Recursion

EECS2030 E: Advanced
Object Oriented Programming
Summer 2025

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Learning Outcomes

This module is designed to help you learn about:

1. How to solve problems ***recursively***
2. Example ***recursions*** on string and arrays
3. Some more advanced example (if time permitted)

Beyond this lecture . . .

- Fantastic resources for sharpening your recursive skills for the exam:

  `http://codingbat.com/java/Recursion-1`

  `http://codingbat.com/java/Recursion-2`

- The *best* approach to learning about recursion is via a functional programming language:

  Haskell Tutorial: `https://www.haskell.org/tutorial/`

Recursion: Principle

- *Recursion* is useful in expressing solutions to problems that can be *recursively* defined:
  - ***Base* Cases:** Small problem instances immediately solvable.
  - ***Recursive* Cases:**
    - Large problem instances *not immediately solvable*.
    - Solve by reusing *solution(s) to strictly smaller problem instances*.
- Similar idea learnt in high school: [*mathematical induction*]
- Recursion can be easily expressed programmatically in Java:

```
m (i) {
  if(i == ...) { /* base case: do something directly */ }
  else {
    m (j);/* recursive call with strictly smaller value */
  }
}
```

- In the body of a method *m*, there might be *a call or calls to m itself*.
- Each such self-call is said to be a *recursive call*.
- Inside the execution of $m(i)$, a recursive call $m(j)$ must be that $j < i$.

Tracing Method Calls via a Stack

- When a method is called, it is ***activated*** (and becomes *active*) and *pushed* onto the stack.
- When the body of a method makes a (helper) method call, that (helper) method is ***activated*** (and becomes *active*) and *pushed* onto the stack.

  ⇒ The stack contains activation records of all *active* methods.
  - *Top* of stack denotes the current point of execution.
  - Remaining parts of stack are (temporarily) ***suspended***.
- When entire body of a method is executed, stack is *popped*.

  ⇒ The current point of execution is returned to the new *top* of stack (which was ***suspended*** and just became ***active***).
- Execution terminates when the stack becomes *empty*.

Recursion: Factorial (1)

- Recall the formal definition of calculating the n factorial:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 & \text{if } n \geq 1 \end{cases}$$

- How do you define the same problem *recursively*?

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

- To solve n!, we combine n and the solution to (n - 1)!.

```
int factorial (int n) {
  int result;
  if(n == 0) { /* base case */ result = 1; }
  else { /* recursive case */
    result = n * factorial (n - 1);
  }
  return result;
}
```


Common Errors of Recursive Methods

- Missing Base Case(s).

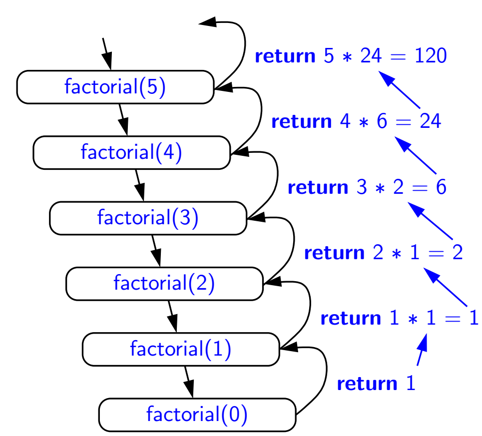
```
int factorial (int n) {
  return n * factorial (n - 1);
}
```

  Base case(s) are meant as points of stopping growing the runtime stack.
- Recursive Calls on Non-Smaller Problem Instances.

```
int factorial (int n) {
  if(n == 0) { /* base case */ return 1; }
  else { /* recursive case */ return n * factorial (n); }
}
```

  Recursive calls on ***strictly smaller*** problem instances are meant for moving gradually towards the base case(s).
- In both cases, a `StackOverflowException` will be thrown.

Recursion: Factorial (2)



Recursion: Factorial (3)

- When running *factorial(5)*, a *recursive call factorial(4)* is made. Call to *factorial(5)* suspended until *factorial(4)* returns a value.
- When running *factorial(4)*, a *recursive call factorial(3)* is made. Call to *factorial(4)* suspended until *factorial(3)* returns a value.

  . . .
- *factorial(0)* returns 1 back to *suspended call factorial(1)*.
- *factorial(1)* receives 1 from *factorial(0)*, multiplies 1 to it, and returns 1 back to the *suspended call factorial(2)*.
- *factorial(2)* receives 1 from *factorial(1)*, multiplies 2 to it, and returns 2 back to the *suspended call factorial(3)*.
- *factorial(3)* receives 2 from *factorial(1)*, multiplies 3 to it, and returns 6 back to the *suspended call factorial(4)*.
- *factorial(4)* receives 6 from *factorial(3)*, multiplies 4 to it, and returns 24 back to the *suspended call factorial(5)*.
- *factorial(5)* receives 24 from *factorial(4)*, multiplies 5 to it, and returns 120 as the result.

Recursion: Factorial (4)

- When the execution of a method (e.g., *factorial(5)*) leads to a nested method call (e.g., *factorial(4)*):
  - The execution of the current method (i.e., *factorial(5)*) is *suspended*, and a structure known as an *activation record* or *activation frame* is created to store information about the progress of that method (e.g., values of parameters and local variables).
  - The nested methods (e.g., *factorial(4)*) may call other nested methods (*factorial(3)*).
  - When all nested methods complete, the activation frame of the *latest suspended* method is re-activated, then continue its execution.
- What kind of data structure does this activation-suspension process correspond to? [LIFO Stack]

Recursion: Fibonacci Sequence (1)

- Can you identify the pattern of a Fibonacci sequence?

$$F = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

- Here is the formal, ***recursive*** definition of calculating the n_{th} number in a Fibonacci sequence (denoted as F_n):

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{if } n > 2 \end{cases}$$

```java
int fib (int n) {
  int result;
  if(n == 1) { /* base case */ result = 1; }
  else if(n == 2) { /* base case */ result = 1; }
  else { /* recursive case */
    result = fib (n - 1) + fib (n - 2);
  }
  return result;
}
```


Recursion: Fibonacci Sequence (2)

```
  fib(5)
= { fib(5) = fib(4) + fib(3); push(fib(5)); suspended: ⟨fib(5)⟩; active: fib(4) }
  fib(4) + fib(3)
= { fib(4) = fib(3) + fib(2); suspended: ⟨fib(4), fib(5)⟩; active: fib(3) }
  ( fib(3) + fib(2) ) + fib(3)
= { fib(3) = fib(2) + fib(1); suspended: ⟨fib(3), fib(4), fib(5)⟩; active: fib(2) }
  (( fib(2) + fib(1) ) + fib(2)) + fib(3)
= { fib(2) returns 1; suspended: ⟨fib(3), fib(4), fib(5)⟩; active: fib(1) }
  (( 1 + fib(1) ) + fib(2)) + fib(3)
= { fib(1) returns 1; suspended: ⟨fib(3), fib(4), fib(5)⟩; active: fib(3) }
  (( 1 + 1 ) + fib(2)) + fib(3)
= { fib(3) returns 1 + 1; pop(); suspended: ⟨fib(4), fib(5)⟩; active: fib(2) }
  (2 + fib(2) ) + fib(3)
= { fib(2) returns 1; suspended: ⟨fib(4), fib(5)⟩; active: fib(4) }
  (2 + 1) + fib(3)
= { fib(4) returns 2 + 1; pop(); suspended: ⟨fib(5)⟩; active: fib(3) }
  3 + fib(3)
= { fib(3) = fib(2) + fib(1); suspended: ⟨fib(3),fib(5)⟩; active: fib(2) }
  3 + ( fib(2) + fib(1))
= { fib(2) returns 1; suspended: ⟨fib(3), fib(5)⟩; active: fib(1) }
  3 + (1 + fib(1) )
= { fib(1) returns 1; suspended: ⟨fib(3), fib(5)⟩; active: fib(3) }
  3 + (1 + 1)
= { fib(3) returns 1 + 1; pop() ; suspended: ⟨fib(5)⟩; active: fib(5) }
  3 + 2
= { fib(5) returns 3 + 2; suspended: ⟨⟩ }
  5
```

Java Library: String

```
public class StringTester {
  public static void main(String[] args) {
    String s = "abcd";
    System.out.println(s.isEmpty()); /* false */
    /* Characters in index range [0, 0) */
    String t0 = s.substring(0, 0);
    System.out.println(t0); /* "" */
    /* Characters in index range [0, 4) */
    String t1 = s.substring(0, 4);
    System.out.println(t1); /* "abcd" */
    /* Characters in index range [1, 3) */
    String t2 = s.substring(1, 3);
    System.out.println(t2); /* "bc" */
    String t3 = s.substring(0, 2) + s.substring(2, 4);
    System.out.println(s.equals(t3)); /* true */
    for(int i = 0; i < s.length(); i ++) {
      System.out.print(s.charAt(i));
    }
    System.out.println();
  }
}
```


Recursion: Palindrome (1)

Problem: A palindrome is a word that reads the same forwards and backwards. Write a method that takes a string and determines whether or not it is a palindrome.

```
System.out.println(isPalindrome(""));  true
System.out.println(isPalindrome("a"));  true
System.out.println(isPalindrome("madam"));  true
System.out.println(isPalindrome("racecar"));  true
System.out.println(isPalindrome("man"));  false
```

Base Case 1: Empty string ⟶ Return *true* immediately.

Base Case 2: String of length 1 ⟶ Return *true* immediately.

Recursive Case: String of length ≥ 2 ⟶

- 1st and last characters match, **and**
- *the rest (i.e., middle) of the string is a palindrome*.

Recursion: Palindrome (2)

```
boolean isPalindrome (String word) {
  if(word.length() == 0 || word.length() == 1) {
    /* base case */
    return true;
  }
  else {
    /* recursive case */
    char firstChar = word.charAt(0);
    char lastChar = word.charAt(word.length() - 1);
    String middle = word.substring(1, word.length() - 1);
    return
        firstChar == lastChar
        /* See the API of java.lang.String.substring. */
        && isPalindrome (middle);
  }
}
```


Recursion: Reverse of String (1)

Problem: The reverse of a string is written backwards. Write a method that takes a string and returns its reverse.

```
System.out.println(reverseOf(""));  /* "" */
System.out.println(reverseOf("a"));  "a"
System.out.println(reverseOf("ab"));  "ba"
System.out.println(reverseOf("abc"));  "cba"
System.out.println(reverseof("abcd"));  "dcba"
```

Base Case 1: Empty string ⟶ Return *empty string*.

Base Case 2: String of length 1 ⟶ Return *that string*.

Recursive Case: String of length ≥ 2 ⟶

1) Head of string (i.e., first character)
2) Reverse of the tail of string (i.e., all but the first character)

Return the concatenation of **2)** and **1)**.

Recursion: Reverse of a String (2)

```
String reverseOf (String s) {
  if(s.isEmpty()) { /* base case 1 */
    return "";
  }
  else if(s.length() == 1) { /* base case 2 */
    return s;
  }
  else { /* recursive case */
    String tail = s.substring(1, s.length());
    String reverseOfTail = reverseOf (tail);
    char head = s.charAt(0);
    return reverseOfTail + head;
  }
}
```


Recursion: Number of Occurrences (1)

Problem: Write a method that takes a string `s` and a character `c`, then count the number of occurrences of `c` in `s`.

```
System.out.println(occurrencesOf("", 'a'));  /* 0 */
System.out.println(occurrencesOf("a", 'a'));  /* 1 */
System.out.println(occurrencesOf("b", 'a'));  /* 0 */
System.out.println(occurrencesOf("baaba", 'a'));  /* 3 */
System.out.println(occurrencesOf("baaba", 'b'));  /* 2 */
System.out.println(occurrencesOf("baaba", 'c'));  /* 0 */
```

Base Case: Empty string ⟶ Return *0*.

Recursive Case: String of length ≥ 1 ⟶

1) Head of `s` (i.e., first character)
2) Number of occurrences of `c` in the tail of `s` (i.e., all but the first character)

If head is equal to `c`, return 1 + **2)**.
If head is not equal to `c`, return 0 + **2)**.

Recursion: Number of Occurrences (2)

```
int occurrencesOf (String s, char c) {
  if(s.isEmpty()) {
    /* Base Case */
    return 0;
  }
  else {
    /* Recursive Case */
    char head = s.charAt(0);
    String tail = s.substring(1, s.length());
    if(head == c) {
      return 1 + occurrencesOf (tail, c);
    }
    else {
      return 0 + occurrencesOf (tail, c);
    }
  }
}
```


Making Recursive Calls on an Array

- Recursive calls denote solutions to *smaller* sub-problems.
- *Naively*, explicitly create a new, smaller array:

```
void m(int[] a) {
  if(a.length == 0) { /* base case */ }
  else if(a.length == 1) { /* base case */ }
  else {
    int[] sub = new int[a.length - 1];
    for(int i = 1; i < a.length; i ++) { sub[i - 1] = a[i]; }
    m(sub) } }
```

- For *efficiency*, we pass the ***reference*** of the same array and specify the *range of indices* to be considered:

```
void m(int[] a, int from, int to) {
  if(from > to) { /* base case */ }
  else if(from == to) { /* base case */ }
  else { m(a, from + 1, to) } }
```

  - `m(a, 0, a.length - 1)` [Initial call; entire array]
  - `m(a, 1, a.length - 1)` [1st r.c. on array of size *a.length* – 1]
  - `m(a, a.length-1, a.length-1)` [Last r.c. on array of size 1]

Recursion: All Positive (1)

Problem: Determine if an array of integers are all positive.

```
System.out.println(allPositive({})); /* true */
System.out.println(allPositive({1, 2, 3, 4, 5})); /* true */
System.out.println(allPositive({1, 2, -3, 4, 5})); /* false */
```

Base Case: Empty array ⟶ Return *true* immediately.
The base case is *true* ∵ we can *not* find a counter-example (i.e., a number *not* positive) from an empty array.
Recursive Case: Non-Empty array ⟶
- 1st element positive, **and**
- *the rest of the array* *is all positive* .

Exercise: Write a method `boolean somePostive(int[] a)` which *recursively* returns *true* if there is some positive number in `a`, and *false* if there are no positive numbers in `a`.
Hint: What to return in the base case of an empty array? [*false*]
∵ No witness (i.e., a positive number) from an empty array

Recursion: All Positive (2)

```
boolean allPositive(int[] a) {
  return allPositiveHelper (a, 0, a.length - 1);
}

boolean allPositiveHelper (int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper (a, from + 1, to);
  }
}
```


Recursion: Is an Array Sorted? (1)

Problem: Determine if an array of integers are sorted in a non-descending order.

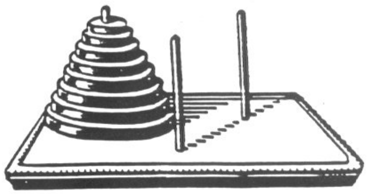
```
System.out.println(isSorted({})); true

System.out.println(isSorted({1, 2, 2, 3, 4})); true

System.out.println(isSorted({1, 2, 2, 1, 3})); false
```

Base Case: Empty array ⟶ Return *true* immediately.
The base case is *true* ∵ we can *not* find a counter-example (i.e., a pair of adjacent numbers that are *not* sorted in a non-descending order) from an empty array.
Recursive Case: Non-Empty array ⟶
- 1st and 2nd elements are sorted in a non-descending order, **and**
- *the rest of the array*, starting from the 2nd element, *are sorted in a non-descending order* .

Recursion: Is an Array Sorted? (2)

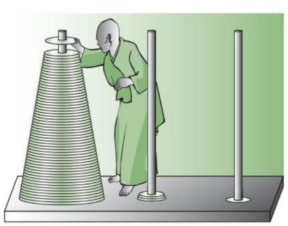
```
boolean isSorted(int[] a) {
  return isSortedHelper (a, 0, a.length - 1);
}

boolean isSortedHelper (int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
      && isSortedHelper (a, from + 1, to);
  }
}
```

Tower of Hanoi: Specification



- ***Given***: A tower of 8 disks, initially stacked in decreasing size on one of 3 pegs
- ***Rules***:
  - Move only one disk at a time.
  - Never move a larger disk onto a smaller one.
- ***Problem***: Transfer the entire tower to one of the other pegs.

Tower of Hanoi: Lengend



*Brahmins at a temple in Benares, India have been carrying out movement of "Sacred Tower of Brahma", consisting of **sixty-four** golden disks, according to the same rules as in the Tower of Hanoi game, and that the completion of the tower would lead to the end of the world.*

Tower of Hanoi: A Recursive Solution

The general, a recursive solution requires 3 steps:

1. Transfer the **n - 1** **smallest** disks to a ***second*** peg.
2. Move the **largest** peg to the ***third*** peg (free of disks).
3. Transfer the **n - 1** **smallest** disks back onto the **largest** disk.

Tower of Hanoi in Java (1)

```
void towerOfHanoi(String[] disks) {
  tohHelper (disks, 0, disks.length - 1, 1, 3);
}
void tohHelper(String[] disks, int from, int to, int ori, int des){
  if(from > to) { }
  else if(from == to) {
    print("move " + disks[to] + " from " + ori + " to " + des);
  }
  else {
    int intermediate = 6 - ori - des;
    tohHelper (disks, from, to - 1, ori, intermediate);
    print("move " + disks[to] + " from " + ori + " to " + des);
    tohHelper (disks, from, to - 1, intermediate, des);
  }
}
```

- `tohHelper(disks, from, to, ori, des)` moves disks $\{disks[from], disks[from+1], \ldots, disks[to]\}$ from peg *ori* to peg *des*.
- Peg id's are 1, 2, and 3 ⇒ The intermediate one is 6 – *ori* – *des*.

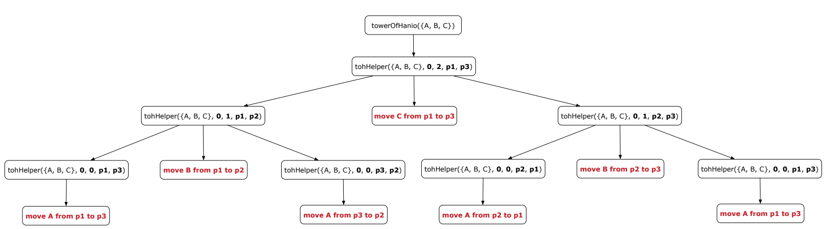
Tower of Hanoi in Java (2)

Say *ds* (disks) is $\{A, B, C\}$, where $A < B < C$.

$\mathit{tohH}(ds, \underbrace{0, 2}_{\{A,B,C\}}, p1, p3) =$

- $\mathit{tohH}(ds, \underbrace{0, 1}_{\{A,B\}}, p1, p2) =$
  - $\mathit{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p1, p3) = \{$ `Move A: p1 to p3`
  - `Move B: p1 to p2`
  - $\mathit{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p3, p2) = \{$ `Move A: p3 to p2`
- `Move C: p1 to p3`
- $\mathit{tohH}(ds, \underbrace{0, 1}_{\{A,B\}}, p2, p3) =$
  - $\mathit{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p2, p1) = \{$ `Move A: p2 to p1`
  - `Move B: p2 to p3`
  - $\mathit{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p1, p3) = \{$ `Move A: p1 to p3`

Tower of Hanoi in Java (3)



Running Time: Tower of Hanoi (1)

- Generalize the problem by considering **n** disks.
- Let ***T(n)*** denote the number of moves required to to transfer **n** disks from one to another under the rules.
- Recall the general solution pattern:
  1. Transfer the **n - 1** **smallest** disks to a ***second*** peg.
  2. Move the **largest** peg to the ***third*** peg (free of disks).
  3. Transfer the **n - 1** **smallest** disks back onto the **largest** disk.
- We end up with the following recurrence relation that allows us to compute ***T(n)*** for any **n** we like:

$$\begin{cases} T(1) & = & 1 \\ T(n) & = & 2 \cdot T(n-1) + 1 \quad \text{where } n > 0 \end{cases}$$

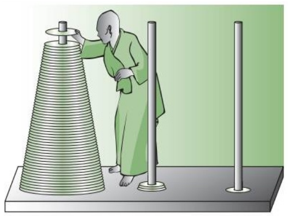
- To solve this recurrence relation, we study the pattern of ***T(n)*** and observe how it reaches the ***base case(s)***.

Running Time: Tower of Hanoi (2)

$$\begin{aligned} T(n) &= \underbrace{2}_{1\ \text{term}} \times T(n-1) + \underbrace{1}_{1\ \text{term}} \\ &= \underbrace{2 \times (2}_{2\ \text{terms}} \times T(n-2) + \underbrace{1) + 1}_{2\ \text{terms}} \\ &= \underbrace{2 \times (2 \times (2}_{3\ \text{terms}} \times T(n-3) + \underbrace{1) + 1) + 1}_{3\ \text{terms}} \\ &= \ldots \\ &= \underbrace{2 \times (2 \times (2 \times (\cdots \times (2 \times}_{n-1\ \text{terms}} \overbrace{T(1)}^{T(n\,-\,(n-1))} \underbrace{+ 1) + \ldots) + 1) + 1) + 1}_{n-1\ \text{terms}} \\ &= 2^{n-1} + (n-1) \end{aligned}$$

$\therefore$ ***T(n)*** is ***O(2^n)***

Tower of Hanoi: Lengend



Brahmins at a temple in Benares, India have been carrying out movement of "Sacred Tower of Brahma", consisting of ***sixty-four*** *golden disks, according to the same rules as in the Tower of Hanoi game, and that the completion of the tower would lead to the end of the world.*

Say one disk can be moved in one second.

Q. How long does it take to finish moving 64 disks ($n = 64$)?

A. 2^{64} seconds ≈ 585 billion years (>> 5 billion centries)!

Beyond this lecture . . .

- Recursions on Arrays: Lab Exercise from EECS2030-F19
- Notes on Recursion:
  `http://www.eecs.yorku.ca/~jackie/teaching/lectures/2024/F/EECS2030/notes/EECS2030_S25_Notes_Recursion.pdf`
- API for `String`:
  `https://docs.oracle.com/javase/8/docs/api/java/lang/String.html`
- Fantastic resources for sharpening your recursive skills for the exam:
  `http://codingbat.com/java/Recursion-1`
  `http://codingbat.com/java/Recursion-2`
- The *best* approach to learning about recursion is via a functional programming language:
  Haskell Tutorial: `https://www.haskell.org/tutorial/`

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