Scanner: Lexical Analysis

Readings: EAC2 Chapter 2

YORK UNIVERSITÉ UNIVERSITY

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Compilers and Interpreters
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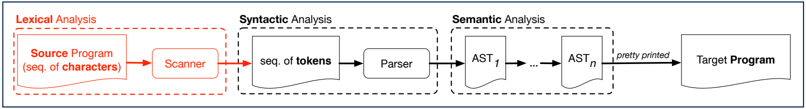
Scanner in Context

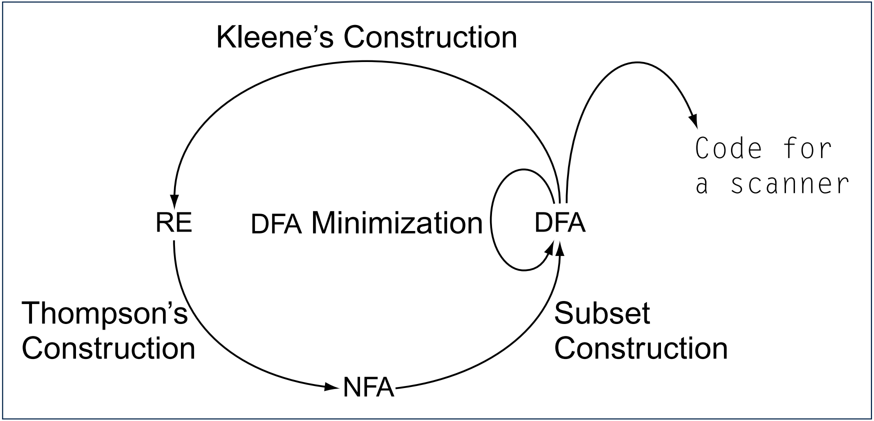
- Recall:



- Treats the input programas as a ***a sequence of characters***
- Applies rules ***recognizing*** character sequences as ***tokens***

  [***lexical*** analysis]
- Upon termination:
  - Reports character sequences not recognizable as tokens
  - Produces a ***a sequence of tokens***
- Only part of compiler touching ***every character*** in input program.
- Tokens *recognizable* by scanner constitute a *regular language*.

Scanner: Formulation & Implementation



Alphabets

An *alphabet* is a ***finite***, ***nonempty*** set of symbols.

- The convention is to write Σ, possibly with a informative subscript, to denote the alphabet in question.
- Use either a ***set enumeration*** or a ***set comprehension*** to define your own alphabet.

  e.g., $\Sigma_{eng} = \{a, b, \ldots, z, A, B, \ldots, Z\}$ [the English alphabet]

  e.g., $\Sigma_{bin} = \{0, 1\}$ [the binary alphabet]

  e.g., $\Sigma_{dec} = \{d \mid 0 \leq d \leq 9\}$ [the decimal alphabet]

  e.g., Σ_{key} [the keyboard alphabet]

Strings (1)

- A ***string*** or a ***word*** is ***finite*** sequence of symbols chosen from some ***alphabet***.

  e.g., Oxford is a string over the English alphabet Σ_{eng}

  e.g., 01010 is a string over the binary alphabet Σ_{bin}

  e.g., 01010.01 is **not** a string over Σ_{bin}

  e.g., 57 is a string over the decimal alphabet Σ_{dec}
- It is **not** correct to say, e.g., $01010 \in \Sigma_{bin}$ [Why?]
- The ***length*** of a string w, denoted as $|w|$, is the number of characters it contains.
  - e.g., $|Oxford| = 6$
  - ϵ is the ***empty string*** ($|\epsilon| = 0$) that may be from any alphabet.
- Given two strings x and y, their ***concatenation***, denoted as xy, is a new string formed by a copy of x followed by a copy of y.
  - e.g., Let $x = 01101$ and $y = 110$, then $xy = 01101110$
  - The empty string ϵ is the ***identity for concatenation***:

    $\epsilon w = w = w\epsilon$ for any string w

Strings (2)

- Given an ***alphabet*** Σ, we write Σ^k, where $k \in \mathbb{N}$, to denote the ***set of strings of length k from Σ***

$$\Sigma^k = \{w \mid \underbrace{w \text{ is a string over } \Sigma}_{\textit{more formal?}} \land |w| = k\}$$

  - e.g., $\{0, 1\}^2 = \{00, 01, 10, 11\}$
  - Given Σ, Σ^0 is $\{\epsilon\}$
- Given Σ, Σ^+ is the ***set of nonempty strings***.

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots = \{w \mid w \in \Sigma^k \land k > 0\} = \bigcup_{k>0} \Sigma^k$$

- Given Σ, Σ^* is the ***set of strings of all possible lengths***.

$$\Sigma^* = \Sigma^+ \cup \{\epsilon\}$$

Review Exercises: Strings

1. What is $|\{a, b, \ldots, z\}^5|$?
2. Enumerate, in a systematic manner, the set $\{a, b, c\}^4$.
3. Explain the difference between Σ and Σ^1.
4. Prove or disprove: $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \Sigma_1^* \subseteq \Sigma_2^*$

Languages

- *A language L over Σ* (where $|\Sigma|$ is finite) is a set of strings s.t.

$$L \subseteq \Sigma^*$$

- When useful, include an informative subscript to denote the ***language*** L in question.
  - e.g., The language of ***compilable*** Java programs

    $$L_{Java} = \{prog \mid prog \in \Sigma_{key}^* \wedge prog \texttt{ compiles in Eclipse}\}$$

    Note. *prog* compiling means **no** ***lexical***, ***syntactical***, or ***type*** errors.
  - e.g., The language of strings with n 0's followed by n 1's ($n \geq 0$)

    $$\{\epsilon, 01, 0011, 000111, \dots\} = \{0^n 1^n \mid n \geq 0\}$$
  - e.g., The language of strings with an equal number of 0's and 1's

    $$\begin{array}{rl} & \{\epsilon, 01, 10, 0011, 0101, 0110, 1100, 1010, 1001, \dots\} \\ = & \{w \mid \texttt{\# of 0's in } w = \texttt{\# of 1's in } w\} \end{array}$$

Review Exercises: Languages

1. Use ***set comprehensions*** to define the following ***languages***. Be as ***formal*** as possible.
   - A language over $\{0, 1\}$ consisting of strings beginning with some 0's (possibly none) followed by at least as many 1's.
   - A language over $\{a, b, c\}$ consisting of strings beginning with some a's (possibly none), followed by some b's and then some c's, s.t. the # of a's is at least as many as the sum of #'s of b's and c's.
2. Explain the difference between the two languages $\{\epsilon\}$ and $\varnothing$.
3. Justify that Σ^*, $\varnothing$, and $\{\epsilon\}$ are all languages over Σ.
4. Prove or disprove: If L is a language over Σ, and $\Sigma_2 \supseteq \Sigma$, then L is also a language over Σ_2.
   Hint: Prove that $\Sigma \subseteq \Sigma_2 \land L \subseteq \Sigma^* \Rightarrow L \subseteq \Sigma_2^*$
5. Prove or disprove: If L is a language over Σ, and $\Sigma_2 \subseteq \Sigma$, then L is also a language over Σ_2.
   Hint: Prove that $\Sigma_2 \subseteq \Sigma \land L \subseteq \Sigma^* \Rightarrow L \subseteq \Sigma_2^*$

Problems

- Given a ***language*** L over some ***alphabet*** Σ, a problem is the ***decision*** on whether or not a given ***string*** w is a member of L.

$$w \in L$$

  Is this equivalent to deciding $w \in \Sigma^*$? [***No***]

  $w \in \Sigma^* \Rightarrow w \in L$ is **not** necessarily true.

- e.g., The Java compiler solves the problem of ***deciding*** if a user-supplied ***string of symbols*** is a **member** of L_{Java}.

Regular Expressions (RE): Introduction

- *Regular expressions* (RegExp's) are:
  - A type of language-defining notation
    - This is ***similar*** to the equally-expressive ***DFA***, ***NFA***, and ϵ-***NFA***.
  - *Textual* and look just like a programming language
    - e.g., Set of strings denoted by $\texttt{01}^* + \texttt{10}^*$? [specify formally]
      $L = \{0x \mid x \in \{1\}^*\} \cup \{1x \mid x \in \{0\}^*\}$
    - e.g., Set of strings denoted by $(\texttt{0}^*\texttt{10}^*\texttt{10}^*)^*\texttt{10}^*$?
      $L = \{w \mid w \texttt{ has odd \# of } 1\texttt{'s}\}$
    - This is ***dissimilar*** to the diagrammatic ***DFA***, ***NFA***, and ϵ-***NFA***.
    - RegExp's can be considered as a "user-friendly" alternative to ***NFA*** for describing software components. [e.g., text search]
    - Writing a RegExp is like writing an algebraic expression, using the defined operators, e.g., `((4 + 3) * 5) % 6`
- Despite the programming convenience they provide, ***RegExp's***, ***DFA***, ***NFA***, and ϵ-***NFA*** are all *provably equivalent*.
  - They are capable of defining **all** and **only** regular languages.

RE: Language Operations (1)

- Given Σ of input alphabets, the simplest RegExp is? [$s \in \Sigma^1$]
  - e.g., Given $\Sigma = \{a, b, c\}$, expression a denotes the language $\{ a \}$ consisting of a single string a.
- Given two languages $L, M \in \Sigma^*$, there are 3 operators for building a *larger language* out of them:

1. ***Union***

$$L \cup M = \{w \mid w \in L \lor w \in M\}$$

   In the textual form, we write $+$ for union.

2. ***Concatenation***

$$LM = \{xy \mid x \in L \land y \in M\}$$

   In the textual form, we write either . or nothing at all for concatenation.

RE: Language Operations (2)

3. ***Kleene Closure*** (or ***Kleene Star***)

$$L^* = \bigcup_{i \geq 0} L^i$$

where

$$\begin{array}{lcl}
L^0 & = & \{\epsilon\} \\
L^1 & = & L \\
L^2 & = & \{x_1 x_2 \mid x_1 \in L \wedge x_2 \in L\} \\
\dots & & \\
L^i & = & \{ \underbrace{x_1 x_2 \dots x_i}_{i \texttt{ concatenations}} \mid x_j \in L \wedge 1 \leq j \leq i\} \\
\dots & &
\end{array}$$

In the textual form, we write $\star$ for closure.

Question: What is $|L^i|$ ($i \in \mathbb{N}$)? [$|L|^i$]

Question: Given that $L = \{0\}^*$, what is L^*? [L]

RE: Construction (1)

We may build *regular expressions* ***recursively***:

- Each (***basic*** or ***recursive***) form of regular expressions denotes a ***language*** (i.e., a set of strings that it accepts).
- ***Base Case***:
  - Constants ϵ and $\varnothing$ are regular expressions.

$$\begin{array}{rcl} L(\ \epsilon\ ) & = & \{\epsilon\} \\ L(\ \varnothing\ ) & = & \varnothing \end{array}$$

  - An input symbol $a \in \Sigma$ is a regular expression.

$$L(\ \boldsymbol{a}\ ) = \{a\}$$

    If we want a regular expression for the language consisting of only the string $w \in \Sigma^*$, we write w as the regular expression.
  - Variables such as **L**, **M**, *etc.*, might also denote languages.

RE: Construction (2)

- ***Recursive Case***: Given that E and F are regular expressions:
  - The union $E + F$ is a regular expression.

$$L(\ \mathbf{E+F}\ ) = L(E) \cup L(F)$$

  - The concatenation EF is a regular expression.

$$L(\ \mathbf{EF}\ ) = L(E)L(F)$$

  - Kleene closure of E is a regular expression.

$$L(\ E^*\ ) = (L(E))^*$$

  - A parenthesized E is a regular expression.

$$L(\ \mathbf{(E)}\ ) = L(E)$$

RE: Construction (3)

Exercises:

- $\varnothing + L$ $\qquad$ [$\varnothing + L = L = \varnothing + L$]
- $\varnothing L$ $\qquad$ [$\varnothing L = \varnothing = L\varnothing$]
- $\varnothing^*$

$$\begin{aligned}\varnothing^* &= \varnothing^0 \cup \varnothing^1 \cup \varnothing^2 \cup \ldots \\ &= \{\epsilon\} \cup \varnothing \cup \varnothing \cup \ldots \\ &= \{\epsilon\}\end{aligned}$$

- $\varnothing^* L$ $\qquad$ [$\varnothing^* L = L = L\varnothing^*$]

RE: Construction (4)

Write a regular expression for the following language

$$\{ w \mid w \texttt{ has alternating } 0\texttt{'s and } 1\texttt{'s} \}$$

- Would $(01)^*$ work? [alternating 10's?]
- Would $(01)^* + (10)^*$ work? [starting and ending with 1?]
- $0(10)^* + (01)^* + (10)^* + 1(01)^*$
- It seems that:
  - 1st and 3rd terms have $(10)^*$ as the common factor.
  - 2nd and 4th terms have $(01)^*$ as the common factor.
- Can we simplify the above regular expression?
- $(\epsilon + 0)(10)^* + (\epsilon + 1)(01)^*$

RE: Review Exercises

Write the regular expressions to describe the following languages:

- $\{ w \mid w \texttt{ ends with } 01 \}$
- $\{ w \mid w \texttt{ contains } 01 \texttt{ as a substring} \}$
- $\{ w \mid w \texttt{ contains no more than three consecutive } 1\texttt{'s} \}$
- $\{ w \mid w \texttt{ ends with } 01 \vee w \texttt{ has an odd \# of 0's} \}$
- $$\left\{ sx.y \;\middle|\; \begin{array}{ll} & s \in \{+, -, \epsilon\} \\ \wedge & x \in \Sigma_{dec}^{*} \\ \wedge & y \in \Sigma_{dec}^{*} \\ \wedge & \neg(x = \epsilon \wedge y = \epsilon) \end{array} \right\}$$
- $$\left\{ xy \;\middle|\; \begin{array}{ll} & x \in \{0, 1\}^{*} \wedge y \in \{0, 1\}^{*} \\ \wedge & x \texttt{ has alternating } 0\texttt{'s and } 1\texttt{'s} \\ \wedge & y \texttt{ has an odd \# } 0\texttt{'s and an odd \# } 1\texttt{'s} \end{array} \right\}$$

RE: Operator Precedence

- In an order of ***decreasing precedence***:
  - Kleene star operator
  - Concatenation operator
  - Union operator
- When necessary, use ***parentheses*** to force the intended order of evaluation.
- e.g.,
  - 10^* vs. $(10)^*$ [10^* is equivalent to $1(0^*)$]
  - $01^* + 1$ vs. $0(1^* + 1)$ [$01^* + 1$ is equivalent to $(0(1^*)) + (1)$]
  - $0 + 1^*$ vs. $(0 + 1)^*$ [$0 + 1^*$ is equivalent to $(0) + (1^*)$]

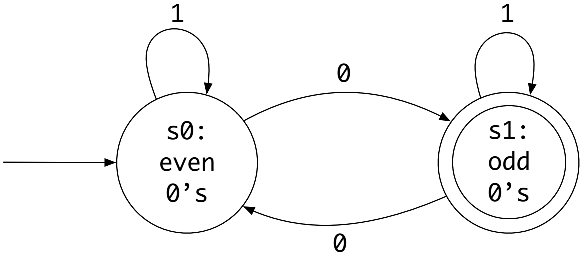
DFA: Deterministic Finite Automata (1.1)

- A *deterministic finite automata (DFA)* is a ***finite state machine*** (***FSM***) that ***accepts*** (or ***recognizes***) a pattern of behaviour.
  - For ***lexical*** analysis, we study patterns of ***strings*** (i.e., how ***alphabet*** symbols are ordered).
  - Unless otherwise specified, we consider strings in $\{0,1\}^*$
  - Each pattern contains the set of satisfying strings.
  - We describe the patterns of strings using set comprehensions:
    - $\{\, w \mid w \texttt{ has an odd number of 0's} \,\}$
    - $\{\, w \mid w \texttt{ has an even number of 1's} \,\}$
    - $\left\{ w \;\middle|\; \begin{array}{ll} & w \neq \epsilon \\ \wedge & w \texttt{ has equal \# of alternating 0's and 1's} \end{array} \right\}$
    - $\{\, w \mid w \texttt{ contains } 01 \texttt{ as a substring} \,\}$
    - $\left\{ w \;\middle|\; \begin{array}{ll} & w \texttt{ has an even number of 0's} \\ \wedge & w \texttt{ has an odd number of 1's} \end{array} \right\}$
- Given a pattern description, we design a ***DFA*** that accepts it.
  - The resulting ***DFA*** can be transformed into an executable program.

DFA: Deterministic Finite Automata (1.2)

- The ***transition diagram*** below defines a DFA which ***accepts***/***recognizes*** exactly the language

$$\{ w \mid w \texttt{ has an odd number of 0's} \}$$

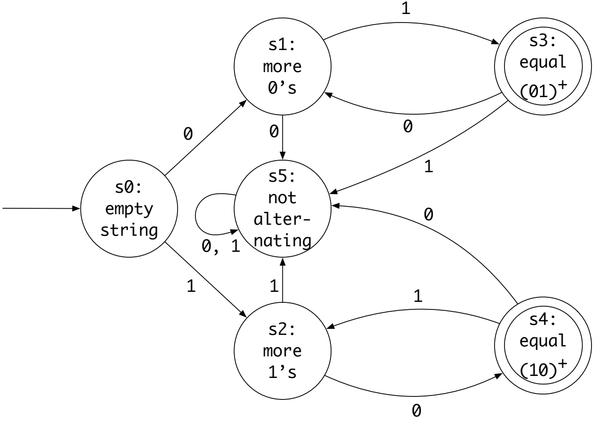


- Each **incoming** or **outgoing** arc (called a ***transition***) corresponds to an input alphabet symbol.
- s_0 with an unlabelled **incoming** transition is the ***start state***.
- s_3 drawn as a double circle is a ***final state***.
- All states have **outgoing** transitions covering $\{0, 1\}$.

DFA: Deterministic Finite Automata (1.3)

The ***transition diagram*** below defines a DFA which ***accepts***/***recognizes*** exactly the language

$$\left\{ w \;\middle|\; \begin{array}{ll} & w \neq \epsilon \\ \wedge & w \texttt{ has equal \# of alternating 0's and 1's} \end{array} \right\}$$



Review Exercises: Drawing DFAs

Draw the transition diagrams for DFAs which accept other example string patterns:

- $\{\, w \mid w \texttt{ has an even number of 1's} \,\}$
- $\{\, w \mid w \texttt{ contains } 01 \texttt{ as a substring} \,\}$
- $\left\{ w \mid \begin{array}{ll} & w \texttt{ has an even number of 0's} \\ \wedge & w \texttt{ has an odd number of 1's} \end{array} \right\}$

DFA: Deterministic Finite Automata (2.1)

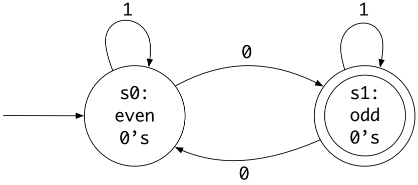
A *deterministic finite automata (DFA)* is a 5-tuple

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

- Q is a finite set of *states*.
- Σ is a finite set of *input symbols* (i.e., the *alphabet*).
- $\delta : (Q \times \Sigma) \rightarrow Q$ is a *transition function*

  δ takes as arguments a state and an input symbol and returns a state.
- $q_0 \in Q$ is the *start state*.
- $F \subseteq Q$ is a set of *final* or *accepting states*.

DFA: Deterministic Finite Automata (2.2)



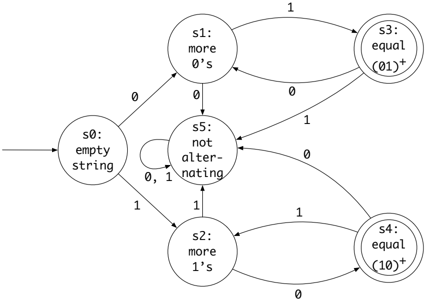
We formalize the above DFA as $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$, where

- $Q = \{s_0, s_1\}$
- $\Sigma = \{0, 1\}$
- $\delta = \{((s_0, 0), s_1), ((s_0, 1), s_0), ((s_1, 0), s_0), ((s_1, 1), s_1)\}$

state \ input	0	1
s_0	s_1	s_0
s_1	s_0	s_1

- $q_0 = s_0$
- $F = \{s_1\}$

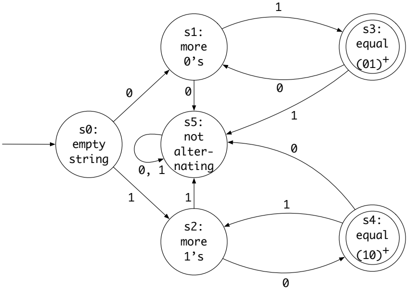
DFA: Deterministic Finite Automata (2.3.1)



We formalize the above DFA as $M = (Q, \Sigma, \delta, q_0, F)$, where

- $Q = \{s_0, s_1, s_2, s_3, s_4, s_5\}$
- $\Sigma = \{0, 1\}$
- $q_0 = s_0$
- $F = \{s_3, s_4\}$

DFA: Deterministic Finite Automata (2.3.2)



- $\delta =$

state \ input	0	1
s_0	s_1	s_2
s_1	s_5	s_3
s_2	s_4	s_5
s_3	s_1	s_5
s_4	s_5	s_2
s_5	s_5	s_5

DFA: Deterministic Finite Automata (2.4)

- Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$:
  - We write *L(M)* to denote the *language of M*: the set of strings that M ***accepts***.
  - A string is ***accepted*** if it results in a sequence of transitions: beginning from the ***start*** state and ending in a ***final*** state.

$$L(M) = \left\{ \begin{array}{l} a_1 a_2 \dots a_n \mid \\ \quad 1 \le i \le n \ \wedge \ a_i \in \Sigma \ \wedge \ \delta(q_{i-1}, a_i) = q_i \ \wedge \ q_n \in F \end{array} \right\}$$

  - M ***rejects*** any string $w \notin L(M)$.
- We may also consider *L(M)* as concatenations of labels from the set of all valid ***paths*** of *M*'s transition diagram; each such path starts with q_0 and ends in a state in F.

DFA: Deterministic Finite Automata (2.5)

- Given a ***DFA*** $M = (Q, \Sigma, \delta, q_0, F)$, we may simplify the definition of *L(M)* by extending δ (which takes an input symbol) to $\hat{\delta}$ (which takes an input string).

  $$\hat{\delta} : (Q \times \Sigma^*) \to Q$$

  We may define $\hat{\delta}$ recursively, using δ!

  $$\begin{array}{lcl} \hat{\delta}(q, \epsilon) & = & q \\ \hat{\delta}(q, xa) & = & \delta(\ \hat{\delta}(q, x), a\ ) \end{array}$$

  where $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$
- A neater definition of $L(M)$: the set of strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ is an ***accepting state***.

  $$L(M) = \{w \mid w \in \Sigma^* \wedge \hat{\delta}(q_0, w) \in F\}$$

- A language L is said to be a *regular language*, if there is some ***DFA M*** such that $L = L(M)$.

Review Exercises: Formalizing DFAs

Formalize DFAs (as 5-tuples) for the other example string patterns mentioned:

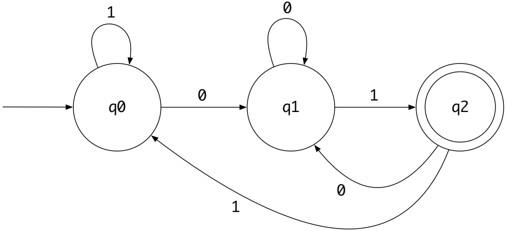
- $\{\, w \mid w \texttt{ has an even number of 0's} \,\}$
- $\{\, w \mid w \texttt{ contains } 01 \texttt{ as a substring} \,\}$
- $\left\{ w \;\middle|\; \begin{array}{ll} & w \texttt{ has an even number of 0's} \\ \wedge & w \texttt{ has an odd number of 1's} \end{array} \right\}$

NFA: Nondeterministic Finite Automata (1.1)

Problem: Design a DFA that accepts the following language:

$$L = \{ \ x01 \mid x \in \{0, 1\}^* \ \}$$

That is, L is the set of strings of 0s and 1s ending with 01.



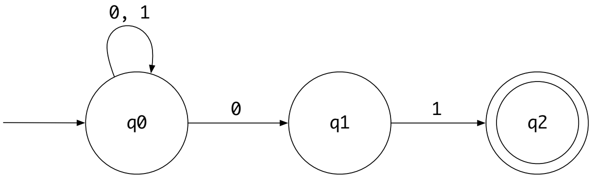
Given an input string w, we may simplify the above DFA by:

- *nondeterministically* treating state q_0 as both:
  - a state *ready* to read the last two input symbols from w
  - a state *not yet ready* to read the last two input symbols from w
- substantially reducing the outgoing transitions from q_1 and q_2

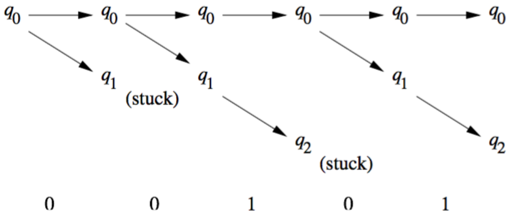
Compare the above DFA with the DFA in slide 39.

NFA: Nondeterministic Finite Automata (1.2)

- A *non-deterministic finite automata (NFA)* that accepts the same language:



- How an NFA determines if an input *00101* should be processed:



NFA: Nondeterministic Finite Automata (2)

- A *nondeterministic finite automata (NFA)*, like a **DFA**, is a ***FSM*** that ***accepts*** (or ***recognizes***) a pattern of behaviour.
- An ***NFA*** being *nondeterministic* means that from a given state, the **same input label** might corresponds to **multiple transitions** that lead to **distinct states**.
  - Each such transition offers an ***alternative path***.
  - Each alternative path is explored in parallel.
  - If **there exists** an alternative path that ***succeeds*** in processing the input string, then we say the ***NFA accepts*** that input string.
  - If **all** alternative paths get stuck at some point and ***fail*** to process the input string, then we say the ***NFA rejects*** that input string.
- ***NFAs*** are often more succinct (i.e., fewer states) and easier to design than **DFAs**.
- However, ***NFAs*** are just as ***expressive*** as are **DFAs**.
  - We can **always** convert an ***NFA*** to a **DFA**.

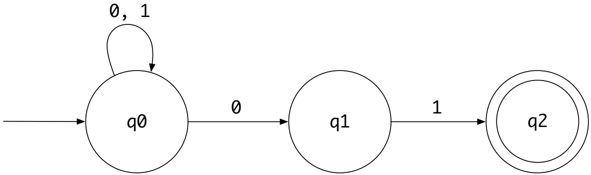
NFA: Nondeterministic Finite Automata (3.1)

- A *nondeterministic finite automata (NFA)* is a 5-tuple

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

  - Q is a finite set of *states*.
  - Σ is a finite set of *input symbols* (i.e., the *alphabet*).
  - $\delta : (Q \times \Sigma) \rightarrow \mathbb{P}(Q)$ is a *transition function*
    - Given a state and an input symbol, δ returns a set of states.
    - Equivalently, we can write: $\delta : (Q \times \Sigma) \nrightarrow Q$ [a <u>partial</u> function]
  - $q_0 \in Q$ is the *start state*.
  - $F \subseteq Q$ is a set of *final* or *accepting states*.
- What is the difference between a ***DFA*** and an ***NFA***?
  - δ of a ***DFA*** returns a <u>single</u> state.
  - δ of an ***NFA*** returns a (possibly empty) <u>set</u> of states.

NFA: Nondeterministic Finite Automata (3.2)



Given an input string 00101:

- **Read 0**: $\delta(\boxed{q_0}, 0) = \{\ \boxed{q_0}, q_1\ \}$
- **Read 0**: $\delta(\boxed{q_0}, 0) \cup \delta(q_1, 0) = \{\ \boxed{q_0}, q_1\ \} \cup \varnothing = \{\ q_0, q_1\ \}$
- **Read 1**: $\delta(\boxed{q_0}, 1) \cup \delta(q_1, 1) = \{\ q_0\ \} \cup \{\ q_2\ \} = \{\ \boxed{q_0}, q_2\ \}$
- **Read 0**: $\delta(\boxed{q_0}, 0) \cup \delta(q_2, 0) = \{\ q_0, q_1\ \} \cup \varnothing = \{\ q_0, \boxed{q_1}\ \}$
- **Read 1**: $\delta(q_0, 1) \cup \delta(\boxed{q_1}, 1) = \{\ q_0, q_1\ \} \cup \{\ q_2\ \} = \{\ q_0, q_1, \boxed{q_2}\ \}$

  $\because \{\ q_0, q_1, q_2\ \} \cap \{\ q_2\ \} \neq \varnothing \therefore$ `00101 is` *`accepted`*

NFA: Nondeterministic Finite Automata (3.3)

- Given a *NFA* $M = (Q, \Sigma, \delta, q_0, F)$, we may simplify the definition of $L(M)$ by extending δ (which takes an input symbol) to $\hat{\delta}$ (which takes an input string).

$$\hat{\delta} : (Q \times \Sigma^*) \to \mathbb{P}(Q)$$

We may define $\hat{\delta}$ recursively, using δ!

$$\begin{array}{rcl} \hat{\delta}(q, \epsilon) & = & \{q\} \\ \hat{\delta}(q, xa) & = & \bigcup\{\delta(q', a) \mid q' \in \hat{\delta}(q, x)\} \end{array}$$

where $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$

- A neater definition of $L(M)$: the set of strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains **at least one** *accepting state*.

$$L(M) = \{w \mid w \in \Sigma^* \wedge \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

DFA ≡ NFA (1)

- For many languages, constructing an accepting ***NFA*** is easier than a ***DFA***.
- From each state of an ***NFA***:
  - Outgoing transitions need **not** cover the entire Σ.
  - From a given state, the same symbol may ***non-deterministically*** lead to multiple states.
- In practice:
  - An ***NFA*** has just as many states as its equivalent DFA does.
  - An ***NFA*** often has fewer transitions than its equivalent ***DFA*** does.
- In the **worst** case:
  - While an ***NFA*** has n states, its equivalent ***DFA*** has 2^n states.
- Nonetheless, an ***NFA*** is still just as ***expressive*** as a ***DFA***.
  - A ***language*** accepted by some ***NFA*** is accepted by some ***DFA***:

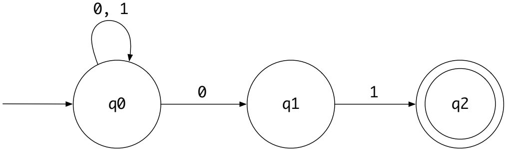
$$\forall N \bullet N \in NFA \Rightarrow (\exists D \bullet D \in DFA \land L(D) = L(N))$$

  - And vice versa, trivially?

$$\forall D \bullet D \in DFA \Rightarrow (\exists N \bullet N \in NFA \land L(D) = L(N))$$

DFA ≡ NFA (2.2): Lazy Evaluation (1)

Given an ***NFA***:



Subset construction (with ***lazy evaluation***) produces a ***DFA*** with δ as:

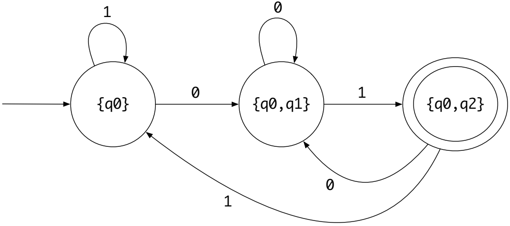
state \ input	0	1
$\{q_0\}$	$\delta(q_0, 0)$ $= \{q_0, q_1\}$	$\delta(q_0, 1)$ $= \{q_0\}$
$\{q_0, q_1\}$	$\delta(q_0, 0) \cup \delta(q_1, 0)$ $= \{q_0, q_1\} \cup \varnothing$ $= \{q_0, q_1\}$	$\delta(q_0, 1) \cup \delta(q_1, 1)$ $= \{q_0\} \cup \{q_2\}$ $= \{q_0, q_2\}$
$\{q_0, q_2\}$	$\delta(q_0, 0) \cup \delta(q_2, 0)$ $= \{q_0, q_1\} \cup \varnothing$ $= \{q_0, q_1\}$	$\delta(q_0, 1) \cup \delta(q_2, 1)$ $= \{q_0\} \cup \varnothing$ $= \{q_0\}$

DFA ≡ NFA (2.2): Lazy Evaluation (2)

Applying ***subset construction*** (with ***lazy evaluation***), we arrive in a ***DFA*** transition table:

state \ input	0	1
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$

We then draw the ***DFA*** accordingly:



Compare the above DFA with the DFA in slide 31.

DFA $\equiv$ NFA (2.2): Lazy Evaluation (3)

- Given an ***NFA*** $N = (Q_N, \Sigma_N, \delta_N, q_0, F_N)$:

ALGORITHM: *ReachableSubsetStates*
 INPUT: $q_0 : Q_N$; **OUTPUT**: $\texttt{Reachable} \subseteq \mathbb{P}(Q_N)$
PROCEDURE:
 $\texttt{Reachable} := \{ \{q_0\} \}$
 $\mathit{ToDiscover} := \{ \{q_0\} \}$
 while $(\mathit{ToDiscover} \neq \varnothing)$ {
  $\texttt{choose}\ S : \mathbb{P}(Q_N)\ \texttt{such that}\ S \in \mathit{ToDiscover}$
  $\texttt{remove}\ S\ \texttt{from}\ \mathit{ToDiscover}$
  $\texttt{NotYetDiscovered} :=$
   $(\ \{\ \{\delta_N(s, 0) \mid s \in S\}\ \} \cup \{\ \{\delta_N(s, 1) \mid s \in S\}\ \}\ ) \setminus \texttt{Reachable}$
  $\texttt{Reachable} := \texttt{Reachable} \cup \texttt{NotYetDiscovered}$
  $\mathit{ToDiscover} := \mathit{ToDiscover} \cup \texttt{NotYetDiscovered}$
 }
 return $\texttt{Reachable}$

- RT of *ReachableSubsetStates*? [$O(2^{|Q_N|})$]
- Often only a small portion of the $|\mathbb{P}(Q_N)|$ ***subset states*** is ***reachable*** from $\{q_0\}$ $\Rightarrow$ ***Lazy Evaluation*** efficient in practice!

ϵ-NFA: Examples (1)

Draw the NFA for the following two languages:

1.

$$\left\{ xy \;\middle|\; \begin{array}{rl} & x \in \{0,1\}^* \\ \wedge & y \in \{0,1\}^* \\ \wedge & x \texttt{ has alternating } 0'\texttt{s and } 1'\texttt{s} \\ \wedge & y \texttt{ has an odd \# } 0'\texttt{s and an odd \# } 1'\texttt{s} \end{array} \right\}$$

2.

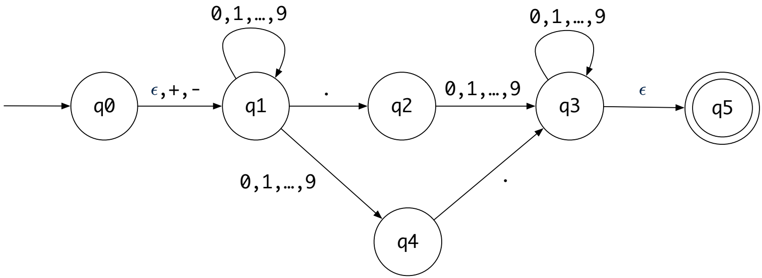
$$\left\{ w : \{0,1\}^* \;\middle|\; \begin{array}{rl} & w \texttt{ has alternating } 0'\texttt{s and } 1'\texttt{s} \\ \vee & w \texttt{ has an odd \# } 0'\texttt{s and an odd \# } 1'\texttt{s} \end{array} \right\}$$

3.

$$\left\{ sx.y \;\middle|\; \begin{array}{rl} & s \in \{+, -, \epsilon\} \\ \wedge & x \in \Sigma_{dec}^* \\ \wedge & y \in \Sigma_{dec}^* \\ \wedge & \neg(x = \epsilon \wedge y = \epsilon) \end{array} \right\}$$

ϵ-NFA: Examples (2)

$$\left\{ sx.y \;\middle|\; \begin{array}{ll} & s \in \{+, -, \epsilon\} \\ \wedge & x \in \Sigma_{dec}^* \\ \wedge & y \in \Sigma_{dec}^* \\ \wedge & \neg(x = \epsilon \wedge y = \epsilon) \end{array} \right\}$$



From q_0 to q_1, reading a sign is **optional**: a *plus* or a *minus*, or *nothing at all* (i.e., ϵ).

ϵ-NFA: Formalization (1)

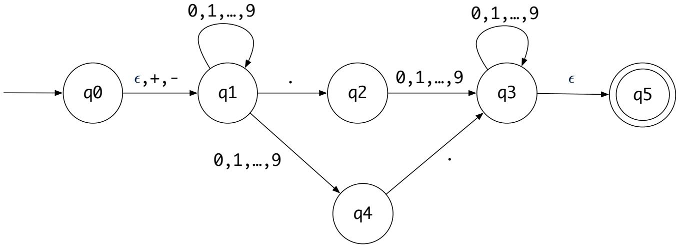
An *ϵ-NFA* is a 5-tuple

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

- Q is a finite set of *states*.
- Σ is a finite set of *input symbols* (i.e., the *alphabet*).
- $\delta : (Q \times (\Sigma \cup \{\epsilon\})) \to \mathbb{P}(Q)$ is a *transition function*

  δ takes as arguments a state and an input symbol, or *an empty string* ϵ, and returns a set of states.
- $q_0 \in Q$ is the *start state*.
- $F \subseteq Q$ is a set of *final* or *accepting states*.

ϵ-NFA: Formalization (2)



Draw a transition table for the above NFA's δ function:

	ϵ	+, -	.	0 .. 9
q_0	$\{q_1\}$	$\{q_1\}$	$\varnothing$	$\varnothing$
q_1	$\varnothing$	$\varnothing$	$\{q_2\}$	$\{q_1, q_4\}$
q_2	$\varnothing$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_3	$\{q_5\}$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_4	$\varnothing$	$\varnothing$	$\{q_3\}$	$\varnothing$
q_5	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

ϵ-NFA: Epsilon-Closures (1)

- Given *ϵ-NFA N*

$$N = (Q, \ \Sigma, \ \delta, \ q_0, \ F)$$

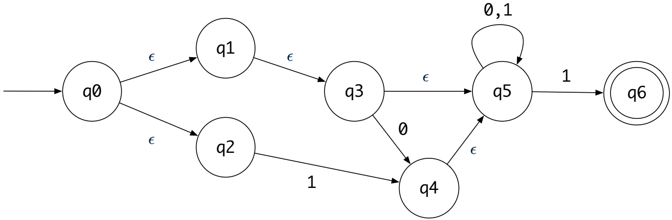
  we define the *epsilon closure* (or *ϵ-closure*) as a function

$$\text{ECLOSE} : Q \rightarrow \mathbb{P}(Q)$$

- For any state $q \in Q$

$$\text{ECLOSE}(q) = \{q\} \cup \bigcup_{p \in \delta(q, \epsilon)} \text{ECLOSE}(p)$$

ϵ-NFA: Epsilon-Closures (2)



$$
\begin{aligned}
& \text{ECLOSE}(q_0) \\
= \; & \{ \; \delta(q_0, \epsilon) = \{q_1, q_2\} \; \} \\
& \{q_0\} \cup \text{ECLOSE}(q_1) \cup \text{ECLOSE}(q_2) \\
= \; & \{ \; \mathit{ECLOSE}(q_1), \; \delta(q_1, \epsilon) = \{q_3\}, \; \mathit{ECLOSE}(q_2), \; \delta(q_2, \epsilon) = \varnothing \; \} \\
& \{q_0\} \cup (\; \{q_1\} \cup \mathit{ECLOSE}(q_3) \;) \cup (\; \{q_2\} \cup \varnothing \;) \\
= \; & \{ \; \mathit{ECLOSE}(q_3), \; \delta(q_3, \epsilon) = \{q_5\} \; \} \\
& \{q_0\} \cup (\; \{q_1\} \cup (\; \{q_3\} \cup \mathit{ECLOSE}(q_5) \;) \;) \cup (\; \{q_2\} \cup \varnothing \;) \\
= \; & \{ \; \mathit{ECLOSE}(q_5), \; \delta(q_5, \epsilon) = \varnothing \; \} \\
& \{q_0\} \cup (\; \{q_1\} \cup (\; \{q_3\} \cup (\; \{q_5\} \cup \varnothing \;) \;) \;) \cup (\; \{q_2\} \cup \varnothing \;)
\end{aligned}
$$

ϵ-NFA: Formalization (3)

- Given a ϵ-*NFA* $M = (Q, \Sigma, \delta, q_0, F)$, we may simplify the definition of *L(M)* by extending δ (which takes an input symbol) to $\hat{\delta}$ (which takes an input string).

$$\hat{\delta} : (Q \times \Sigma^*) \to \mathbb{P}(Q)$$

We may define $\hat{\delta}$ recursively, using δ!

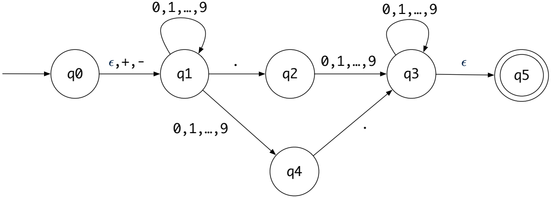
$$\begin{array}{lcl} \hat{\delta}(q, \epsilon) & = & \mathtt{ECLOSE}(q) \\ \hat{\delta}(q, xa) & = & \bigcup\{\, \mathtt{ECLOSE}(q'') \mid q'' \in \delta(q', a) \wedge q' \in \hat{\delta}(q, x) \,\} \end{array}$$

where $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$

- Then we define $L(M)$ as the set of strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains **at least one** *accepting state*.

$$L(M) = \{w \mid w \in \Sigma^* \wedge \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

ϵ-NFA: Formalization (4)



Given an input string 5.6:

$\hat{\delta}(q_0, \epsilon) = \text{ECLOSE}(q_0) = \{q_0, q_1\}$

- **Read 5**: $\delta(q_0, 5) \cup \delta(q_1, 5) = \varnothing \cup \{q_1, q_4\} = \{\ q_1, q_4\ \}$

  $\hat{\delta}(q_0, 5) = \text{ECLOSE}(q_1) \cup \text{ECLOSE}(q_4) = \{q_1\} \cup \{q_4\} = \{q_1, q_4\}$
- **Read .**: $\delta(q_1, .) \cup \delta(q_4, .) = \{q_2\} \cup \{q_3\} = \{\ q_2, q_3\ \}$

  $\hat{\delta}(q_0, 5.) = \text{ECLOSE}(q_2) \cup \text{ECLOSE}(q_3) = \{q_2\} \cup \{q_3, q_5\} = \{q_2, q_3, q_5\}$
- **Read 6**: $\delta(q_2, 6) \cup \delta(q_3, 6) \cup \delta(q_5, 6) = \{q_3\} \cup \{q_3\} \cup \varnothing = \{\ q_3\ \}$

  $\hat{\delta}(q_0, 5.6) = \text{ECLOSE}(q_3) = \{q_3, q_5\}$ [5.6 is *accepted*]

DFA $\equiv$ ϵ-NFA: Extended Subset Const. (1)

Subset construction (with ***lazy evaluation*** and *epsilon closures*) produces a ***DFA*** transition table.

	$d \in 0..9$	$s \in \{+, -\}$	.
$\{q_0, q_1\}$	$\{q_1, q_4\}$	$\{q_1\}$	$\{q_2\}$
$\{q_1, q_4\}$	$\{q_1, q_4\}$	$\varnothing$	$\{q_2, q_3, q_5\}$
$\{q_1\}$	$\{q_1, q_4\}$	$\varnothing$	$\{q_2\}$
$\{q_2\}$	$\{q_3, q_5\}$	$\varnothing$	$\varnothing$
$\{q_2, q_3, q_5\}$	$\{q_3, q_5\}$	$\varnothing$	$\varnothing$
$\{q_3, q_5\}$	$\{q_3, q_5\}$	$\varnothing$	$\varnothing$

For example, $\delta(\{q_0, q_1\}, d)$ is calculated as follows: $[d \in 0..9]$

$$
\begin{aligned}
& \bigcup\{\text{ECLOSE}(q) \mid q \in \delta(q_0, d) \cup \delta(q_1, d)\} \\
= \; & \bigcup\{\text{ECLOSE}(q) \mid q \in \varnothing \cup \{q_1, q_4\}\} \\
= \; & \bigcup\{\text{ECLOSE}(q) \mid q \in \{q_1, q_4\}\} \\
= \; & \text{ECLOSE}(q_1) \cup \text{ECLOSE}(q_4) \\
= \; & \{q_1\} \cup \{q_4\} \\
= \; & \{q_1, q_4\}
\end{aligned}
$$

DFA $\equiv$ ϵ-NFA: Extended Subset Const. (2)

Given an ϵ=*NFA* $N = (Q_N, \Sigma_N, \delta_N, q_0, F_N)$, by applying the ***extended subset construction*** to it, the resulting *DFA* $D = (Q_D, \Sigma_D, \delta_D, q_{D_{start}}, F_D)$ is such that:

$$
\begin{array}{lcl}
\Sigma_D & = & \Sigma_N \\
q_{D_{start}} & = & \mathtt{ECLOSE}(q_0) \\
F_D & = & \{ \ S \mid S \subseteq Q_N \wedge S \cap F_N \neq \varnothing \ \} \\
Q_D & = & \{ \ S \mid S \subseteq Q_N \wedge (\exists w \bullet w \in \Sigma^* \Rightarrow S = \hat{\delta}_N(q_0, w)) \ \} \\
\delta_D(S, a) & = & \bigcup \{ \ \mathtt{ECLOSE}(s') \mid s \in S \wedge s' \in \delta_N(s, a) \ \}
\end{array}
$$

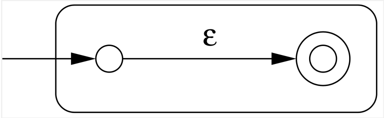
Regular Expression to ϵ-NFA

- Just as we construct each complex *regular expression* recursively, we define its equivalent ϵ-*NFA* *recursively*.
- Given a regular expression R, we construct an ϵ-NFA E, such that $L(R) = L(E)$, with
  - Exactly **one** accept state.
  - No incoming arc to the start state.
  - No outgoing arc from the accept state.

Regular Expression to ϵ-NFA

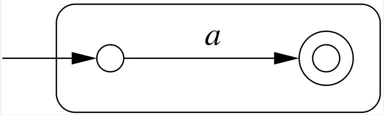
Base Cases:

- ϵ



- $\varnothing$

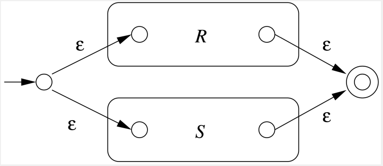
- a $[a \in \Sigma]$



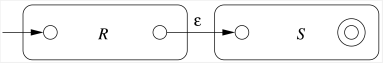
Regular Expression to ϵ-NFA

Recursive Cases: [*R* and *S* are RE's]

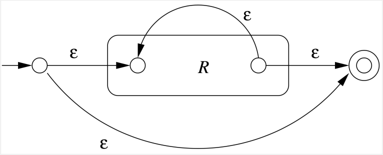
- $R + S$



- RS

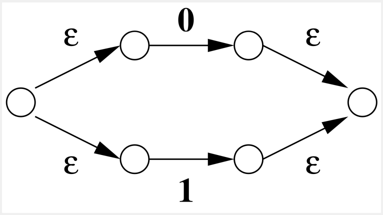


- R^*

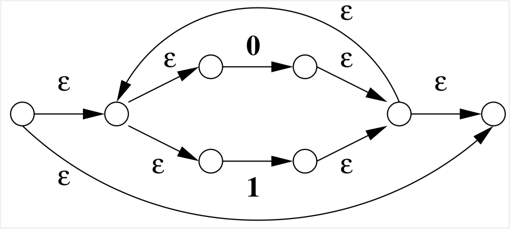


Regular Expression to ϵ-NFA: Examples (1.1)

- $0 + 1$

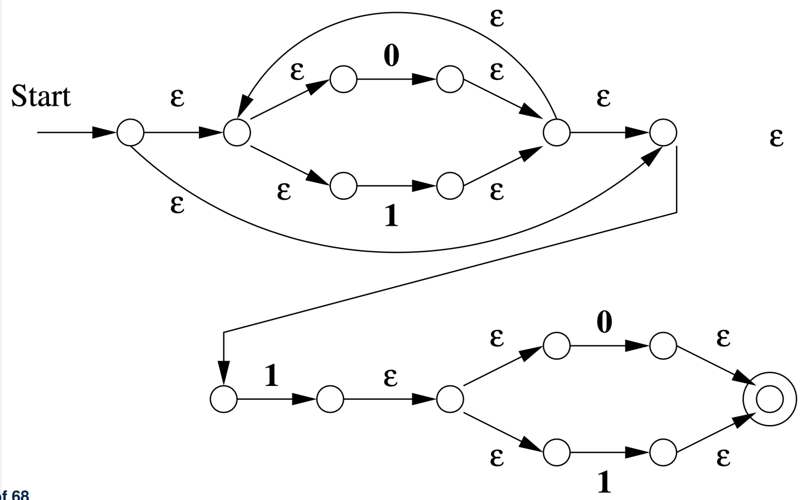


- $(0 + 1)^*$



Regular Expression to ϵ-NFA: Examples (1.2)

- $(0 + 1)^* 1 (0 + 1)$



Minimizing DFA: Motivation

- Recall: $\boxed{\text{Regular Expresion}} \longrightarrow \boxed{\epsilon\text{-NFA}} \longrightarrow \boxed{\text{DFA}}$
- DFA produced by the ***extended subset construction*** (with ***lazy evaluation***) may **not** be ***minimum*** on its size of state.
- When the required size of memory is sensitive
  (e.g., processor's cache memory),
  the fewer number of DFA states, the better.

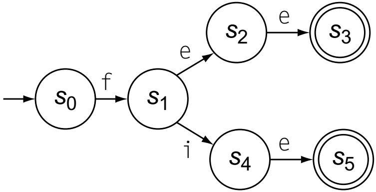
Minimizing DFA: Algorithm

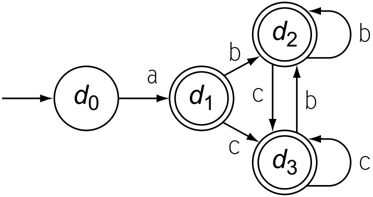
```
ALGORITHM: MinimizeDFAStates
  INPUT: DFA M = (Q, Σ, δ, q0, F)
  OUTPUT: M′ s.t. minimum |Q| and equivalent behaviour as M
PROCEDURE:
  P := ∅ /* refined partition so far */
  T := { F, Q − F } /* last refined partition */
  while (P ≠ T):
    P := T
    T := ∅
    for(p ∈ P):
      find the maximal S ⊂ p s.t. splittable(p, S)
      if S ≠ ∅ then
        T := T ∪ {S, p − S}
      else
        T := T ∪ {p}
      end
```

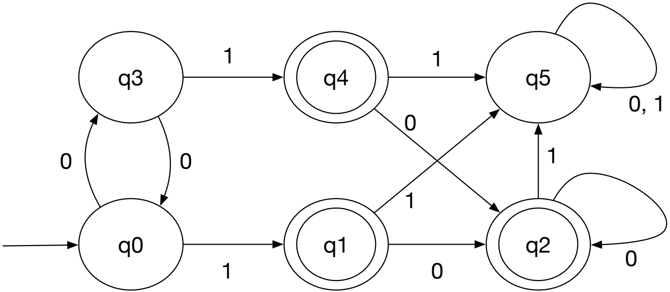
splittable(p, S) holds iff there is $c \in \Sigma$ s.t.

1. $S \subset p$ (or equivalently: $p - S \neq \varnothing$)
2. Transitions via c lead all $s \in S$ to states in ***same partition*** $p1$ ($p1 \neq p$).

Minimizing DFA: Examples







Exercises: Minimize the DFA from **here**; Q1 & Q2, p59, EAC2.

Exercise: Regular Expression to Minimized DFA

Given regular expression `r[0..9]+` which specifies the pattern of a register name, derive the equivalent DFA with the minimum number of states. Show <u>all</u> steps.

Implementing DFA as Scanner

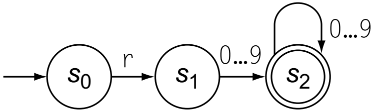
- The source language has a list of ***syntactic categories***:
  e.g., keyword `while` [`while`]
  e.g., identifiers [`[a-zA-Z][a-zA-Z0-9_]*`]
  e.g., white spaces [`[ \t\r]+`]
- A compiler's *scanner* must recognize ***words*** from **all** syntactic categories of the source language.
  - Each syntactic category is specified via a ***regular expression***.

$$\underbrace{r_1}_{\texttt{syn. cat. } 1} + \underbrace{r_1}_{\texttt{syn. cat. } 2} + \ldots + \underbrace{r_n}_{\texttt{syn. cat. } n}$$

  - Overall, a scanner should be implemented based on the ***minimized DFA*** accommodating all syntactic categories.
- Principles of a scanner:
  - Returns one ***word*** at a time
  - Each returned word is the ***longest possible*** that matches a ***pattern***
  - A **priority** may be specified among patterns
    (e.g., `new` is a keyword, not identifier)

Implementing DFA: Table-Driven Scanner (1)

- Consider the ***syntactic category*** of `register` names.
- Specified as a *regular expression* : `r[0..9]+`
- Afer conversion to ϵ-NFA, then to DFA, then to ***minimized DFA***:



- The following tables encode knowledge about the above DFA:

Classifier (`CharCat`)

`r`	`0,1,2,...,9`	`EOF`	**Other**
Register	*Digit*	*Other*	*Other*

Transition (δ)

	Register	***Digit***	***Other***
s_0	s_1	s_e	s_e
s_1	s_e	s_2	s_e
s_2	s_e	s_2	s_e
s_e	s_e	s_e	s_e

Token Type (`Type`)

s_0	s_1	s_2	s_e
invalid	*invalid*	*register*	*invalid*

Implementing DFA: Table-Driven Scanner (2)

The scanner then is implemented via a 4-stage skeleton:

```
NextWord()
  -- Stage 1:  Initialization
  state := s0 ; word := ε
  initialize an empty stack s ; s.push(bad)
  -- Stage 2:  Scanning Loop
  while (state ≠ se)
    NextChar(char) ; word := word + char
    if state ∈ F then reset stack s end
    s.push(state)
    cat := CharCat[char]
    state := δ[state, cat]
  -- Stage 3:  Rollback Loop
  while (state ∉ F ∧ state ≠ bad)
    state := s.pop()
    truncate word
  -- Stage 4:  Interpret and Report
  if state ∈ F then return Type[state]
  else return invalid
  end
```

Index (1)

Index (2)

Index (3)

Index (4)

Index (5)

Index (6)