Math Review: Logic, Sets, Relations

EECS4302 A:
Compilers and Interpreters
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Background for Self-Study

- Topics of ***sets*** and ***relations*** were covered in EECS1019/1090.
- Slide 3 to Slide 23 contain what you should recall.

Propositional Logic (1)

- A *proposition* is a statement of claim that must be of either *true* or *false*, but not both.
- Basic logical operands are of type Boolean: *true* and *false*.
- We use logical operators to construct compound statements.
  - Unary logical operator: negation ($\neg$)

p	$\neg p$
true	*false*
false	*true*

  - Binary logical operators: conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), equivalence ($\equiv$), and if-and-only-if ($\iff$).

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \iff q$	$p \equiv q$
true	*true*	*true*	*true*	*true*	*true*	*true*
true	*false*	*false*	*true*	*false*	*false*	*false*
false	*true*	*false*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*true*	*true*	*true*

Propositional Logic: Implication (1)

- Written as $p \Rightarrow q$ [pronounced as "p implies q"]
  - We call p the antecedent, assumption, or premise.
  - We call q the consequence or conclusion.
- Compare the *truth* of $p \Rightarrow q$ to whether a contract is *honoured*:
  - antecedent/assumption/premise $p \approx$ promised terms [e.g., salary]
  - consequence/conclusion $q \approx$ obligations [e.g., duties]
- When the promised terms are met, then the contract is:
  - *honoured* if the obligations fulfilled. [$(true \Rightarrow true) \iff true$]
  - *breached* if the obligations violated. [$(true \Rightarrow false) \iff false$]
- When the promised terms are not met, then:
  - Fulfilling the obligation (q) or not ($\neg q$) does *not breach* the contract.

p	q	$p \Rightarrow q$
false	*true*	*true*
false	*false*	*true*

Propositional Logic: Implication (2)

There are alternative, equivalent ways to expressing $p \Rightarrow q$:

- q **if** p

  q is *true* if p is *true*

- p **only if** q

  If p is *true*, then for $p \Rightarrow q$ to be *true*, it can only be that q is also *true*.
  Otherwise, if p is *true* but q is *false*, then $(true \Rightarrow false) \equiv false$.

Note. To prove $p \equiv q$, prove $p \iff q$ (pronounced: "p if and only if q"):

  - p **if** q [$p \Leftarrow q \equiv q \Rightarrow p$]
  - p **only if** q [$p \Rightarrow q$]

- p is **sufficient** for q [similar to q **if** p]

  For q to be *true*, it is sufficient to have p being *true*.

- q is **necessary** for p [similar to p **only if** q]

  If p is *true*, then it is necessarily the case that q is also *true*.
  Otherwise, if p is *true* but q is *false*, then $(true \Rightarrow false) \equiv false$.

- q **unless** $\neg p$ [When is $p \Rightarrow q$ *true*?]

  If q is *true*, then $p \Rightarrow q$ *true* regardless of p.
  If q is *false*, then $p \Rightarrow q$ cannot be *true* unless p is *false*.

Propositional Logic: Implication (3)

Given an implication $p \Rightarrow q$, we may construct its:

- **Inverse**: $\neg p \Rightarrow \neg q$ [negate antecedent and consequence]
- **Converse**: $q \Rightarrow p$ [swap antecedent and consequence]
- **Contrapositive**: $\neg q \Rightarrow \neg p$ [inverse of converse]

Propositional Logic (2)

- **Axiom**: Definition of $\Rightarrow$

$$p \Rightarrow q \equiv \neg p \vee q$$

- **Theorem**: Identity of $\Rightarrow$

$$true \Rightarrow p \equiv p$$

- **Theorem**: Zero of $\Rightarrow$

$$false \Rightarrow p \equiv true$$

- **Axiom**: De Morgan

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned}$$

- **Axiom**: Double Negation

$$p \equiv \neg(\neg p)$$

- **Theorem**: Contrapositive

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

Predicate Logic (1)

- A *predicate* is a ***universal*** or ***existential*** statement about objects in some universe of disclosure.
- Unlike propositions, predicates are typically specified using ***variables***, each of which declared with some ***range*** of values.
- We use the following symbols for common numerical ranges:
  - $\mathbb{Z}$: the set of integers [$-\infty, \dots, -1, 0, 1, \dots, +\infty$]
  - $\mathbb{N}$: the set of natural numbers [$0, 1, \dots, +\infty$]
- Variable(s) in a predicate may be ***quantified***:
  - *Universal quantification* :

    All values that a variable may take satisfy certain property.
    e.g., Given that i is a natural number, i is *always* non-negative.
  - *Existential quantification* :

    Some value that a variable may take satisfies certain property.
    e.g., Given that i is an integer, i *can be* negative.

Predicate Logic (2.1): Universal Q. (∀)

- A *universal quantification* has the form $(\forall X \bullet R \Rightarrow P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***For all*** (combinations of) values of variables listed in X that satisfies R, it is the case that P is satisfied.
  - $\forall i \bullet i \in \mathbb{N} \Rightarrow i \geq 0$ [*true*]
  - $\forall i \bullet i \in \mathbb{Z} \Rightarrow i \geq 0$ [*false*]
  - $\forall i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \Rightarrow i < j \vee i > j$ [*false*]
- *Proof Strategies*
  1. How to prove $(\forall X \bullet R \Rightarrow P)$ ***true***?
     - **Hint**. When is $R \Rightarrow P$ ***true***? [*true* ⇒ *true*, *false* ⇒ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.
  2. How to prove $(\forall X \bullet R \Rightarrow P)$ ***false***?
     - **Hint**. When is $R \Rightarrow P$ ***false***? [*true* ⇒ *false*]
     - Give a ***witness/counterexample*** of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.

Predicate Logic (2.2): Existential Q. (∃)

- An *existential quantification* has the form $(\exists X \bullet R \wedge P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***There exist*** (a combination of) values of variables listed in X that satisfy both R and P.
  - $\exists i \bullet i \in \mathbb{N} \wedge i \geq 0$ [*true*]
  - $\exists i \bullet i \in \mathbb{Z} \wedge i \geq 0$ [*true*]
  - $\exists i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \wedge (i < j \vee i > j)$ [*true*]
- *Proof Strategies*
  1. How to prove $(\exists X \bullet R \wedge P)$ ***true***?
     - **Hint**. When is $R \wedge P$ ***true***? [*true* ∧ *true*]
     - Give a ***witness*** of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
  2. How to prove $(\exists X \bullet R \wedge P)$ ***false***?
     - **Hint**. When is $R \wedge P$ ***false***? [*true* ∧ *false*, *false* ∧ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.

Predicate Logic (3): Exercises

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 0$.

  All 10 integers between 1 and 10 are greater than 0.
- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 1$.

  Integer 1 (a witness/counterexample) in the range between 1 and 10 is *not* greater than 1.
- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 1$.

  Integer 2 (a witness) in the range between 1 and 10 is greater than 1.
- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 10$?

  All integers in the range between 1 and 10 are *not* greater than 10.

Predicate Logic (4): Switching Quantifications

Conversions between ∀ and ∃:

$$(\forall X \bullet R \Rightarrow P) \iff \neg(\exists X \bullet R \wedge \neg P)$$
$$(\exists X \bullet R \wedge P) \iff \neg(\forall X \bullet R \Rightarrow \neg P)$$

Set of Tuples

Given n sets $S_1, S_2, \dots, S_n$, a ***cross/Cartesian product*** of theses sets is a set of n-tuples.

Each ***n-tuple*** $(e_1, e_2, \dots, e_n)$ contains n elements, each of which a member of the corresponding set.

$$S_1 \times S_2 \times \dots \times S_n = \{(e_1, e_2, \dots, e_n) \mid e_i \in S_i \land 1 \le i \le n\}$$

e.g., $\{a, b\} \times \{2, 4\} \times \{\$, \&\}$ is a set of triples:

$$\begin{aligned} & \{a, b\} \times \{2, 4\} \times \{\$, \&\} \\ = \; & \{\ (e_1, e_2, e_3) \mid e_1 \in \{a, b\} \land e_2 \in \{2, 4\} \land e_3 \in \{\$, \&\}\ \} \\ = \; & \left\{ \begin{array}{l} (a, 2, \$), (a, 2, \&), (a, 4, \$), (a, 4, \&), \\ (b, 2, \$), (b, 2, \&), (b, 4, \$), (b, 4, \&) \end{array} \right\} \end{aligned}$$

Relations (1): Constructing a Relation

A *relation* is a set of mappings, each being an ***ordered pair*** that maps a member of set S to a member of set T.

e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$

- $\varnothing$ is the ***minimum*** relation (i.e., an empty relation).
- $\boxed{S \times T}$ is the ***maximum*** relation (say r_1) between S and T, mapping from each member of S to each member in T:

$$\{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

- $\{(x, y) \mid (x, y) \in S \times T \land x \neq 1\}$ is a relation (say r_2) that maps only some members in S to every member in T:

$$\{(2, a), (2, b), (3, a), (3, b)\}$$

Relations (2.1): Set of Possible Relations

- We use the ***power set*** operator to express the set of ***all possible relations*** on S and T:

$$\mathbb{P}(S \times T)$$

  Each member in $\mathbb{P}(S \times T)$ is a relation.

- To declare a relation variable r, we use the colon (`:`) symbol to mean ***set membership***:

$$r : \mathbb{P}(S \times T)$$

- Or alternatively, we write:

$$r : S \leftrightarrow T$$

  where the set $S \leftrightarrow T$ is synonymous to the set $\mathbb{P}(S \times T)$

Relations (2.2): Exercise

Enumerate $\{a, b\} \leftrightarrow \{1, 2, 3\}$.

- **Hints**:
  - You may enumerate all relations in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$ via their *cardinalities*: 0, 1, ..., $|\{a, b\} \times \{1, 2, 3\}|$.
  - What's the *maximum* relation in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$?

$$\{\ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\ \}$$

- The answer is a set containing ***all*** of the following relations:
  - Relation with cardinality 0: $\varnothing$
  - How many relations with cardinality 1? $[\ \binom{|\{a,b\} \times \{1,2,3\}|}{1} = 6\ ]$
  - How many relations with cardinality 2? $[\ \binom{|\{a,b\} \times \{1,2,3\}|}{2} = \frac{6 \times 5}{2!} = 15\ ]$

  ...

  - Relation with cardinality $|\{a, b\} \times \{1, 2, 3\}|$:

$$\{\ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\ \}$$

Relations (3.1): Domain, Range, Inverse

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- **domain** of *r*: set of first-elements from *r*
  - Definition: $\mathrm{dom}(r) = \{ \, d \mid (d, r') \in r \, \}$
  - e.g., $\mathrm{dom}(r) = \{a, b, c, d, e, f\}$
- **range** of *r*: set of second-elements from *r*
  - Definition: $\mathrm{ran}(r) = \{ \, r' \mid (d, r') \in r \, \}$
  - e.g., $\mathrm{ran}(r) = \{1, 2, 3, 4, 5, 6\}$
- **inverse** of *r*: a relation like *r* with elements swapped
  - Definition: $r^{-1} = \{ \, (r', d) \mid (d, r') \in r \, \}$
  - e.g., $r^{-1} = \{(1, a), (2, b), (3, c), (4, a), (5, b), (6, c), (1, d), (2, e), (3, f)\}$

Relations (3.2): Image

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

relational image of *r* over set *s*: sub-range of *r* mapped by *s*.

- Definition: $r[s] = \{ \, r' \mid (d, r') \in r \land d \in s \, \}$
- e.g., $r[\{a, b\}] = \{1, 2, 4, 5\}$

Relations (3.3): Restrictions

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- **domain restriction** of *r* over set *ds*: sub-relation of *r* with domain *ds*.
  - Definition: $ds \lhd r = \{ \, (d, r') \mid (d, r') \in r \land d \in ds \, \}$
  - e.g., $\{a, b\} \lhd r = \{(\mathbf{a}, 1), (\mathbf{b}, 2), (\mathbf{a}, 4), (\mathbf{b}, 5)\}$
- **range restriction** of *r* over set *rs*: sub-relation of *r* with range *rs*.
  - Definition: $r \rhd rs = \{ \, (d, r') \mid (d, r') \in r \land r' \in rs \, \}$
  - e.g., $r \rhd \{1, 2\} = \{(a, \mathbf{1}), (b, \mathbf{2}), (d, \mathbf{1}), (e, \mathbf{2})\}$

Relations (3.4): Subtractions

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- **domain subtraction** of *r* over set *ds*: sub-relation of *r* with domain <u>not</u> *ds*.
  - Definition: $ds \mathbin{⩤} r = \{ \, (d, r') \mid (d, r') \in r \land d \notin ds \, \}$
  - e.g., $\{a, b\} \mathbin{⩤} r = \{(\mathbf{c}, 3), (\mathbf{c}, 6), (\mathbf{d}, 1), (\mathbf{e}, 2), (\mathbf{f}, 3)\}$
- **range subtraction** of *r* over set *rs*: sub-relation of *r* with range <u>not</u> *rs*.
  - Definition: $r \mathbin{⩥} rs = \{ \, (d, r') \mid (d, r') \in r \land r' \notin rs \, \}$
  - e.g., $r \mathbin{⩥} \{1, 2\} = \{(c, \mathbf{3}), (a, \mathbf{4}), (b, \mathbf{5}), (c, \mathbf{6}), (f, \mathbf{3})\}$

Functions (1): Functional Property

- A ***relation*** r on sets S and T (i.e., $r \in S \leftrightarrow T$) is also a *function* if it satisfies the ***functional property***:

$$\texttt{isFunctional}(r) \iff \forall s, t_1, t_2 \bullet (s \in S \land t_1 \in T \land t_2 \in T) \Rightarrow ((s, t_1) \in r \land (s, t_2) \in r \Rightarrow t_1 = t_2)$$

  - That is, in a ***function***, it is forbidden for a member of S to map to more than one members of T.
  - Equivalently, in a ***function***, two distinct members of T cannot be mapped by the same member of S.
- e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$, which of the following ***relations*** satisfy the above ***functional property***?
  - $S \times T$ [No]
    Witness 1: $(1, a), (1, b)$; ***Witness*** 2: $(2, a), (2, b)$; ***Witness*** 3: $(3, a), (3, b)$.
  - $(S \times T) \setminus \{(x, y) \mid (x, y) \in S \times T \land x = 1\}$ [No]
    Witness 1: $(2, a), (2, b)$; ***Witness*** 2: $(3, a), (3, b)$
  - $\{(1, a), (2, b), (3, a)\}$ [Yes]
  - $\{(1, a), (2, b)\}$ [Yes]

Functions (2.1): Total vs. Partial

Given a **relation** $r \in S \leftrightarrow T$

- r is a ***partial function*** if it satisfies the ***functional property***:

$$\boxed{r \in S \nrightarrow T} \iff (\texttt{isFunctional}(r) \land \mathrm{dom}(r) \subseteq S)$$

Remark. $r \in S \nrightarrow T$ means there **may (or may not) be** $s \in S$ s.t. $r(s)$ is ***undefined*** (i.e., $r[\{s\}] = \varnothing$).
  - e.g., $\{\ \{(\mathbf{2}, a), (\mathbf{1}, b)\}, \{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\}\ \} \subseteq \{1, 2, 3\} \nrightarrow \{a, b\}$
- r is a ***total function*** if there is a mapping for each $s \in S$:

$$\boxed{r \in S \rightarrow T} \iff (\texttt{isFunctional}(r) \land \mathrm{dom}(r) = S)$$

Remark. $r \in S \rightarrow T$ implies $r \in S \nrightarrow T$, but not vice versa. Why?
  - e.g., $\{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\} \in \{1, 2, 3\} \rightarrow \{a, b\}$
  - e.g., $\{(\mathbf{2}, a), (\mathbf{1}, b)\} \notin \{1, 2, 3\} \rightarrow \{a, b\}$

Functions (2.2): Relation Image vs. Function Application

- Recall: A ***function*** is a ***relation***, but a ***relation*** is not necessarily a ***function***.
- Say we have a ***partial function*** $f \in \{1, 2, 3\} \nrightarrow \{a, b\}$:

$$f = \{(\mathbf{3}, a), (\mathbf{1}, b)\}$$

  - With f wearing the ***relation*** hat, we can invoke *relational images*:

$$f[\{3\}] = \{a\} \qquad f[\{1\}] = \{b\} \qquad f[\{2\}] = \varnothing$$

Remark. $\Rightarrow |f[\{v\}]| \leq 1 \because$
  - each member in $\mathrm{dom}(f)$ is mapped to at most one member in $\mathrm{ran}(f)$
  - each input set $\{v\}$ is a **singleton** set
  - With f wearing the ***function*** hat, we can invoke *functional applications*:

$$f(3) = a \qquad f(1) = b \qquad f(2) \ \texttt{is} \ \textbf{undefined}$$

Index (1)

Index (2)

Index (3)