 5*
5	5	)	$ 0 (3 *Pair* 5	— *none* —	*shift 8*
6	8	(	$ 0 (3 *Pair* 5) 8	(*Pair*)	*reduce 4*
7	2	(	$ 0 *Pair* 2	*Pair*	*reduce 3*
8	1	(	$ 0 *List* 1	— *none* —	*shift 3*
9	3	)	$ 0 *List* 1 (3	— *none* —	*shift 7*
10	7	eof	$ 0 *List* 1 (3) 7	( )	*reduce 5*
11	4	eof	$ 0 *List* 1 *Pair* 4	*List Pair*	*reduce 2*
12	1	eof	$ 0 *List* 1	*List*	*accept*

BUP: Example Tracing (2.3)

Consider the steps of performing BUP on input `())`:

Iteration	State	word	Stack	Handle	Action
initial	—	(	$ 0	— *none* —	—
1	0	(	$ 0	— *none* —	*shift 3*
2	3	)	$ 0 (3	— *none* —	*shift 7*
3	7	)	$ 0 (3) 7	— *none* —	*error*

LR(1) Items: Definition

- In **LR(1)** parsing, ***Action*** and ***Goto*** tabeles encode legitimate ways (w.r.t. a CFG) for finding ***handles*** (for ***reductions***).
- In a ***table***-driven **LR(1)** parser, the table-construction algorithm represents each potential ***handle*** (for a ***reduction***) with an **LR(1)** item e.g.,

$$[A \rightarrow \beta \bullet \gamma,\ \mathtt{a}]$$

  where:
  - A ***production rule*** $\boxed{A \rightarrow \beta\gamma}$ is currently being applied.
  - A ***terminal symbol*** $\boxed{\mathtt{a}}$ servers as a ***lookahead symbol***.
  - A ***placeholder*** $\boxed{\bullet}$ indicates the parser's ***stack top***.
    - ✓ The parser's ***stack*** contains β ("left context").
    - ✓ γ is yet to be matched.
    - Upon matching $\beta\gamma$, if $\mathtt{a}$ matches the current word, then we "replace" $\beta\gamma$ (and their associated states) with A (and its associated state).

LR(1) Items: Scenarios

An *LR(1) item* can denote:

1. **POSSIBILITY** $[A \rightarrow \bullet\beta\gamma,\ \mathtt{a}]$
   - In the current parsing context, an A would be valid.
   - $\bullet$ represents the position of the parser's ***stack top***
   - Recognizing a β next would be one step towards discovering an A.
2. **PARTIAL COMPLETION** $[A \rightarrow \beta \bullet \gamma,\ \mathtt{a}]$
   - The parser has progressed from $[A \rightarrow \bullet\beta\gamma,\ \mathtt{a}]$ by recognizing β.
   - Recognizing a γ next would be one step towards discovering an A.
3. **COMPLETION** $[A \rightarrow \beta\gamma\bullet,\ \mathtt{a}]$
   - Parser has progressed from $[A \rightarrow \bullet\beta\gamma,\ \mathtt{a}]$ by recognizing $\beta\gamma$.
   - $\beta\gamma$ found in a context where an A followed by $\mathtt{a}$ would be valid.
   - If the current input word matches $\mathtt{a}$, then:
     - Current ***complet item*** is a *handle*.
     - Parser can ***reduce*** $\beta\gamma$ to A
     - Accordingly, in the ***stack***, $\beta\gamma$ (and their associated states) are replaced with A (and its associated state).

LR(1) Items: Example (1.1)

Consider the following grammar for parentheses:

1	*Goal*	$\rightarrow$	*List*
2	*List*	$\rightarrow$	*List Pair*
3		|	*Pair*
4	*Pair*	$\rightarrow$	$\underline{(}$ *Pair* $\underline{)}$
5		|	$\underline{(}$ $\underline{)}$

Initial State: $[Goal \rightarrow \bullet List,\ \texttt{eof}]$

Desired Final State: $[Goal \rightarrow List\bullet,\ \texttt{eof}]$

Intermediate States: Subset Construction

Q. Derive all *LR(1) items* for the above grammar.

- **FOLLOW**($List$) = {eof, (}   **FOLLOW**($Pair$) = {eof, (,)}
- For each production $A \rightarrow \beta$, given **FOLLOW**(A), *LR(1) items* are:

$$\{\ [A \rightarrow \bullet\beta\gamma,\ \texttt{a}] \mid a \in \textbf{FOLLOW}(A)\ \}$$
$$\cup$$
$$\{\ [A \rightarrow \beta\bullet\gamma,\ \texttt{a}] \mid a \in \textbf{FOLLOW}(A)\ \}$$
$$\cup$$
$$\{\ [A \rightarrow \beta\gamma\bullet,\ \texttt{a}] \mid a \in \textbf{FOLLOW}(A)\ \}$$

LR(1) Items: Example (1.2)

Q. Given production $A \rightarrow \beta$ (e.g., $Pair \rightarrow$ `(` $Pair$ `)`), how many *LR(1) items* can be generated?

- The current parsing progress (on matching the RHS) can be:
  1. $\bullet$`(` $Pair$ `)`
  2. `(` $\bullet Pair$ `)`
  3. `(` $Pair\bullet$ `)`
  4. `(` $Pair$ `)`$\bullet$
- Lookahead symbol following $Pair$? **FOLLOW**($Pair$) = {`eof`, `(`, `)`}
- All possible *LR(1) items* related to $Pair \rightarrow$ `(` $Pair$ `)`?

  ✓ [$\bullet$`(` $Pair$ `)`, `eof`] [$\bullet$`(` $Pair$ `)`, `(`] [$\bullet$`(` $Pair$ `)`, `)`]
  ✓ [`(` $\bullet Pair$ `)`, `eof`] [`(` $\bullet Pair$ `)`, `(`] [`(` $\bullet Pair$ `)`, `)`]
  ✓ [`(` $Pair\bullet$ `)`, `eof`] [`(` $Pair\bullet$ `)`, `(`] [`(` $Pair\bullet$ `)`, `)`]
  ✓ [`(` $Pair$ `)`$\bullet$, `eof`] [`(` $Pair$ `)`$\bullet$, `(`] [`(` $Pair$ `)`$\bullet$, `)`]

A. How many in general (in terms of A and β)?

$$\underbrace{|\beta| + 1}_{\texttt{possible positions of } \bullet} \times \underbrace{|\textbf{FOLLOW}(A)|}_{\texttt{possible lookahead symbols}}$$

LR(1) Items: Example (1.3)

A. There are 33 *LR(1) items* in the parentheses grammar.

$[Goal \rightarrow \bullet\, List, \texttt{eof}]$		
$[Goal \rightarrow List\, \bullet, \texttt{eof}]$		
$[List \rightarrow \bullet\, List\ \ Pair, \texttt{eof}]$	$[List \rightarrow \bullet\, List\ \ Pair, \underline{(}\,]$	
$[List \rightarrow List\, \bullet\, Pair, \texttt{eof}]$	$[List \rightarrow List\, \bullet\, Pair, \underline{(}\,]$	
$[List \rightarrow List\ \ Pair\, \bullet, \texttt{eof}]$	$[List \rightarrow List\ \ Pair\, \bullet, \underline{(}\,]$	
$[List \rightarrow \bullet\, Pair, \texttt{eof}\,]$	$[List \rightarrow \bullet\, Pair, \underline{(}\,]$	
$[List \rightarrow Pair\, \bullet, \texttt{eof}\,]$	$[List \rightarrow Pair\, \bullet, \underline{(}\,]$	
$[Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \texttt{eof}\,]$	$[Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{)}]$	$[Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{(}]$
$[Pair \rightarrow \underline{(}\ \bullet\, Pair\ \underline{)}, \texttt{eof}\,]$	$[Pair \rightarrow \underline{(}\ \bullet\, Pair\ \underline{)}, \underline{)}]$	$[Pair \rightarrow \underline{(}\ \bullet\, Pair\ \underline{)}, \underline{(}]$
$[Pair \rightarrow \underline{(}\ Pair\, \bullet\ \underline{)}, \texttt{eof}\,]$	$[Pair \rightarrow \underline{(}\ Pair\, \bullet\ \underline{)}, \underline{)}]$	$[Pair \rightarrow \underline{(}\ Pair\, \bullet\ \underline{)}, \underline{(}]$
$[Pair \rightarrow \underline{(}\ Pair\ \underline{)}\, \bullet, \texttt{eof}\,]$	$[Pair \rightarrow \underline{(}\ Pair\ \underline{)}\, \bullet, \underline{)}]$	$[Pair \rightarrow \underline{(}\ Pair\ \underline{)}\, \bullet, \underline{(}]$
$[Pair \rightarrow \bullet\, \underline{(}\ \ \underline{)}, \texttt{eof}]$	$[Pair \rightarrow \bullet\, \underline{(}\ \ \underline{)}, \underline{(}]$	$[Pair \rightarrow \bullet\, \underline{(}\ \ \underline{)}, \underline{)}]$
$[Pair \rightarrow \underline{(}\ \bullet\ \underline{)}, \texttt{eof}]$	$[Pair \rightarrow \underline{(}\ \bullet\ \underline{)}, \underline{(}]$	$[Pair \rightarrow \underline{(}\ \bullet\ \underline{)}, \underline{)}]$
$[Pair \rightarrow \underline{(}\ \ \underline{)}\, \bullet, \texttt{eof}]$	$[Pair \rightarrow \underline{(}\ \ \underline{)}\, \bullet, \underline{(}]$	$[Pair \rightarrow \underline{(}\ \ \underline{)}\, \bullet, \underline{)}]$

LR(1) Items: Example (2)

Consider the following grammar for expressions:

0	$Goal \rightarrow Expr$			6	$Term' \rightarrow$	x $Factor\ Term'$
1	$Expr \rightarrow Term\ Expr'$			7	$\mid$	$\div$ $Factor\ Term'$
2	$Expr' \rightarrow + \ Term\ Expr'$			8	$\mid$	ϵ
3	$\mid$ - $Term\ Expr'$			9	$Factor \rightarrow$	($Expr$)
4	$\mid \epsilon$			10	$\mid$	num
5	$Term \rightarrow Factor\ Term'$			11	$\mid$	name

Q. Derive all *LR(1) items* for the above grammar.
Hints. First compute **FOLLOW** for each non-terminal:

	Expr	*Expr'*	*Term*	*Term'*	*Factor*
FOLLOW	eof,)	eof,)	eof, +, -,)	eof, +, -,)	eof, +, -, x, ÷,)

Tips. Ignore ϵ ***production*** such as $Expr' \rightarrow \epsilon$
since the **FOLLOW** sets already take them into consideration.

Canonical Collection ($\mathcal{CC}$) vs. LR(1) items

1	*Goal*	$\rightarrow$	*List*
2	*List*	$\rightarrow$	*List Pair*
3		|	*Pair*
4	*Pair*	$\rightarrow$	(*Pair*)
5		|	( )

Recall:

LR(1) Items: 33 items

Initial State: $[Goal \rightarrow \bullet List,\ \texttt{eof}]$

Desired Final State: $[Goal \rightarrow List \bullet,\ \texttt{eof}]$

- The *canonical collection* [Example of $\mathcal{CC}$]

  $$\mathcal{CC} = \{cc_0, cc_1, cc_2, \ldots, cc_n\}$$

  denotes the set of **valid subset states** of a **LR(1) parser**.
  - Each $cc_i \in \mathcal{CC}$ $(0 \leq i \leq n)$ is a set of ***LR(1) items***.
  - $\mathcal{CC} \subseteq \mathbb{P}($***LR(1) items***$)$ $|\mathcal{CC}|$? [$|\mathcal{CC}| \leq 2^{|\textit{LR(1) items}|}$]
- To model a **LR(1) parser**, we use techniques analogous to how an ϵ-NFA is converted into a DFA (subset construction and ϵ-closure).
- **Analogies.**
  - ✓ ***LR(1) items*** $\approx$ states of source *NFA*
  - ✓ $\mathcal{CC}$ $\approx$ subset states of target *DFA*

Constructing $\mathcal{CC}$: The *closure* Procedure (1)

```
1   ALGORITHM: closure
2     INPUT: CFG G = (V, Σ, R, S), a set s of LR(1) items
3     OUTPUT: a set of LR(1) items
4   PROCEDURE:
5     lastS := ∅
6     while (lastS ≠ s):
7       lastS := s
8       for [A → ··· • C δ, a] ∈ s:
9         for C → γ ∈ R:
10          for b ∈ FIRST(δa):
11            s := s ∪ { [ C → •γ, b] }
12    return s
```

- **Line 8**: $[A \rightarrow \cdots \bullet\ C\ \delta,\ a] \in s$ indicates that the parser's next task is to match $C\ \delta$ with a lookahead symbol a.
- **Line 9**: <u>Given</u>: matching γ can reduce to C
- **Line 10**: <u>Given</u>: $b \in$ **FIRST**(δa) is a valid lookahead symbol after reducing γ to C
- **Line 11**: Add a new item $[\ C \rightarrow \bullet\gamma,\ b]$ into s.
- **Line 6**: Termination is guaranteed.

  ∵ Each iteration adds ≥ 1 item to s (otherwise $lastS \neq s$ is *false*).

Constructing $\mathcal{CC}$: The *closure* Procedure (2.1)

1	*Goal*	$\rightarrow$	*List*	
2	*List*	$\rightarrow$	*List Pair*	
3		|	*Pair*	
4	*Pair*	$\rightarrow$	(*Pair*)	
5		|	( )	

Initial State: $[Goal \rightarrow \bullet List,\ \texttt{eof}]$

Calculate cc_0 = ***closure***$(\{\ [Goal \rightarrow \bullet List,\ \texttt{eof}]\ \})$.

Constructing $\mathcal{CC}$: The *goto* Procedure (1)

```
1   ALGORITHM: goto
2     INPUT: a set s of LR(1) items, a symbol x
3     OUTPUT: a set of LR(1) items
4   PROCEDURE:
5     moved := ∅
6     for item ∈ s:
7       if item = [α → β • xδ, a] then
8         moved := moved ∪ { [α → βx • δ, a] }
9       end
10    return closure(moved)
```

Line 7: Given: item $[\alpha \rightarrow \beta \bullet x\delta,\ \mathtt{a}]$ (where *x* is the next to match)
Line 8: Add $[\alpha \rightarrow \beta x \bullet \delta,\ \mathtt{a}]$ (indicating $\mathtt{x}$ is matched) to *moved*
Line 10: Calculate and return ***closure***(*moved*) as the "***next subset state***" from *s* with a "transition" $\mathtt{x}$.

Constructing $\mathcal{CC}$: The *goto* Procedure (2)

1	$Goal \rightarrow List$
2	$List \rightarrow List\ Pair$
3	$\mid\ Pair$
4	$Pair \rightarrow \underline{(}\ Pair\ \underline{)}$
5	$\mid\ \underline{(}\ \underline{)}$

$$cc_0 = \left\{ \begin{array}{ccc} [Goal \rightarrow \bullet\, List, \texttt{eof}] & [List \rightarrow \bullet\, List\ Pair, \texttt{eof}] & [List \rightarrow \bullet\, List\ Pair, \underline{(}] \\ [List \rightarrow \bullet\, Pair, \texttt{eof}] & [List \rightarrow \bullet\, Pair, \underline{(}] & [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \texttt{eof}] \\ [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{(}] & [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \texttt{eof}] & [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \underline{(}] \end{array} \right\}$$

Calculate $goto(cc_0, \texttt{(})$. ["next state" from cc_0 taking $\texttt{(}$]

Constructing $\mathcal{CC}$: The Algorithm (1)

```
ALGORITHM: BuildCC
  INPUT: a grammar G = (V, Σ, R, S), goal production S → S′
  OUTPUT:
    (1) a set CC = {cc_0, cc_1, ..., cc_n} where cc_i ⊆ G′s LR(1) items
    (2) a transition function
PROCEDURE:
  cc_0 := closure({[S → •S′, eof]})
  CC := {cc_0}
  processed := {cc_0}
  lastCC := ∅
  while(lastCC ≠ CC):
    lastCC := CC
    for cc_i s.t. cc_i ∈ CC ∧ cc_i ∉ processed:
      processed := processed ∪ {cc_i}
      for x s.t. [··· → ··· • x ...] ∈ cc_i
        temp := goto(cc_i, x)
        if temp ∉ CC then
          CC := CC ∪ {temp}
        end
        δ := δ ∪ (cc_i, x, temp)
```

Constructing $\mathcal{CC}$: The Algorithm (2.1)

1	*Goal*	$\rightarrow$	*List*
2	*List*	$\rightarrow$	*List Pair*
3		$\mid$	*Pair*
4	*Pair*	$\rightarrow$	$\underline{(}$ *Pair* $\underline{)}$
5		$\mid$	$\underline{(}$ $\underline{)}$

- Calculate $\mathcal{CC} = \{cc_0, cc_1, \ldots, cc_{11}\}$
- Calculate the transition function $\delta : \mathcal{CC} \times (\Sigma \cup V) \rightarrow \mathcal{CC}$

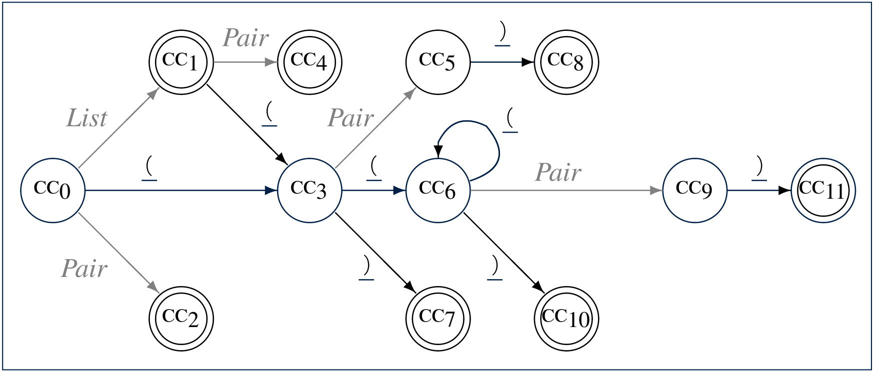
Constructing $\mathcal{CC}$: The Algorithm (2.2)

Resulting transition table:

Iteration	Item	*Goal*	*List*	*Pair*	(	)	eof
0	CC_0	∅	CC_1	CC_2	CC_3	∅	∅
1	CC_1	∅	∅	CC_4	CC_3	∅	∅
	CC_2	∅	∅	∅	∅	∅	∅
	CC_3	∅	∅	CC_5	CC_6	CC_7	∅
2	CC_4	∅	∅	∅	∅	∅	∅
	CC_5	∅	∅	∅	∅	CC_8	∅
	CC_6	∅	∅	CC_9	CC_6	CC_{10}	∅
	CC_7	∅	∅	∅	∅	∅	∅
3	CC_8	∅	∅	∅	∅	∅	∅
	CC_9	∅	∅	∅	∅	CC_{11}	∅
	CC_{10}	∅	∅	∅	∅	∅	∅
4	CC_{11}	∅	∅	∅	∅	∅	∅

Constructing $\mathcal{CC}$: The Algorithm (2.3)

Resulting DFA for the parser:



Constructing $\mathcal{CC}$: The Algorithm (2.4.1)

Resulting canonical collection $\mathcal{CC}$: [Def. of $\mathcal{CC}$]

$$cc_0 = \left\{ \begin{array}{ccc} [Goal \rightarrow \bullet\, List, \texttt{eof}] & [List \rightarrow \bullet\, List\ Pair, \texttt{eof}] & [List \rightarrow \bullet\, List\ Pair, \underline{(}] \\ [List \rightarrow \bullet\, Pair, \texttt{eof}] & [List \rightarrow \bullet\, Pair, \underline{(}] & [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \texttt{eof}] \\ [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{(}] & [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \texttt{eof}] & [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \underline{(}] \end{array} \right\}$$

$$cc_1 = \left\{ \begin{array}{ccc} [Goal \rightarrow List\, \bullet, \texttt{eof}] & [List \rightarrow List\, \bullet\, Pair, \texttt{eof}] & [List \rightarrow List\, \bullet\, Pair, \underline{(}] \\ [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \texttt{eof}] & [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{(}] & [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \texttt{eof}] \\ & [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \underline{(}] & \end{array} \right\}$$

$$cc_2 = \left\{ [List \rightarrow Pair\, \bullet, \texttt{eof}] \quad [List \rightarrow Pair\, \bullet, \underline{(}] \right\}$$

$$cc_3 = \left\{ \begin{array}{ccc} [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{)}] & [Pair \rightarrow \underline{(}\, \bullet\, Pair\ \underline{)}, \texttt{eof}] & [Pair \rightarrow \underline{(}\, \bullet\, Pair\ \underline{)}, \underline{(}] \\ [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \underline{)}] & [Pair \rightarrow \underline{(}\, \bullet\, \underline{)}, \texttt{eof}] & [Pair \rightarrow \underline{(}\, \bullet\, \underline{)}, \underline{(}] \end{array} \right\}$$

$$cc_4 = \left\{ [List \rightarrow List\ Pair\, \bullet, \texttt{eof}] \quad [List \rightarrow List\ Pair\, \bullet, \underline{(}] \right\}$$

$$cc_5 = \left\{ [Pair \rightarrow \underline{(}\ Pair\, \bullet\, \underline{)}, \texttt{eof}] \quad [Pair \rightarrow \underline{(}\ Pair\, \bullet\, \underline{)}, \underline{(}] \right\}$$

$$cc_6 = \left\{ \begin{array}{cc} [Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{)}] & [Pair \rightarrow \underline{(}\, \bullet\, Pair\ \underline{)}, \underline{)}] \\ [Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \underline{)}] & [Pair \rightarrow \underline{(}\, \bullet\, \underline{)}, \underline{)}] \end{array} \right\}$$

$$cc_7 = \left\{ [Pair \rightarrow \underline{(}\ \underline{)}\, \bullet, \texttt{eof}] \quad [Pair \rightarrow \underline{(}\ \underline{)}\, \bullet, \underline{(}] \right\}$$

$$cc_8 = \left\{ [Pair \rightarrow \underline{(}\ Pair\ \underline{)}\, \bullet, \texttt{eof}] \quad [Pair \rightarrow \underline{(}\ Pair\ \underline{)}\, \bullet, \underline{(}] \right\}$$

$$cc_9 = \left\{ [Pair \rightarrow \underline{(}\ Pair\, \bullet\, \underline{)}, \underline{)}] \right\}$$

$$cc_{10} = \left\{ [Pair \rightarrow \underline{(}\ \underline{)}\, \bullet, \underline{)}] \right\}$$

$$cc_{11} = \left\{ [Pair \rightarrow \underline{(}\ Pair\ \underline{)}\, \bullet, \underline{)}] \right\}$$

Constructing *Action* and *Goto* Tables (1)

```
1  ALGORITHM: BuildActionGotoTables
2    INPUT:
3      (1) a grammar G = (V, Σ, R, S)
4      (2) goal production S → S′
5      (3) a canonical collection CC = {cc_0, cc_1, ..., cc_n}
6      (4) a transition function δ : CC × Σ → CC
7    OUTPUT: Action Table & Goto Table
8  PROCEDURE:
9    for cc_i ∈ CC:
10     for item ∈ cc_i:
11       if item = [A → β • xγ, a] ∧ δ(cc_i, x) = cc_j then
12         Action[i, x] := shift j
13       elseif item = [A → β•, a] then
14         Action[i, a] := reduce A → β
15       elseif item = [S → S′•, eof] then
16         Action[i, eof] := accept
17       end
18     for v ∈ V:
19       if δ(cc_i, v) = cc_j then
20         Goto[i, v] = j
21       end
```

- **L12, 13**: Next valid step in discovering A is to match terminal symbol x.
- **L14, 15**: Having recognized β, if current word matches lookahead a, reduce β to A.
- **L16, 17**: Accept if input exhausted and what's recognized reducible to start var. S.
- **L20, 21**: Record consequence of a reduction to non-terminal v from state i

Constructing *Action* and *Goto* Tables (2)

Resulting **Action** and **Goto** tables:

State	*Action* **Table**			*Goto* **Table**	
	eof	(	)	***List***	***Pair***
0		s 3		1	2
1	acc	s 3			4
2	r 3	r 3			
3		s 6	s 7		5
4	r 2	r 2			
5			s 8		
6		s 6	s 10		9
7	r 5	r 5			
8	r 4	r 4			
9			s 11		
10			r 5		
11			r 4		

BUP: Discovering Ambiguity (1)

1	$Goal$	$\rightarrow$	$Stmt$
2	$Stmt$	$\rightarrow$	`if expr then` $Stmt$
3		$\mid$	`if expr then` $Stmt$ `else` $Stmt$
4		$\mid$	`assign`

- Calculate $\mathcal{CC} = \{cc_0, cc_1, \ldots, \}$
- Calculate the transition function $\delta : \mathcal{CC} \times \Sigma \rightarrow \mathcal{CC}$

BUP: Discovering Ambiguity (2.1)

Resulting transition table:

	Item	*Goal*	*Stmt*	if	expr	then	else	assign	eof
0	cc_0	∅	cc_1	cc_2	∅	∅	∅	cc_3	∅
1	cc_1	∅	∅	∅	∅	∅	∅	∅	∅
	cc_2	∅	∅	∅	cc_4	∅	∅	∅	∅
	cc_3	∅	∅	∅	∅	∅	∅	∅	∅
2	cc_4	∅	∅	∅	∅	cc_5	∅	∅	∅
3	cc_5	∅	cc_6	cc_7	∅	∅	∅	cc_8	∅
4	cc_6	∅	∅	∅	∅	∅	cc_9	∅	∅
	cc_7	∅	∅	∅	cc_{10}	∅	∅	∅	∅
	cc_8	∅	∅	∅	∅	∅	∅	∅	∅
5	cc_9	∅	cc_{11}	cc_2	∅	∅	∅	cc_3	∅
	cc_{10}	∅	∅	∅	∅	cc_{12}	∅	∅	∅
6	cc_{11}	∅	∅	∅	∅	∅	∅	∅	∅
	cc_{12}	∅	cc_{13}	cc_7	∅	∅	∅	cc_8	∅
7	cc_{13}	∅	∅	∅	∅	∅	cc_{14}	∅	∅
8	cc_{14}	∅	cc_{15}	cc_7	∅	∅	∅	cc_8	∅
9	cc_{15}	∅	∅	∅	∅	∅	∅	∅	∅

BUP: Discovering Ambiguity (2.2.1)

Resulting canonical collection $\mathcal{CC}$:

$$cc_0 = \left\{ \begin{array}{ll} [Goal \rightarrow \bullet\ Stmt, \texttt{eof}] & [Stmt \rightarrow \bullet\ \texttt{if expr then}\ Stmt, \texttt{eof}] \\ [Stmt \rightarrow \bullet\ \texttt{assign}, \texttt{eof}] & [Stmt \rightarrow \bullet\ \texttt{if expr then}\ Stmt\ \texttt{else}\ Stmt, \texttt{eof}] \end{array} \right\}$$

$$cc_1 = \left\{ [Goal \rightarrow Stmt\ \bullet, \texttt{eof}] \right\}$$

$$cc_2 = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if}\ \bullet\ \texttt{expr then}\ Stmt, \texttt{eof}], \\ [Stmt \rightarrow \texttt{if}\ \bullet\ \texttt{expr then}\ Stmt\ \texttt{else}\ Stmt, \texttt{eof}] \end{array} \right\}$$

$$cc_3 = \left\{ [Stmt \rightarrow \texttt{assign}\ \bullet, \texttt{eof}] \right\}$$

$$cc_4 = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if expr}\ \bullet\ \texttt{then}\ Stmt, \texttt{eof}], \\ [Stmt \rightarrow \texttt{if expr}\ \bullet\ \texttt{then}\ Stmt\ \texttt{else}\ Stmt, \texttt{eof}] \end{array} \right\}$$

$$cc_5 = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if expr then}\ \bullet\ Stmt, \texttt{eof}], \\ [Stmt \rightarrow \texttt{if expr then}\ \bullet\ Stmt\ \texttt{else}\ Stmt, \texttt{eof}], \\ [Stmt \rightarrow\ \bullet\ \texttt{if expr then}\ Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow\ \bullet\ \texttt{assign}, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow\ \bullet\ \texttt{if expr then}\ Stmt\ \texttt{else}\ Stmt, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

$$cc_6 = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if expr then}\ Stmt\ \bullet, \texttt{eof}], \\ [Stmt \rightarrow \texttt{if expr then}\ Stmt\ \bullet\ \texttt{else}\ Stmt, \texttt{eof}] \end{array} \right\}$$

$$cc_7 = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if}\ \bullet\ \texttt{expr then}\ Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \texttt{if}\ \bullet\ \texttt{expr then}\ Stmt\ \texttt{else}\ Stmt, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

BUP: Discovering Ambiguity (2.2.2)

Resulting canonical collection $\mathcal{CC}$:

$$CC_8 = \{[Stmt \rightarrow \texttt{assign} \bullet, \{\texttt{eof}, \texttt{else}\}]\}$$

$$CC_9 = \begin{Bmatrix} [Stmt \rightarrow \texttt{if expr then } Stmt \texttt{ else } \bullet \ Stmt, \texttt{eof}], \\ [Stmt \rightarrow \bullet \texttt{ if expr then } Stmt, \texttt{eof}], \\ [Stmt \rightarrow \bullet \texttt{ if expr then } Stmt \texttt{ else } Stmt, \texttt{eof}], \\ [Stmt \rightarrow \bullet \texttt{ assign}, \texttt{eof}] \end{Bmatrix}$$

$$CC_{10} = \begin{Bmatrix} [Stmt \rightarrow \texttt{if expr} \bullet \texttt{ then } Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \texttt{if expr} \bullet \texttt{ then } Stmt \texttt{ else } Stmt, \{\texttt{eof}, \texttt{else}\}] \end{Bmatrix}$$

$$CC_{11} = \{[Stmt \rightarrow \texttt{if expr then } Stmt \texttt{ else } Stmt \ \bullet, \texttt{eof}]\}$$

$$CC_{12} = \begin{Bmatrix} [Stmt \rightarrow \texttt{if expr then} \bullet \ Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \texttt{if expr then} \bullet \ Stmt \texttt{ else } Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \bullet \texttt{if expr then } Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \bullet \texttt{if expr then } Stmt \texttt{ else } Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \bullet \texttt{assign}, \{\texttt{eof}, \texttt{else}\}] \end{Bmatrix}$$

$$CC_{13} = \begin{Bmatrix} [Stmt \rightarrow \texttt{if expr then } Stmt \bullet, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \texttt{if expr then } Stmt \bullet \texttt{ else } Stmt, \{\texttt{eof}, \texttt{else}\}] \end{Bmatrix}$$

$$CC_{14} = \begin{Bmatrix} [Stmt \rightarrow \texttt{if expr then } Stmt \texttt{ else } \bullet \ Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \bullet \texttt{ if expr then } Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \bullet \texttt{ if expr then } Stmt \texttt{ else } Stmt, \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \bullet \texttt{ assign}, \{\texttt{eof}, \texttt{else}\}] \end{Bmatrix}$$

BUP: Discovering Ambiguity (3)

- Consider cc_{13}

$$cc_{13} = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if expr then}\ Stmt\ \bullet\ , \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \texttt{if expr then}\ Stmt\ \bullet\ \texttt{else}\ Stmt, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

Q. What does it mean if the current word to consume is `else`?
A. We can either ***shift*** (then expecting to match another *Stmt*) or ***reduce*** to a *Stmt*.
Action[13, `else`] cannot hold ***shift*** and ***reduce*** simultaneously.
⇒ This is known as the *shift-reduce conflict*.

- Consider another scenario:

$$cc_i = \left\{ \begin{array}{l} [A \rightarrow \gamma\delta\bullet,\ \texttt{a}], \\ [B \rightarrow \gamma\delta\bullet,\ \texttt{a}] \end{array} \right\}$$

Q. What does it mean if the current word to consume is `a`?
A. We can either ***reduce*** to A or ***reduce*** to B.
Action[i, a] cannot hold A and B simultaneously.
⇒ This is known as the *reduce-reduce conflict*.

Index (1)

Index (2)

Index (3)

Index (4)

Index (5)

Index (6)

Index (7)

Index (8)