Specifying & Refining a Bridge Controller

MEB: Chapter 2

EECS3342 E: System Specification and Refinement
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Learning Outcomes

This module is designed to help you understand:

- What a ***Requirement Document*** (***RD***) is
- What a ***refinement*** is
- Writing ***formal specifications***
  - (Static) contexts: constants, axioms, theorems
  - (Dynamic) machines: variables, invariants, events, guards, actions
- ***Proof Obligations*** (***POs***) associated with proving:
  - ***refinements***
  - system ***properties***
- Applying ***inference rules*** of the ***sequent calculus***

Recall: Correct by Construction

- Directly reasoning about **source code** (written in a programming language) is too complicated to be feasible.
- Instead, given a ***requirements document***, prior to **implementation**, we develop ***models*** through a series of ***refinement*** steps:
  - Each model formalizes an ***external observer***'s perception of the system.
  - Models are "sorted" with ***increasing levels of accuracy*** w.r.t. the system.
  - The ***first model***, though the most ***abstract***, can already be proved satisfying some ***requirements***.
  - Starting from the ***second model***, each model is analyzed and proved ***correct*** relative to two criteria:
    1. Some ***requirements*** (i.e., R-descriptions)
    2. ***Proof Obligations*** (***POs***) related to the ***preceding model*** being ***refined by*** the ***current model*** (via "extra" ***state*** variables and ***events***).
  - The ***last model*** (which is *correct by construction*) should be **sufficiently close** to be transformed into a **working program** (e.g., in C).

State Space of a Model

- A model's *state space* is the set of **all** configurations:
  - Each ***configuration*** assigns values to **constants** & **variables**, subject to:
    - ***axiom*** (e.g., typing constraints, assumptions)
    - ***invariant*** properties/theorems
  - Say an initial model of a bank system with two constants and a variable:

    $c \in \mathbb{N}1 \land L \in \mathbb{N}1 \land accounts \in String \nrightarrow \mathbb{Z}$ /* typing constraint */

    $\forall id \bullet id \in \mathrm{dom}(accounts) \Rightarrow -c \leq accounts(id) \leq L$ /* desired property */

    Q. What is the ***state space*** of this initial model?

    A. All valid combinations of c, L, and $accounts$.
    - Configuration 1: $(c = 1,000, L = 500,000, b = \varnothing)$
    - Configuration 2: $(c = 2,375, L = 700,000, b = \{("id1", 500), ("id2", 1,250)\})$

      . . . [Challenge: ***Combinatorial Explosion***]
  - Model Concreteness ↑ ⇒ (State Space ↑ ∧ Verification Difficulty ↑)
- A model's ***complexity*** should be guided by those properties intended to be verified against that model.

  ⇒ ***Infeasible*** to prove **all** desired properties on a model.

  ⇒ ***Feasible*** to distribute desired properties over a list of ***refinements***.

Roadmap of this Module

- We will walk through the *development process* of constructing ***models*** of a control system regulating cars on a bridge.

  Such controllers exemplify a ***reactive system***.

  (with **sensors** and **actuators**)

- Always stay on top of the following roadmap:
  1. A ***Requirements Document*** (***RD***) of the bridge controller
  2. A brief overview of the ***refinement strategy***
  3. An initial, the most ***abstract*** model
  4. A subsequent ***model*** representing the ***1st refinement***
  5. A subsequent ***model*** representing the ***2nd refinement***
  6. A subsequent ***model*** representing the ***3rd refinement***

Requirements Document: Mainland, Island

Imagine you are asked to build a bridge (as an alternative to ferry) connecting the downtown and Toronto Island.

Page Source: `https://soldbyshane.com/area/toronto-islands/`

Requirements Document: E-Descriptions

Each ***E-Description*** is an **atomic** ***specification*** of a ***constraint*** or an ***assumption*** of the system's working environment.

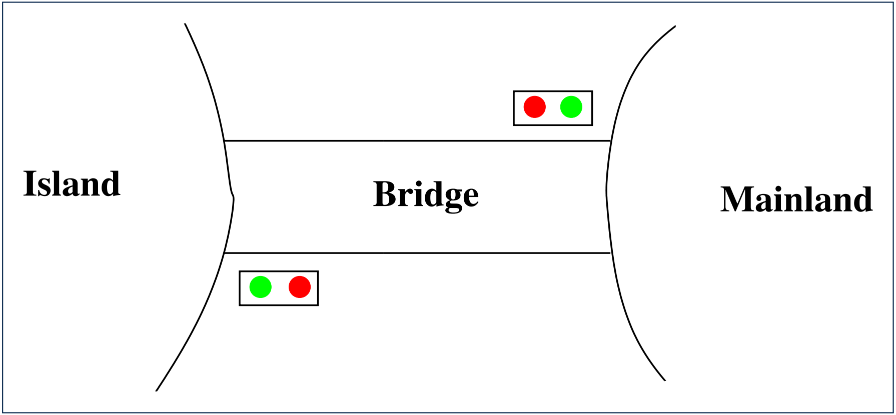
ENV1	The system is equipped with two traffic lights with two colors: green and red.
ENV2	The traffic lights control the entrance to the bridge at both ends of it.
ENV3	Cars are not supposed to pass on a red traffic light, only on a green one.
ENV4	The system is equipped with four sensors with two states: on or off.
ENV5	The sensors are used to detect the presence of a car entering or leaving the bridge: "on" means that a car is willing to enter the bridge or to leave it.

Requirements Document: R-Descriptions

Each ***R-Description*** is an **atomic** ***specification*** of an intended ***functionality*** or a desired ***property*** of the working system.

REQ1	The system is controlling cars on a bridge connecting the mainland to an island.
REQ2	The number of cars on bridge and island is limited.
REQ3	The bridge is one-way or the other, not both at the same time.

Requirements Document: Visual Summary of Equipment Pieces



Refinement Strategy

- Before diving into details of the ***models***, we first clarify the adopted ***design strategy of progressive refinements***.

  0. The ***initial model*** (m_0) will address the intended functionality of a limited number of cars on the island and bridge.
  [**REQ2**]

  1. A ***1st refinement*** (m_1 which ***refines*** m_0) will address the intended functionality of the *bridge being one-way*.
  [**REQ1**, **REQ3**]

  2. A ***2nd refinement*** (m_2 which ***refines*** m_1) will address the environment constraints imposed by *traffic lights*.
  [**ENV1, ENV2, ENV3**]

  3. A ***final, 3rd refinement*** (m_3 which ***refines*** m_2) will address the environment constraints imposed by *sensors* and the *architecture*: controller, environment, communication channels.
  [**ENV4, ENV5**]

- Recall *Correct by Construction*:

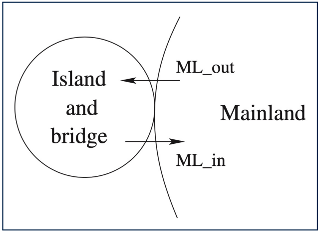
  From each ***model*** to its ***refinement***, only a manageable amount of details are added, making it ***feasible*** to conduct **analysis** and **proofs**.

Model m_0: Abstraction

- In this most ***abstract*** perception of the bridge controller, we do **not** even consider the bridge, traffic lights, and sensors!
- Instead, we focus on this single ***requirement***:

REQ2	The number of cars on bridge and island is limited.

- **Analogies**:
  - Observe the system from the sky: island and bridge appear only as a compound.



  - "***Zoom in***" on the system as ***refinements*** are introduced.

Model m_0: State Space

1. The ***static*** part is fixed and may be seen/imported.

   A ***constant*** d denotes the maximum number of cars allowed to be on the ***island-bridge compound*** at any time.

   (whereas cars on the mainland is unbounded)

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$

   Remark. ***Axioms*** are assumed true and may be used to prove theorems.

2. The ***dynamic*** part changes as the system ***evolves***.

   A ***variable*** n denotes the actual number of cars, at a given moment, in the ***island-bridge compound***.

variables: n

invariants:
inv0_1 : $n \in \mathbb{N}$
inv0_2 : $n \leq d$

   Remark. ***Invariants*** should be (subject to **proofs**):

   - ***Established*** when the system is first initialized
   - ***Preserved***/***Maintained*** after any ***enabled event***'s actions take effect

Model m_0: State Transitions via Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- At any given ***state*** (a valid ***configuration*** of constants/variables):
  - An event is said to be ***enabled*** if its guard evaluates to ***true***.
  - An event is said to be ***disabled*** if its guard evaluates to ***false***.
  - An ***enabled*** event makes a ***state transition*** if it occurs and its ***actions*** take effect.
- *1st* ***event***: A car exits mainland (and enters the island-bridge compound).

```
ML_out
  begin
    n := n + 1
  end
```

Correct Specification? Say $d = 2$.
Witness: ***Event Trace*** $\langle init, ML\_out, ML\_out, ML\_out \rangle$

- *2nd* ***event***: A car enters mainland (and exits the island-bridge compound).

```
ML_in
  begin
    n := n − 1
  end
```

Correct Specification? Say $d = 2$.
Witness: ***Event Trace*** $\langle init, ML\_in \rangle$

Model m_0: Actions vs. Before-After Predicates

- When an enabled event *e* occurs there are two notions of ***state***:
  - ***Before-/Pre-State***: Configuration just ***before*** *e*'s actions take effect
  - ***After-/Post-State***: Configuration just ***after*** *e*'s actions take effect

  Remark. When an enabled event occurs, its ***action(s)*** cause a **transition** from the ***pre-state*** to the ***post-state***.
- As examples, consider ***actions*** of m_0's two events:

Events	ML_out $n := n + 1$	ML_in $n := n - 1$
before–after predicates	$n' = n + 1$	$n' = n - 1$

  - An event ***action*** "$n := n + 1$" is not a variable assignment; instead, it is a ***specification***: "*n becomes n + 1 (when the state transition completes)*".
  - The ***before-after predicate*** (***BAP***) "$n' = n + 1$" expresses that n' (the ***post-state*** value of *n*) is one more than *n* (the ***pre-state*** value of *n*).
- When we express ***proof obligations*** (***POs***) associated with ***events***, we use ***BAP***.

Design of Events: Invariant Preservation

- Our design of the two events

```
ML_out
  begin
    n := n + 1
  end
```

```
ML_in
  begin
    n := n − 1
  end
```

  only specifies how the ***variable*** n should be updated.

- Remember, ***invariants*** are conditions that should never be ***violated***!

invariants:
  inv0_1 : $n \in \mathbb{N}$
  inv0_2 : $n \leq d$

- By simulating the system as an ***ASM***, we discover ***witnesses*** (i.e., event traces) of the ***invariants*** not being preserved all the time.

$$\exists s \bullet s \in \text{STATE SPACE} \Rightarrow \neg invariants(s)$$

- We formulate such a commitment to preserving ***invariants*** as a ***proof obligation (PO)*** rule (a.k.a. a ***verification condition (VC)*** rule).

Sequents: Syntax and Semantics

- We formulate each *PO/VC* rule as a (horizontal or vertical) ***sequent***:

$$H \vdash G \qquad \begin{array}{c} H \\ \vdash \\ G \end{array}$$

  - The symbol $\vdash$ is called the ***turnstile***.
  - H is a set of predicates forming the ***hypotheses***/***assumptions***.
    [assumed as ***true***]
  - G is a set of predicates forming the ***goal***/***conclusion***.
    [claimed to be ***provable*** from H]
- Informally:
  - $H \vdash G$ is ***true*** if G can be proved by assuming H.
    [i.e., We say "H ***entails*** G" or "H ***yields*** G"]
  - $H \vdash G$ is ***false*** if G cannot be proved by assuming H.
- Formally: $H \vdash G \iff (H \Rightarrow G)$

  Q. What does it mean when H is empty (i.e., no hypotheses)?

  A. $\vdash G \equiv$ ***true*** $\vdash G$ [Why not $\vdash G \equiv$ ***false*** $\vdash G$?]

PO of Invariant Preservation: Sketch

- Here is a sketch of the PO/VC rule for *invariant preservation*:

Axioms ***Invariants*** Satisfied at ***Pre-State*** Guards of the Event $\vdash$ ***Invariants*** Satisfied at ***Post-State***	INV

- Informally, this is what the above PO/VC *requires to prove*:

Assuming **all** axioms, invariants, and the event's guards hold at the ***pre-state***,

after the ***state transition*** is made by the event,

all invariants hold at the ***post-state***.

PO of Invariant Preservation: Components

constants: d

variables: n

ML_out
  begin
    $n := n + 1$
  end

axioms:
  axm0_1 : $d \in \mathbb{N}$

invariants:
  inv0_1 : $n \in \mathbb{N}$
  inv0_2 : $n \leq d$

ML_in
  begin
    $n := n - 1$
  end

- c: list of ***constants*** $\langle d \rangle$
- $A(c)$: list of ***axioms*** $\langle \textbf{axm0\_1} \rangle$
- ***v*** and ***v'***: list of ***variables*** in ***pre***- and ***post***-states $\boldsymbol{v} \mathrel{\widehat{=}} \langle n \rangle$, $\boldsymbol{v'} \mathrel{\widehat{=}} \langle n' \rangle$
- $I(c, v)$: list of ***invariants*** $\langle \textbf{inv0\_1}, \textbf{inv0\_2} \rangle$
- $G(c, v)$: the ***event***'s list of guards

  $G(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \textbf{true} \rangle$, $G(\langle d \rangle, \langle n \rangle)$ of $ML\_in \mathrel{\widehat{=}} \langle \textbf{true} \rangle$
- $E(c, v)$: effect of the ***event***'s actions i.t.o. what variable values ***become***

  $E(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \boldsymbol{n} + 1 \rangle$, $E(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \boldsymbol{n} - 1 \rangle$
- $v' = E(c, v)$: ***before-after predicate*** formalizing E's actions

  BAP of ML_out: $\langle \boldsymbol{n'} \rangle = \langle \boldsymbol{n} + 1 \rangle$, BAP of ML_in: $\langle \boldsymbol{n'} \rangle = \langle \boldsymbol{n} - 1 \rangle$

Rule of Invariant Preservation: Sequents

- Based on the components (c, $A(c)$, v, $I(c, v)$, $E(c, v)$), we are able to formally state the ***PO/VC Rule of Invariant Preservation***:

$$\frac{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ G(c, \mathbf{v}) \\ \vdash \\ I_i(c, \mathbf{E(c, v)}) \end{array}}{} \quad \underline{\text{INV}}$$

  where I_i denotes a single invariant condition

  - Accordingly, how many ***sequents*** to be proved? [# events × # invariants]
  - We have <u>two</u> ***sequents*** generated for ***event*** *ML_out* of model m_0:

$$\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array} \quad \underline{\text{ML\_out/}\mathbf{inv0\_1}\text{/INV}} \qquad \begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \leq d \end{array} \quad \underline{\text{ML\_out/}\mathbf{inv0\_2}\text{/INV}}$$

  <u>Exercise</u>. Write the ***POs of invariant preservation*** for event *ML_in*.

- Before claiming that a ***model*** is *correct*, outstanding ***sequents*** associated with <u>all</u> ***POs*** must be <u>proved/discharged</u>.

Inference Rules: Syntax and Semantics

- An *inference rule (IR)* has the following form:

$$\frac{A}{C}\ \mathbf{L}$$

Formally: $A \Rightarrow C$ is an axiom.

Informally: To prove C, it is sufficient to prove A instead.

Informally: C is the case, assuming that A is the case.

  - L is a name label for referencing the ***inference rule*** in proofs.
  - ***A*** is a **set** of sequents known as ***antecedents*** of rule L.
  - ***C*** is a **single** sequent known as ***consequent*** of rule L.
- Let's consider ***inference rules (IRs)*** with two different flavours:

$$\frac{H1 \vdash G}{H1, H2 \vdash G}\ \mathbf{MON} \qquad \frac{}{n \in \mathbb{N} \vdash n+1 \in \mathbb{N}}\ \mathbf{P2}$$

  - IR **MON**: To prove $H1, H2 \vdash G$, it suffices to prove $H1 \vdash G$ instead.
  - IR **P2**: $n \in \mathbb{N} \vdash n+1 \in \mathbb{N}$ is an ***axiom***.

[proved automatically without further justifications]

Proof of Sequent: Steps and Structure

- To prove the following sequent (related to ***invariant preservation***):

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \underline{\text{ML\_out/}\textbf{inv0\_1}\text{/INV}}$$

1. Apply a ***inference rule***, which ***transforms*** some "outstanding" **sequent** to one or more other **sequents** to be proved instead.
2. Keep applying ***inference rules*** until **all** ***transformed*** **sequents** are ***axioms*** that do **not** require any further justifications.

- Here is a *formal proof* of ML_out/**inv0_1**/INV, by applying IRs **MON** and **P2**:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

Example Inference Rules (1)

$$\frac{}{\vdash 0 \in \mathbb{N}} \quad \textbf{P1}$$

1st Peano axiom: 0 is a natural number.

$$\frac{}{n \in \mathbb{N} \vdash n + 1 \in \mathbb{N}} \quad \textbf{P2}$$

2nd Peano axiom: $n + 1$ is a natural number, assuming that n is a natural number.

$$\frac{}{0 < n \vdash n - 1 \in \mathbb{N}} \quad \textbf{P2'}$$

$n - 1$ is a natural number, assuming that n is positive.

$$\frac{}{n \in \mathbb{N} \vdash 0 \leq n} \quad \textbf{P3}$$

3rd Peano axiom: n is non-negative, assuming that n is a natural number.

Example Inference Rules (2)

$$\frac{}{n < m \ \vdash \ n + 1 \leq m} \ \textbf{INC}$$

$n + 1$ is less than or equal to m, assuming that n is strictly less than m.

$$\frac{}{n \leq m \ \vdash \ n - 1 < m} \ \textbf{DEC}$$

$n - 1$ is strictly less than m, assuming that n is less than or equal to m.

Example Inference Rules (3)

$$\frac{H1 \vdash G}{H1, H2 \vdash G} \quad \textbf{MON}$$

To prove a goal under certain hypotheses,
it suffices to prove it under less hypotheses.

$$\frac{H, P \vdash R \qquad H, Q \vdash R}{H, P \vee Q \vdash R} \quad \textbf{OR\_L}$$

Proof by Cases:
To prove a goal under a disjunctive assumption,
it suffices to prove **independently**
the same goal, twice, under each disjunct.

$$\frac{H \vdash P}{H \vdash P \vee Q} \quad \textbf{OR\_R1}$$

To prove a disjunction,
it suffices to prove the left disjunct.

$$\frac{H \vdash Q}{H \vdash P \vee Q} \quad \textbf{OR\_R2}$$

To prove a disjunction,
it suffices to prove the right disjunct.

Revisiting Design of Events: *ML_out*

- Recall that we already proved ***PO*** ML_out/**inv0_1**/INV :

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \\ \hline \end{array} \quad \textbf{MON} \quad \begin{array}{|l|}\hline n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \\ \hline \end{array} \quad \textbf{P2}$$

$\therefore$ ***ML_out/inv0_1/INV*** succeeds in being discharged.

- How about the other ***PO*** ML_out/**inv0_2**/INV for the same event?

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \leq d \\ \hline \end{array} \quad \textbf{MON} \quad \begin{array}{|l|}\hline n \leq d \\ \vdash \\ n + 1 \leq d \\ \hline \end{array} \quad \textbf{?}$$

$\therefore$ ***ML_out/inv0_2/INV*** fails to be discharged.

Revisiting Design of Events: *ML_in*

- How about the ***PO*** ML_in/**inv0_1**/INV for *ML_in*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{?}$$

∴ ***ML_in/inv0_1/INV*** fails to be discharged.

- How about the other ***PO*** ML_in/**inv0_2**/INV for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n - 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \lor n - 1 = d \end{array}} \quad \textbf{OR\_1} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \end{array}} \quad \textbf{DEC}$$

∴ ***ML_in/inv0_2/INV*** succeeds in being discharged.

Fixing the Design of Events

- Proofs of ***ML_out/inv0_2/INV*** and ***ML_in/inv0_1/INV*** fail due to the two events being *enabled when they should not*.
- Having this feedback, we add proper ***guards*** to *ML_out* and *ML_in*:

```
ML_out
  when
    n < d
  then
    n := n + 1
  end
```

```
ML_in
  when
    n > 0
  then
    n := n − 1
  end
```

- Having changed both events, updated ***sequents*** will be generated for the PO/VC rule of ***invariant preservation***.
- All ***sequents*** ($\{ML\_out,\ ML\_in\} \times \{\textbf{inv0\_1},\ \textbf{inv0\_2}\}$) now ***provable***?

Revisiting Fixed Design of Events: *ML_out*

- How about the ***PO*** ML_out/**inv0_1**/INV for *ML_out*:

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n < d \\ \vdash \\ n + 1 \in \mathbb{N} \\ \hline \end{array} \quad \textbf{MON} \quad \begin{array}{|l|}\hline n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \\ \hline \end{array} \quad \textbf{P2}$$

  ∴ ***ML_out/inv0_1/INV*** still succeeds in being discharged!

- How about the other ***PO*** ML_out/**inv0_2**/INV for the same event?

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n < d \\ \vdash \\ n + 1 \leq d \\ \hline \end{array} \quad \textbf{MON} \quad \begin{array}{|l|}\hline n < d \\ \vdash \\ n + 1 \leq d \\ \hline \end{array} \quad \textbf{INC}$$

  ∴ ***ML_out/inv0_2/INV*** now succeeds in being discharged!

Revisiting Fixed Design of Events: *ML_in*

- How about the ***PO*** ML_in/**inv0_1**/INV for *ML_in*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n > 0 \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n > 0 \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{P2'}$$

$\therefore$ ***ML_in/inv0_1/INV*** now succeeds in being discharged!

- How about the other ***PO*** ML_in/**inv0_2**/INV for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n > 0 \\ \vdash \\ n - 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \vee n - 1 = d \end{array}} \quad \textbf{OR\_1} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \end{array}} \quad \textbf{DEC}$$

$\therefore$ ***ML_in/inv0_2/INV*** still succeeds in being discharged!

Initializing the Abstract System m_0

- Discharging the four ***sequents*** proved that both ***invariant*** conditions are ***preserved*** between occurrences/interleavings of ***events*** `ML_out` and `ML_in`.
- But how are the ***invariants established*** in the first place?

  Analogy. Proving P via ***mathematical induction***, two cases to prove:
  - $P(1), P(2), \ldots$ [***base*** cases ≈ ***establishing*** inv.]
  - $P(n) \Rightarrow P(n+1)$ [***inductive*** cases ≈ ***preserving*** inv.]
- Therefore, we specify how the ***ASM***'s ***initial state*** looks like:

```
init
  begin
    n := 0
  end
```

✓ The IB compound, once ***initialized***, has no cars.

✓ Initialization always possible: guard is ***true***.

✓ There is no ***pre-state*** for *init*.

∴ The RHS of := must not involve variables.

∴ The RHS of := may only involve constants.

✓ There is only the ***post-state*** for *init*.

∴ Before-***After Predicate***: $n' = 0$

PO of Invariant Establishment

```
init
  begin
    n := 0
  end
```

✓ An ***reactive system***, once ***initialized***, should never terminate.

✓ Event *init* cannot "preserve" the ***invariants***.

∵ State before its occurrence (***pre-state***) does not exist.

✓ Event *init* only required to ***establish*** invariants for the first time

- A new formal component is needed:
  - $K(c)$: effect of ***init***'s actions i.t.o. what variable values ***become***

    e.g., $K(\langle d \rangle)$ of *init* $\widehat{=}$ $\langle 0 \rangle$
  - $v' = K(c)$: ***before-after predicate*** formalizing *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{n'} \rangle = \langle 0 \rangle$
- Accordingly, PO of ***invariant establishment*** is formulated as a ***sequent***:

Axioms ⊢ *Invariants* Satisfied at ***Post-State***	INV	$A(c)$ ⊢ $I_i(c, \mathbf{K(c)})$	INV

Discharging PO of Invariant Establishment

- How many ***sequents*** to be proved? [# invariants]
- We have <u>two</u> ***sequents*** generated for ***event*** *init* of model m_0:

$$\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \in \mathbb{N} \end{array} \quad \underline{\text{init/}\textbf{inv0\_1}\text{/INV}} \qquad \begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \leq d \end{array} \quad \underline{\text{init/}\textbf{inv0\_2}\text{/INV}}$$

- Can we discharge the ***PO*** init/**inv0_1**/INV ?

$$\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \in \mathbb{N} \end{array} \quad \textbf{MON} \quad \begin{array}{l} \\ \vdash \\ 0 \in \mathbb{N} \end{array} \quad \textbf{P1}$$

∴ ***init/inv0_1/INV*** <u>succeeds</u> in being discharged.

- Can we discharge the ***PO*** init/**inv0_2**/INV ?

$$\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \leq d \end{array} \quad \textbf{P3}$$

∴ ***init/inv0_2/INV*** <u>succeeds</u> in being discharged.

System Property: Deadlock Freedom

- So far we have proved that our initial model m_0 is s.t. all ***invariant conditions*** are:
  - Established when system is first initialized via *init*
  - Preserved whenevner there is a ***state transition***

    (via an enabled event: *ML_out* or *ML_in*)
- However, whenever ***event occurrences*** are conditional (i.e., ***guards*** stronger than ***true***), there is a possibility of ***deadlock*** :
  - A state where ***guards*** of all events evaluate to ***false***
  - When a ***deadlock*** happens, none of the ***events*** is ***enabled***.

    ⇒ The system is blocked and not reactive anymore!
- We express this ***non-blocking*** property as a new requirement:

REQ4	Once started, the system should work for ever.

PO of Deadlock Freedom (1)

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- Recall some of the formal components we discussed:
  - c: list of ***constants*** $\langle d \rangle$
  - $A(c)$: list of ***axioms*** $\langle \mathbf{axm0\_1} \rangle$
  - ***v*** and ***v'***: list of ***variables*** in ***pre***- and ***post***-states $\mathbf{v} \mathrel{\widehat{=}} \langle n \rangle$, $\mathbf{v'} \mathrel{\widehat{=}} \langle n' \rangle$
  - $I(c, v)$: list of ***invariants*** $\langle \mathbf{inv0\_1}, \mathbf{inv0\_2} \rangle$
  - $G(c, v)$: the event's list of ***guards***

$$G(\langle d \rangle, \langle n \rangle) \text{ of } ML\_out \mathrel{\widehat{=}} \langle n < d \rangle,\ G(\langle d \rangle, \langle n \rangle) \text{ of } ML\_in \mathrel{\widehat{=}} \langle n > 0 \rangle$$

- A system is ***deadlock-free*** if <u>at least one</u> of its ***events*** is ***enabled***:

Axioms *Invariants* Satisfied at *Pre-State* $\vdash$ Disjunction of the guards satisfied at *Pre-State*	<u>DLF</u>	$A(c)$ $I(c, \mathbf{v})$ $\vdash$ $G_1(c, \mathbf{v}) \vee \cdots \vee G_m(c, \mathbf{v})$	<u>DLF</u>

To prove about deadlock freedom
- An event's effect of state transition is <u>**not**</u> relevant.
- Instead, the evaluation of <u>all</u> events' ***guards*** at the ***pre-state*** is relevant.

PO of Deadlock Freedom (2)

- ***Deadlock freedom*** is not necessarily a desired property.

  $\Rightarrow$ When it is (like m_0), then the generated ***sequents*** must be discharged.

- Applying the PO of ***deadlock freedom*** to the initial model m_0:

$$\begin{array}{|l|}\hline A(c) \\ I(c, \mathbf{v}) \\ \vdash \\ G_1(c, \mathbf{v}) \lor \cdots \lor G_m(c, \mathbf{v}) \\ \hline \end{array} \ \underline{\text{DLF}} \qquad \begin{array}{|l|}\hline d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \ \underline{\text{DLF}}$$

  Our bridge controller being ***deadlock-free*** means that cars can ***always*** enter (via *ML_out*) or leave (via *ML_in*) the island-bridge compound.

- Can we formally discharge this ***PO*** for our ***initial model*** m_0?

Example Inference Rules (4)

$$\frac{}{H, P \vdash P} \quad \textbf{HYP}$$

A goal is proved if it can be assumed.

$$\frac{}{\bot \vdash P} \quad \textbf{FALSE\_L}$$

Assuming ***false*** ($\bot$),
anything can be proved.

$$\frac{}{P \vdash \top} \quad \textbf{TRUE\_R}$$

true ($\top$) is proved,
regardless of the assumption.

$$\frac{}{P \vdash E = E} \quad \textbf{EQ}$$

An expression being equal to itself is proved,
regardless of the assumption.

Example Inference Rules (5)

$$\frac{H(\boldsymbol{F}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{F})}{H(\boldsymbol{E}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{E})} \quad \textbf{EQ\_LR}$$

To prove a goal $P(\boldsymbol{E})$ assuming $H(\boldsymbol{E})$,
where both P and H depend on expression $\boldsymbol{E}$,
it suffices to prove $P(\boldsymbol{F})$ assuming $H(\boldsymbol{F})$,
where both P and H depend on expresion $\boldsymbol{F}$,
given that $\boldsymbol{E}$ is equal to $\boldsymbol{F}$.

$$\frac{H(\boldsymbol{E}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{E})}{H(\boldsymbol{F}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{F})} \quad \textbf{EQ\_RL}$$

To prove a goal $P(\boldsymbol{F})$ assuming $H(\boldsymbol{F})$,
where both P and H depend on expression $\boldsymbol{F}$,
it suffices to prove $P(\boldsymbol{E})$ assuming $H(\boldsymbol{E})$,
where both P and H depend on expresion $\boldsymbol{E}$,
given that $\boldsymbol{E}$ is equal to $\boldsymbol{F}$.

Discharging PO of DLF: Exercise

$A(c)$ $I(c, \mathbf{v})$ $\vdash$ $G_1(c, \mathbf{v}) \vee \cdots \vee G_m(c, \mathbf{v})$	$\underline{\text{DLF}}$	$d \in \mathbb{N}$ $n \in \mathbb{N}$ $n \leq d$ $\vdash$ $n < d \vee n > 0$	??

Discharging PO of DLF: First Attempt

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \end{array}}$$

$\equiv$

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{MON}\; \boxed{\begin{array}{l} n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{OR\_L}\; \left\{ \begin{array}{l} \boxed{\begin{array}{l} n < d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{OR\_R1}\; \boxed{\begin{array}{l} n < d \\ \vdash \\ n < d \end{array}} \;\textbf{HYP} \\ \boxed{\begin{array}{l} n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{EQ\_LR, MON}\; \boxed{\begin{array}{l} \\ \vdash \\ d < d \lor d > 0 \end{array}} \;\textbf{OR\_R2}\; \boxed{\begin{array}{l} \\ \vdash \\ d > 0 \end{array}} \;\textbf{?} \end{array} \right.$$

Why Did the DLF PO Fail to Discharge?

- In our first attempt, proof of the 2nd case failed: $\vdash \ d > 0$
- This ***unprovable*** sequent gave us a good hint:
  - For the model under consideration (m_0) to be ***deadlock-free***, it is required that $d > 0$. [≥ 1 car allowed in the IB compound]
  - But current ***specification*** of m_0 ***not*** strong enough to entail this:
    - $\neg(d > 0) \equiv d \leq 0$ is possible for the current model
    - Given **axm0_1** : $d \in \mathbb{N}$

    $\Rightarrow$ $d = 0$ is allowed by m_0 which causes a ***deadlock***.
- Recall the *init* event and the two ***guarded*** events:

```
init
  begin
    n := 0
  end
```

```
ML_out
  when
    n < d
  then
    n := n + 1
  end
```

```
ML_in
  when
    n > 0
  then
    n := n − 1
  end
```

When $d = 0$, the disjunction of guards evaluates to ***false***: $0 < 0 \vee 0 > 0$

$\Rightarrow$ As soon as the system is initialized, it ***deadlocks immediately***

as no car can either enter or leave the IR compound!!

Fixing the Context of Initial Model

- Having understood the failed proof, we add a proper ***axiom*** to m_0:

axioms:
  axm0_2 : $d > 0$

- We have effectively elaborated on **REQ2**:

REQ2	The number of cars on bridge and island is limited but positive.

- Having changed the context, an updated ***sequent*** will be generated for the PO/VC rule of ***deadlock freedom***.
- Is this new sequent now ***provable***?

Discharging PO of DLF: Second Attempt

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array}$$

$\equiv$

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{MON}\; \begin{array}{|l|}\hline d > 0 \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{OR\_L}\; \left\{ \begin{array}{l} \begin{array}{|l|}\hline d > 0 \\ n < d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{OR\_R1}\; \begin{array}{|l|}\hline d > 0 \\ n < d \\ \vdash \\ n < d \\ \hline \end{array} \;\textbf{HYP} \\ \begin{array}{|l|}\hline d > 0 \\ n = d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{EQ\_LR, MON}\; \begin{array}{|l|}\hline d > 0 \\ \vdash \\ d < d \lor d > 0 \\ \hline \end{array} \;\textbf{OR\_R2}\; \begin{array}{|l|}\hline d > 0 \\ \vdash \\ d > 0 \\ \hline \end{array} \;\textbf{HYP} \end{array} \right.$$

Initial Model: Summary

- The final version of our ***initial model*** m_0 is ***provably correct*** w.r.t.:
  - Establishment of ***Invariants***
  - Preservation of ***Invariants***
  - ***Deadlock*** Freedom
- Here is the final ***specification*** of m_0:

constants: d

axioms:
  axm0_1 : $d \in \mathbb{N}$
  axm0_2 : $d > 0$

variables: n

invariants:
  inv0_1 : $n \in \mathbb{N}$
  inv0_2 : $n \leq d$

```
init
  begin
    n := 0
  end
```

```
ML_out
  when
    n < d
  then
    n := n + 1
  end
```

```
ML_in
  when
    n > 0
  then
    n := n - 1
  end
```

Index (1)

Index (2)

Index (3)

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