Introduction

MEB: Prologue, Chapter 1

EECS3342 E: System Specification and Refinement
Fall 2024

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Learning Outcomes

This module is designed to help you understand:

- What a *safety-critical* system is
- **Code of Ethics** for Professional Engineers
- What a ***Formal Method*** Is
- ***Verification*** vs. ***Validation***
- ***Model***-Based System Development

What is a Safety-Critical System (SCS)?

- A *safety-critical system (**SCS**)* is a system whose ***failure*** or ***malfunction*** has one (or more) of the following consequences:
  - death or serious injury to **people**
  - loss or severe damage to **equipment/property**
  - harm to the **environment**
- Based on the above definition, do you know of any systems that are ***safety-critical***?

Professional Engineers: Code of Ethics

- *Code of Ethics* is a basic guide for ***professional conduct*** and imposes duties on practitioners, with respect to **society**, **employers**, **clients**, **colleagues** (including employees and subordinates), the **engineering profession** and him or herself.
- It is the duty of a practitioner to act at all times with,
  1. ***fairness*** and ***loyalty*** to the practitioner's associates, employers, clients, subordinates and employees;
  2. ***fidelity*** (i.e., dedication, faithfulness) to public needs;
  3. devotion to ***high ideals*** of personal honour and professional integrity;
  4. ***knowledge*** of developments in the area of professional engineering relevant to any services that are undertaken; and
  5. ***competence*** in the performance of any professional engineering services that are undertaken.
- Consequence of misconduct?
  - ***suspension*** or ***termination*** of professional licenses
  - civil ***law suits***

Source: PEO's Code of Ethics

Developing Safety-Critical Systems

Industrial standards in various domains list *acceptance criteria* for **mission**- or **safety**-critical systems that practitioners need to comply with: e.g.,

- ***Aviation*** Domain: ***RTCA DO-178C*** "*Software Considerations in Airborne Systems and Equipment Certification*"
- ***Nuclear*** Domain: ***IEEE 7-4.3.2*** "*Criteria for Digital Computers in Safety Systems of Nuclear Power Generating Stations*"

Two important criteria are:

1. System ***requirements*** are precise and complete
2. System ***implementation*** conforms to the requirements

But how do we accomplish these criteria?

Safety-Critical vs. Mission-Critical?

- ***Critical***:

  A task whose successful completion ensures the success of a larger, more complex operation.

  e.g., Success of a pacemaker $\Rightarrow$ Regulated heartbeats of a patient
- ***Safety***:

  Being free from danger/injury to or loss of human lives.
- ***Mission***:

  An operation or task assigned by a higher authority.

  Q. Formally relate being ***safety***-critical and ***mission***-critical.

  A.
  - ***safety***-critical $\Rightarrow$ ***mission***-critical
  - ***mission***-critical $\not\Rightarrow$ ***safety***-critical
- Relevant industrial standard: ***RTCA DO-178C*** (replacing RTCA DO-178B in 2012) "*Software Considerations in Airborne Systems and Equipment Certification*"

Source: Article from OpenSystems

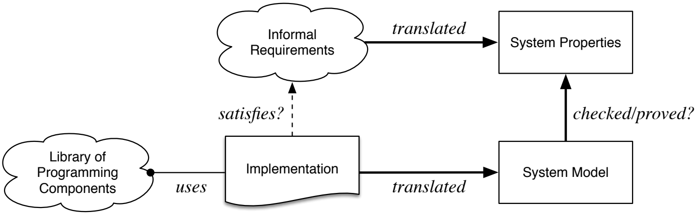
Using Formal Methods for Certification

- A ***formal method (FM)*** is a ***mathematically rigorous*** technique for the specification, development, and verification of software and hardware systems.
- ***DO-333*** "*Formal methods supplement to DO-178C and DO-278A*" advocates the use of formal methods:

  *The use of **formal methods** is motivated by the expectation that, as in other engineering disciplines, performing appropriate **mathematical analyses** can contribute to establishing the **correctness** and **robustness** of a design.*

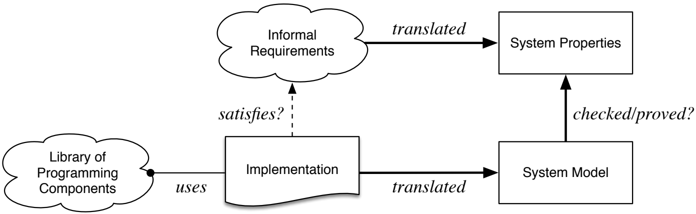
- FMs, because of their mathematical basis, are capable of:
  - *Unambiguously* describing software system requirements.
  - Enabling *precise* communication between engineers.
  - Providing ***verification (towards certification) evidence*** of:
    - A ***formal*** representation of the system being *healthy*.
    - A ***formal*** representation of the system *satisfying* *safety properties*.

Verification: Building the Product Right?



- ***Implementation*** built via ***reusable programming components***.
- *Goal* : *Implementation Satisfies Intended Requirements*
- To verify this, we ***formalize*** them as a ***system model*** and a set of (e.g., safety) ***properties***, using the specification language of a <u>theorem prover</u> (EECS3342) or a <u>model checker</u> (EECS4315).
- Two Verification Issues:
  1. Library components may ***not behave as intended***.
  2. Successful checks/proofs ensure that we *built the product right*, with respect to the <u>informal</u> requirements. **But**...

Validation: Building the Right Product?



- Successful checks/proofs $\not\Rightarrow$ We ***built the right product***.
- The target of our checks/proofs may not be valid:

  The requirements may be ***ambiguous***, ***incomplete***, or ***contradictory***.
- Solution: ***Precise Documentation*** [EECS4312]

Catching Defects – When?

- To minimize *development costs*, minimize *software defects*.
- Software Development Cycle:

  Requirements → *Design* → *Implementation* → Release

  Q. Design or Implementation Phase?

  Catch defects *as early as possible*.

Design and architecture	Implementation	Integration testing	Customer beta test	Postproduct release
1X*	5X	10X	15X	30X

  ∵ The cost of fixing defects *increases exponentially* as software progresses through the development lifecycle.

- Discovering *defects* after **release** costs up to 30 times more than catching them in the **design** phase.
- Choice of a *design language*, amendable to ***formal verification***, is therefore critical for your project.

Source: IBM Report

Model-Based System Development

- ***Modelling*** and ***formal reasoning*** should be performed **before** implementing/coding a system.
  - A system's ***model*** is its ***abstraction***, filtering irrelevant details.

    A system ***model*** means as much to a software engineer as a ***blueprint*** means to an architect.
  - A system may have a list of ***models***, "sorted" by **accuracy**:

    $$\langle m_0, m_1, \ldots, \boxed{m_i}, \boxed{m_j}, \ldots, m_n \rangle$$

    - The list starts by the most ***abstract*** model with least details.
    - A more ***abstract*** model $\boxed{m_i}$ is said to be ***refined by*** its subsequent, more ***concrete*** model $\boxed{m_j}$.
    - The list ends with the most ***concrete/refined*** model with most details.
  - It is far easier to reason about:
    - a system's ***abstract*** models (rather than its full ***implementation***)
    - ***refinement steps*** between subsequent models
- The final product is ***correct by construction***.

Learning through Case Studies

- We will study example ***models of programs/codes***, as well as ***proofs*** on them, drawn from various application domains:
  - REACTIVE Systems [sensors vs. actuators]
  - DISTRIBUTED Systems [(geographically) distributed parties]
- What you learn in this course will allow you to explore example in other application domains:
  - SEQUENTIAL Programs [single thread of control]
  - CONCURRENT Programs [interleaving processes]
- The **Rodin Platform** will be used to:
  - Construct system ***models*** using the Even-B notation.
  - Prove ***properties*** and ***refinements*** using ***classical logic*** (propositional and predicate calculus) and ***set theory***.

Index (1)

Review of Math

MEB: Chapter 9

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Learning Outcomes of this Lecture

This module is designed to help you **review**:

- Propositional Logic
- Predicate Logic
- Sets, Relations, and Functions

Propositional Logic (1)

- A *proposition* is a statement of claim that must be of either *true* or *false*, but not both.
- Basic logical operands are of type Boolean: *true* and *false*.
- We use logical operators to construct compound statements.
  - Unary logical operator: negation ($\neg$)

p	$\neg p$
true	*false*
false	*true*

  - Binary logical operators: conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), equivalence ($\equiv$), and if-and-only-if ($\Longleftrightarrow$).

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Longleftrightarrow q$	$p \equiv q$
true	*true*	*true*	*true*	*true*	*true*	*true*
true	*false*	*false*	*true*	*false*	*false*	*false*
false	*true*	*false*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*true*	*true*	*true*

Propositional Logic: Implication (1)

- Written as $p \Rightarrow q$ [pronounced as "p implies q"]
  - We call p the antecedent, assumption, or premise.
  - We call q the consequence or conclusion.
- Compare the *truth* of $p \Rightarrow q$ to whether a contract is *honoured*:
  - antecedent/assumption/premise $p \approx$ promised terms [e.g., salary]
  - consequence/conclusion $q \approx$ obligations [e.g., duties]
- When the promised terms are met, then the contract is:
  - *honoured* if the obligations fulfilled. [$(true \Rightarrow true) \iff true$]
  - *breached* if the obligations violated. [$(true \Rightarrow false) \iff false$]
- When the promised terms are not met, then:
  - Fulfilling the obligation (q) or not ($\neg q$) does *not breach* the contract.

p	q	$p \Rightarrow q$
false	*true*	*true*
false	*false*	*true*

Propositional Logic: Implication (2)

There are alternative, equivalent ways to expressing $p \Rightarrow q$:

- *q* **if** *p*

  q is *true* if *p* is *true*

- *p* **only if** *q*

  If *p* is *true*, then for $p \Rightarrow q$ to be *true*, it can only be that *q* is also *true*.
  Otherwise, if *p* is *true* but *q* is *false*, then $(\textit{true} \Rightarrow \textit{false}) \equiv \textit{false}$.

  Note. To prove $p \equiv q$, prove $p \iff q$ (pronounced: "p if and only if q"):

  - *p* **if** *q* [$q \Rightarrow p$]
  - *p* **only if** *q* [$p \Rightarrow q$]

- *p* is **sufficient** for *q*

  For *q* to be *true*, it is sufficient to have *p* being *true*.

- *q* is **necessary** for *p* [similar to *p* **only if** *q*]

  If *p* is *true*, then it is necessarily the case that *q* is also *true*.
  Otherwise, if *p* is *true* but *q* is *false*, then $(\textit{true} \Rightarrow \textit{false}) \equiv \textit{false}$.

- *q* **unless** $\neg p$ [When is $p \Rightarrow q$ *true*?]

  If *q* is *true*, then $p \Rightarrow q$ *true* regardless of *p*.
  If *q* is *false*, then $p \Rightarrow q$ cannot be *true* unless *p* is *false*.

Propositional Logic: Implication (3)

Given an implication $p \Rightarrow q$, we may construct its:

- **Inverse**: $\neg p \Rightarrow \neg q$ [negate antecedent and consequence]
- **Converse**: $q \Rightarrow p$ [swap antecedent and consequence]
- **Contrapositive**: $\neg q \Rightarrow \neg p$ [inverse of converse]

Propositional Logic (2)

- **Axiom**: Definition of $\Rightarrow$

$$p \Rightarrow q \equiv \neg p \vee q$$

- **Theorem**: Identity of $\Rightarrow$

$$true \Rightarrow p \equiv p$$

- **Theorem**: Zero of $\Rightarrow$

$$false \Rightarrow p \equiv true$$

- **Axiom**: De Morgan

$$\begin{array}{rcl} \neg(p \wedge q) & \equiv & \neg p \vee \neg q \\ \neg(p \vee q) & \equiv & \neg p \wedge \neg q \end{array}$$

- **Axiom**: Double Negation

$$p \equiv \neg\ (\neg\ p)$$

- **Theorem**: Contrapositive

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

Predicate Logic (1)

- A *predicate* is a ***universal*** or ***existential*** statement about objects in some universe of disclosure.
- Unlike propositions, predicates are typically specified using ***variables***, each of which declared with some ***range*** of values.
- We use the following symbols for common numerical ranges:
  - $\mathbb{Z}$: the set of integers $[\,-\infty, \ldots, -1, 0, 1, \ldots, +\infty\,]$
  - $\mathbb{N}$: the set of natural numbers $[\,0, 1, \ldots, +\infty\,]$
- Variable(s) in a predicate may be ***quantified***:
  - *Universal quantification* :
    All values that a variable may take satisfy certain property.
    e.g., Given that i is a natural number, i is *always* non-negative.
  - *Existential quantification* :
    Some value that a variable may take satisfies certain property.
    e.g., Given that i is an integer, i *can be* negative.

Predicate Logic (2.1): Universal Q. ($\forall$)

- A *universal quantification* has the form $(\forall X \bullet R \Rightarrow P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***For all*** (combinations of) values of variables listed in X that satisfies R, it is the case that P is satisfied.
  - $\forall i \bullet i \in \mathbb{N} \Rightarrow i \geq 0$ [*true*]
  - $\forall i \bullet i \in \mathbb{Z} \Rightarrow i \geq 0$ [*false*]
  - $\forall i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \Rightarrow i < j \vee i > j$ [*false*]
- *Proof Strategies*
  1. How to prove $(\forall X \bullet R \Rightarrow P)$ ***true***?
     - **Hint.** When is $R \Rightarrow P$ ***true***? [*true* $\Rightarrow$ *true*, *false* $\Rightarrow$ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.
  2. How to prove $(\forall X \bullet R \Rightarrow P)$ ***false***?
     - **Hint.** When is $R \Rightarrow P$ ***false***? [*true* $\Rightarrow$ *false*]
     - Give a ***witness/counterexample*** of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.

Predicate Logic (2.2): Existential Q. ($\exists$)

- An *existential quantification* has the form $(\exists X \bullet R \land P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***There exist*** (a combination of) values of variables listed in X that satisfy both R and P.
  - $\exists i \bullet i \in \mathbb{N} \land i \geq 0$ [*true*]
  - $\exists i \bullet i \in \mathbb{Z} \land i \geq 0$ [*true*]
  - $\exists i, j \bullet i \in \mathbb{Z} \land j \in \mathbb{Z} \land (i < j \lor i > j)$ [*true*]
- *Proof Strategies*
  1. How to prove $(\exists X \bullet R \land P)$ ***true***?
     - **Hint.** When is $R \land P$ ***true***? [*true* $\land$ *true*]
     - Give a ***witness*** of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
  2. How to prove $(\exists X \bullet R \land P)$ ***false***?
     - **Hint.** When is $R \land P$ ***false***? [*true* $\land$ *false*, *false* $\land$ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.

Predicate Logic (3): Exercises

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \Rightarrow x > 0$.

  All 10 integers between 1 and 10 are greater than 0.

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \Rightarrow x > 1$.

  Integer 1 (a witness/counterexample) in the range between 1 and 10 is *not* greater than 1.

- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \land x > 1$.

  Integer 2 (a witness) in the range between 1 and 10 is greater than 1.

- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \land x > 10$?

  All integers in the range between 1 and 10 are *not* greater than 10.

Predicate Logic (4): Switching Quantifications

Conversions between $\forall$ and $\exists$:

$$(\forall X \bullet R \Rightarrow P) \iff \neg(\exists X \bullet R \land \neg P)$$
$$(\exists X \bullet R \land P) \iff \neg(\forall X \bullet R \Rightarrow \neg P)$$

Sets: Definitions and Membership

- A **set** is a collection of objects.
  - Objects in a set are called its *elements* or *members*.
  - *Order* in which elements are arranged does not matter.
  - An element can appear *at most once* in the set.
- We may define a set using:
  - ***Set Enumeration***: Explicitly list all members in a set.
    e.g., $\{1, 3, 5, 7, 9\}$
  - ***Set Comprehension***: Implicitly specify the condition that all members satisfy.
    e.g., $\{x \mid 1 \leq x \leq 10 \land x \texttt{ is an odd number}\}$
- An empty set (denoted as $\{\}$ or $\varnothing$) has no members.
- We may check if an element is a *member* of a set:
  e.g., $5 \in \{1, 3, 5, 7, 9\}$ [*true*]
  e.g., $4 \notin \{x \mid x \leq 1 \leq 10, x \texttt{ is an odd number}\}$ [*true*]
- The number of elements in a set is called its *cardinality*.
  e.g., $|\varnothing| = 0$, $|\{x \mid x \leq 1 \leq 10, x \texttt{ is an odd number}\}| = 5$

Set Relations

Given two sets S_1 and S_2:

- S_1 is a ***subset*** of S_2 if every member of S_1 is a member of S_2.

$$S_1 \subseteq S_2 \iff (\forall x \bullet x \in S1 \Rightarrow x \in S2)$$

- S_1 and S_2 are ***equal*** iff they are the subset of each other.

$$S_1 = S_2 \iff S_1 \subseteq S_2 \land S_2 \subseteq S_1$$

- S_1 is a ***proper subset*** of S_2 if it is a strictly smaller subset.

$$S_1 \subset S_2 \iff S_1 \subseteq S_2 \land |S1| < |S2|$$

Set Relations: Exercises

$? \subseteq S$ always holds [$\varnothing$ and S]

$? \subset S$ always fails [S]

$? \subset S$ holds for some S and fails for some S [$\varnothing$]

$S_1 = S_2 \Rightarrow S_1 \subseteq S_2$? [Yes]

$S_1 \subseteq S_2 \Rightarrow S_1 = S_2$? [No]

Set Operations

Given two sets S_1 and S_2:

- ***Union*** of S_1 and S_2 is a set whose members are in either.

$$S_1 \cup S_2 = \{x \mid x \in S_1 \lor x \in S_2\}$$

- ***Intersection*** of S_1 and S_2 is a set whose members are in both.

$$S_1 \cap S_2 = \{x \mid x \in S_1 \land x \in S_2\}$$

- ***Difference*** of S_1 and S_2 is a set whose members are in S_1 but not S_2.

$$S_1 \setminus S_2 = \{x \mid x \in S_1 \land x \notin S_2\}$$

Power Sets

The *power set* of a set S is a *set* of all S's *subsets*.

$$\mathbb{P}(S) = \{s \mid s \subseteq S\}$$

The power set contains subsets of *cardinalities* 0, 1, 2, ..., $|S|$.
e.g., $\mathbb{P}(\{1,2,3\})$ is a set of sets, where each member set s has cardinality 0, 1, 2, or 3:

$$\left\{ \begin{array}{l} \varnothing, \\ \{1\},\ \{2\},\ \{3\}, \\ \{1,2\},\ \{2,3\},\ \{3,1\}, \\ \{1,2,3\} \end{array} \right\}$$

Exercise: What is $\mathbb{P}(\{1,2,3,4,5\}) \setminus \mathbb{P}(\{1,2,3\})$?

Set of Tuples

Given n sets $S_1, S_2, \ldots, S_n$, a ***cross/Cartesian product*** of theses sets is a set of n-tuples.

Each ***n-tuple*** $(e_1, e_2, \ldots, e_n)$ contains n elements, each of which a member of the corresponding set.

$$S_1 \times S_2 \times \cdots \times S_n = \{(e_1, e_2, \ldots, e_n) \mid e_i \in S_i \land 1 \le i \le n\}$$

e.g., $\{a, b\} \times \{2, 4\} \times \{\$, \&\}$ is a set of triples:

$$\begin{array}{rl} & \{a, b\} \times \{2, 4\} \times \{\$, \&\} \\ = & \{\ (e_1, e_2, e_3) \mid e_1 \in \{a, b\} \land e_2 \in \{2, 4\} \land e_3 \in \{\$, \&\}\ \} \\ = & \left\{ \begin{array}{l} (a, 2, \$), (a, 2, \&), (a, 4, \$), (a, 4, \&), \\ (b, 2, \$), (b, 2, \&), (b, 4, \$), (b, 4, \&) \end{array} \right\} \end{array}$$

Relations (1): Constructing a Relation

A *relation* is a set of mappings, each being an ***ordered pair*** that maps a member of set S to a member of set T.

e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$

- $\varnothing$ is the ***minimum*** relation (i.e., an empty relation).
- $\boxed{S \times T}$ is the ***maximum*** relation (say r_1) between S and T, mapping from each member of S to each member in T:

$$\{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

- $\{(x, y) \mid (x, y) \in S \times T \wedge x \neq 1\}$ is a relation (say r_2) that maps only some members in S to every member in T:

$$\{(2, a), (2, b), (3, a), (3, b)\}$$

Relations (2.1): Set of Possible Relations

- We use the ***power set*** operator to express the set of ***all possible relations*** on S and T:

$$\mathbb{P}(S \times T)$$

  Each member in $\mathbb{P}(S \times T)$ is a relation.

- To declare a relation variable r, we use the colon ($:$) symbol to mean ***set membership***:

$$r : \mathbb{P}(S \times T)$$

- Or alternatively, we write:

$$r : S \leftrightarrow T$$

  where the set $S \leftrightarrow T$ is synonymous to the set $\mathbb{P}(S \times T)$

Relations (2.2): Exercise

Enumerate $\{a, b\} \leftrightarrow \{1, 2, 3\}$.

- **Hints**:
  - You may enumerate all relations in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$ via their *cardinalities*: 0, 1, . . . , $|\{a, b\} \times \{1, 2, 3\}|$.
  - What's the *maximum* relation in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$?

$$\{ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) \}$$

- The answer is a set containing ***all*** of the following relations:
  - Relation with cardinality 0: $\varnothing$
  - How many relations with cardinality 1? $[\ \binom{|\{a,b\}\times\{1,2,3\}|}{1} = 6\ ]$
  - How many relations with cardinality 2? $[\ \binom{|\{a,b\}\times\{1,2,3\}|}{2} = \frac{6\times5}{2!} = 15\ ]$

  . . .

  - Relation with cardinality $|\{a, b\} \times \{1, 2, 3\}|$:

$$\{ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) \}$$

Relations (3.1): Domain, Range, Inverse

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- $\boxed{\textit{domain} \text{ of } r}$: set of first-elements from r
  - Definition: $\mathrm{dom}(r) = \{ d \mid (d, r') \in r \}$
  - e.g., $\mathrm{dom}(r) = \{a, b, c, d, e, f\}$
  - ASCII syntax: `dom(r)`
- $\boxed{\textit{range} \text{ of } r}$: set of second-elements from r
  - Definition: $\mathrm{ran}(r) = \{ r' \mid (d, r') \in r \}$
  - e.g., $\mathrm{ran}(r) = \{1, 2, 3, 4, 5, 6\}$
  - ASCII syntax: `ran(r)`
- $\boxed{\textit{inverse} \text{ of } r}$: a relation like r with elements swapped
  - Definition: $r^{-1} = \{ (r', d) \mid (d, r') \in r \}$
  - e.g., $r^{-1} = \{(1, a), (2, b), (3, c), (4, a), (5, b), (6, c), (1, d), (2, e), (3, f)\}$
  - ASCII syntax: `r~`

Relations (3.2): Image

Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

$\boxed{\textbf{\textit{relational image}} \text{ of } r \text{ over set } s}$: sub-range of *r* mapped by *s*.

- Definition: $r[s] = \{\ r' \mid (d, r') \in r \wedge d \in s\ \}$
- e.g., $r[\{a, b\}] = \{1, 2, 4, 5\}$
- ASCII syntax: `r[s]`

Relations (3.3): Restrictions

Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

- **domain restriction** of *r* over set *ds*: sub-relation of *r* with domain *ds*.
  - Definition: $ds \lhd r = \{\ (d, r') \mid (d, r') \in r \land d \in ds\ \}$
  - e.g., $\{a, b\} \lhd r = \{(\mathbf{a}, 1), (\mathbf{b}, 2), (\mathbf{a}, 4), (\mathbf{b}, 5)\}$
  - ASCII syntax: `ds <| r`
- **range restriction** of *r* over set *rs*: sub-relation of *r* with range *rs*.
  - Definition: $r \rhd rs = \{\ (d, r') \mid (d, r') \in r \land r' \in rs\ \}$
  - e.g., $r \rhd \{1, 2\} = \{(a, \mathbf{1}), (b, \mathbf{2}), (d, \mathbf{1}), (e, \mathbf{2})\}$
  - ASCII syntax: `r |> rs`

Relations (3.4): Subtractions

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- ***domain subtraction*** of r over set ds : sub-relation of r with domain not ds.
  - Definition: $ds \lhd r = \{ (d, r') \mid (d, r') \in r \land d \notin ds \}$
  - e.g., $\{a, b\} \lhd r = \{(\mathbf{c}, 3), (\mathbf{c}, 6), (\mathbf{d}, 1), (\mathbf{e}, 2), (\mathbf{f}, 3)\}$
  - ASCII syntax: `ds <<| r`
- ***range subtraction*** of r over set rs : sub-relation of r with range not rs.
  - Definition: $r \rhd rs = \{ (d, r') \mid (d, r') \in r \land r' \notin rs \}$
  - e.g., $r \rhd \{1, 2\} = \{(c, \mathbf{3}), (a, \mathbf{4}), (b, \mathbf{5}), (c, \mathbf{6}), (f, \mathbf{3})\}$
  - ASCII syntax: `r |>> rs`

Relations (3.5): Overriding

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

overriding **of** r **with relation** t: a relation which agrees with t within $\mathrm{dom}(t)$, and agrees with r outside $\mathrm{dom}(t)$

- Definition: $r \mathbin{\lhd\!\!\!+} t = \{\ (d, r') \mid (d, r') \in t \lor ((d, r') \in r \land d \notin \mathrm{dom}(t))\ \}$
- e.g.,

$$
\begin{array}{ll}
 & r \mathbin{\lhd\!\!\!+} \{(a, 3), (c, 4)\} \\
= & \underbrace{\{(a, 3), (c, 4)\}}_{\{(d, r') \mid (d, r') \in t\}} \cup \underbrace{\{(b, 2), (b, 5), (d, 1), (e, 2), (f, 3)\}}_{\{(d, r') \mid (d, r') \in r \land d \notin \mathrm{dom}(t)\}} \\
= & \{(a, 3), (c, 4), (b, 2), (b, 5), (d, 1), (e, 2), (f, 3)\}
\end{array}
$$

- ASCII syntax: `r <+ t`

Relations (4): Exercises

1. *Define $r[s]$ in terms of other relational operations.*

   Answer: $r[s] = \mathrm{ran}(s \lhd r)$

   e.g.,

$$r[\underbrace{\{a, b\}}_{s}] = \mathrm{ran}(\underbrace{\{(\mathbf{a}, 1), (\mathbf{b}, 2), (\mathbf{a}, 4), (\mathbf{b}, 5)\}}_{\{a,b\} \lhd r}) = \{1, 2, 4, 5\}$$

2. *Define $r \Leftarrow t$ in terms of other relational operators.*

   Answer: $r \Leftarrow t = t \cup (\mathrm{dom}(t) \ntriangleleft r)$

   e.g.,

$$\begin{aligned} & r \Leftarrow \underbrace{\{(a, 3), (c, 4)\}}_{t} \\ = \quad & \underbrace{\{(a, 3), (c, 4)\}}_{t} \cup \underbrace{\{(b, 2), (b, 5), (d, 1), (e, 2), (f, 3)\}}_{\underbrace{\mathrm{dom}(t)}_{\{a,c\}} \ntriangleleft r} \\ = \quad & \{(a, 3), (c, 4), (b, 2), (b, 5), (d, 1), (e, 2), (f, 3)\} \end{aligned}$$

Functions (1): Functional Property

- A ***relation*** r on sets S and T (i.e., $r \in S \leftrightarrow T$) is also a *function* if it satisfies the ***functional property***:

$$isFunctional(r)$$
$$\iff$$
$$\forall s, t_1, t_2 \bullet (s \in S \land t_1 \in T \land t_2 \in T) \Rightarrow ((s, t_1) \in r \land (s, t_2) \in r \Rightarrow t_1 = t_2)$$

  - That is, in a ***function***, it is forbidden for a member of S to map to more than one members of T.
  - Equivalently, in a ***function***, two distinct members of T cannot be mapped by the same member of S.
- e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$, which of the following ***relations*** satisfy the above ***functional property***?
  - $S \times T$ [No]

    Witness 1: $(1, a), (1, b)$; ***Witness*** 2: $(2, a), (2, b)$; ***Witness*** 3: $(3, a), (3, b)$.
  - $(S \times T) \setminus \{(x, y) \mid (x, y) \in S \times T \land x = 1\}$ [No]

    Witness 1: $(2, a), (2, b)$; ***Witness*** 2: $(3, a), (3, b)$
  - $\{(1, a), (2, b), (3, a)\}$ [Yes]
  - $\{(1, a), (2, b)\}$ [Yes]

Functions (2.1): Total vs. Partial

Given a **relation** $r \in S \leftrightarrow T$

- r is a ***partial function*** if it satisfies the ***functional property***:

  $$\boxed{r \in S \nrightarrow T} \iff (\texttt{isFunctional(r)} \land \mathrm{dom}(r) \subseteq S)$$

  Remark. $r \in S \nrightarrow T$ means there **may (or may not) be** $s \in S$ s.t. $r(s)$ is ***undefined*** (i.e., $r[\{s\}] = \varnothing$).

  - e.g., $\{\ \{(\mathbf{2}, a), (\mathbf{1}, b)\}, \{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\}\ \} \subseteq \{1, 2, 3\} \nrightarrow \{a, b\}$
  - ASCII syntax: `r : +->`

- r is a ***total function*** if there is a mapping for each $s \in S$:

  $$\boxed{r \in S \rightarrow T} \iff (\texttt{isFunctional(r)} \land \mathrm{dom}(r) = S)$$

  Remark. $r \in S \rightarrow T$ implies $r \in S \nrightarrow T$, but not vice versa. Why?

  - e.g., $\{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\} \in \{1, 2, 3\} \rightarrow \{a, b\}$
  - e.g., $\{(\mathbf{2}, a), (\mathbf{1}, b)\} \notin \{1, 2, 3\} \rightarrow \{a, b\}$
  - ASCII syntax: `r : -->`

Functions (2.2): Relation Image vs. Function Application

- Recall: A ***function*** is a ***relation***, but a ***relation*** is not necessarily a ***function***.
- Say we have a ***partial function*** $f \in \{1,2,3\} \nrightarrow \{a,b\}$:

$$f = \{(\mathbf{3},a),(\mathbf{1},b)\}$$

  - With f wearing the ***relation*** hat, we can invoke *relational images*:

$$\begin{array}{rcl} f[\{3\}] & = & \{a\} \\ f[\{1\}] & = & \{b\} \\ f[\{2\}] & = & \varnothing \end{array}$$

    Remark. $\Rightarrow |f[\{v\}]| \le 1 \because$
    - each member in $\mathrm{dom}(f)$ is mapped to at most one member in $\mathrm{ran}(f)$
    - each input set $\{v\}$ is a **singleton** set
  - With f wearing the ***function*** hat, we can invoke *functional applications*:

$$\begin{array}{rcl} f(3) & = & a \\ f(1) & = & b \\ f(2) & \texttt{is} & \textit{undefined} \end{array}$$

Functions (2.3): Modelling Decision

An organization has a system for keeping **track** of its employees as to where they are on the premises (e.g., ``Zone A, Floor 23''). To achieve this, each employee is issued with an active badge which, when scanned, synchronizes their current positions to a central database.

Assume the following two sets:

- *Employee* denotes the **set** of all employees working for the organization.
- *Location* denotes the **set** of all valid locations in the organization.

1. Is it appropriate to *model/formalize* such a **track** functionality as a ***relation*** (i.e., $where\_is \in Employee \leftrightarrow Location$)?
   Answer. No – an employee cannot be at distinct locations simultaneously.
   e.g., $where\_is[Alan] = \{$ ``Zone A, Floor 23'', ``Zone C, Floor 46'' $\}$
2. How about a ***total function*** (i.e., $where\_is \in Employee \rightarrow Location$)?
   Answer. No – in reality, not necessarily all employees show up.
   e.g., $where\_is(Mark)$ should be ***undefined*** if Mark happens to be on vacation.
3. How about a ***partial function*** (i.e., $where\_is \in Employee \nrightarrow Location$)?
   Answer. Yes – this addresses the inflexibility of the total function.

Functions (3.1): Injective Functions

Given a ***function*** f (either partial or total):

- f is ***injective***/***one-to-one***/***an injection*** if f does **not** map more than one members of S to a single member of T.

  $isInjective(f)$
  $\iff$
  $\forall s_1, s_2, t \bullet (s_1 \in S \wedge s_2 \in S \wedge t \in T) \Rightarrow ((s_1, t) \in f \wedge (s_2, t) \in f \Rightarrow s_1 = s_2)$

- If f is a ***partial injection***, we write: $\boxed{f \in S \nrightarrowtail T}$
  - e.g., $\{\varnothing, \{(1, \mathbf{a})\}, \{(2, \mathbf{a}), (3, \mathbf{b})\}\} \subseteq \{1,2,3\} \nrightarrowtail \{a, b\}$
  - e.g., $\{(1, \mathbf{b}), (2, a), (3, \mathbf{b})\} \notin \{1,2,3\} \nrightarrowtail \{a, b\}$ [total, not inj.]
  - e.g., $\{(1, \mathbf{b}), (3, \mathbf{b})\} \notin \{1,2,3\} \nrightarrowtail \{a, b\}$ [partial, not inj.]
  - ASCII syntax: `f : >+>`
- If f is a ***total injection***, we write: $\boxed{f \in S \rightarrowtail T}$
  - e.g., $\{1,2,3\} \rightarrowtail \{a, b\} = \varnothing$
  - e.g., $\{(2, d), (1, a), (3, c)\} \in \{1,2,3\} \rightarrowtail \{a, b, c, d\}$
  - e.g., $\{(\mathbf{2}, d), (\mathbf{1}, c)\} \notin \{1,2,3\} \rightarrowtail \{a, b, c, d\}$ [not total, inj.]
  - e.g., $\{(2, \mathbf{d}), (1, c), (3, \mathbf{d})\} \notin \{1,2,3\} \rightarrowtail \{a, b, c, d\}$ [total, not inj.]
  - ASCII syntax: `f : >->`

Functions (3.2): Surjective Functions

Given a ***function*** f (either partial or total):

- f is ***surjective***/***onto***/***a surjection*** if f maps to all members of T.

$$\mathtt{isSurjective}(f) \iff \mathrm{ran}(f) = T$$

- If f is a ***partial surjection***, we write: $\boxed{f \in S \twoheadrightarrow\!\!\!\!\!\!+\; T}$
  - e.g., $\{\ \{(1,\mathbf{b}),(2,\mathbf{a})\},\{(1,\mathbf{b}),(2,\mathbf{a}),(3,\mathbf{b})\}\ \} \subseteq \{1,2,3\} \mathrel{+\!\!\!\twoheadrightarrow} \{a,b\}$
  - e.g., $\{(2,\mathbf{a}),(1,\mathbf{a}),(3,\mathbf{a})\ \} \notin \{1,2,3\} \mathrel{+\!\!\!\twoheadrightarrow} \{a,b\}$ [total, not sur.]
  - e.g., $\{(2,\mathbf{b}),(1,\mathbf{b})\} \notin \{1,2,3\} \mathrel{+\!\!\!\twoheadrightarrow} \{a,b\}$ [partial, not sur.]
  - ASCII syntax: `f : +->>`
- If f is a ***total surjection***, we write: $\boxed{f \in S \twoheadrightarrow T}$
  - e.g., $\{\ \{(2,a),(1,b),(3,a)\},\{(2,b),(1,a),(3,b)\}\ \} \subseteq \{1,2,3\} \twoheadrightarrow \{a,b\}$
  - e.g., $\{(\mathbf{2},a),(\mathbf{3},b)\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$ [not total, sur.]
  - e.g., $\{(2,\mathbf{a}),(3,\mathbf{a}),(1,\mathbf{a})\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$ [total., not sur]
  - ASCII syntax: `f : -->>`

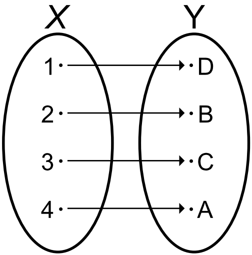
Functions (3.3): Bijective Functions

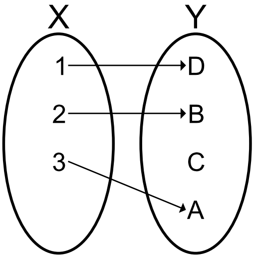
Given a function f:

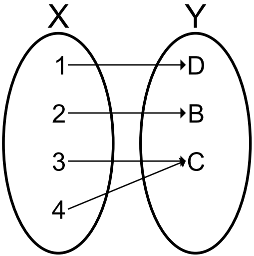
f is ***bijective***/***a bijection***/*one-to-one correspondence* if f is ***total***, ***injective***, and ***surjective***.

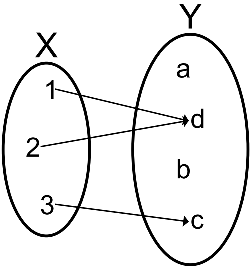
- e.g., $\{1,2,3\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b\} = \varnothing$
- e.g., $\{\ \{(1,a),(2,b),(3,c)\},\{(2,a),(3,b),(1,c)\}\ \} \subseteq \{1,2,3\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$
- e.g., $\{(\mathbf{2},b),(\mathbf{3},c),(\mathbf{4},a)\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$
  [not total, inj., sur.]
- e.g., $\{(1,\mathbf{a}),(2,b),(3,c),(4,\mathbf{a})\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$
  [total, not inj., sur.]
- e.g., $\{(1,\mathbf{a}),(2,\mathbf{c})\} \notin \{1,2\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$
  [total, inj., not sur.]
- ASCII syntax: `f : >->>`

Functions (4.1): Exercises









Functions (4.2): Modelling Decisions

1. Should an array `a` declared as "`String[] a`" be *modelled/formalized* as a *partial* function (i.e., $a \in \mathbb{Z} \nrightarrow String$) or a *total* function (i.e., $a \in \mathbb{Z} \rightarrow String$)?

   Answer. $a \in \mathbb{Z} \rightarrow String$ is not appropriate as:
   - Indices are non-negative (i.e., $a(i)$, where $i < 0$, is ***undefined***).
   - Each array size is finite: not all positive integers are valid indices.
2. What does it mean if an **array** is *modelled/formalized* as a partial ***injection*** (i.e., $a \in \mathbb{Z} \nrightarrowtail String$)?

   Answer. It means that the array does **not** contain any duplicates.
3. Can an integer array "`int[] a`" be *modelled/formalized* as a partial ***surjection*** (i.e., $a \in \mathbb{Z} \twoheadrightarrow\!\!\!\!\!\!\!\!\!\not\quad \mathbb{Z}$)?

   Answer. Yes, if `a` stores all 2^{32} integers (i.e., $[-2^{31}, 2^{31} - 1]$).
4. Can a string array "`String[] a`" be *modelled/formalized* as a partial ***surjection*** (i.e., $a \in \mathbb{Z} \twoheadrightarrow\!\!\!\!\!\!\!\!\!\not\quad String$)?

   Answer. No $\because$ # possible strings is ∞.
5. Can an integer array "`int[]`" storing all 2^{32} values be *modelled/formalized* as a ***bijection*** (i.e., $a \in \mathbb{Z} \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathbb{Z}$)?

   Answer. No, because it cannot be ***total*** (as discussed earlier).

Beyond this lecture . . .

- For the $where\_is \in Employee \nrightarrow Location$ model, what does it mean when it is:
  - ***Injective*** [$where\_is \in Employee \rightarrowtail\!\!\!\!\!\!\!\nrightarrow Location$]
  - ***Surjective*** [$where\_is \in Employee \twoheadrightarrow\!\!\!\!\!\!\!\nrightarrow Location$]
  - ***Bijective*** [$where\_is \in Employee \rightarrowtail\!\!\!\!\!\!\!\twoheadrightarrow Location$]
- Review examples discussed in your earlier math courses on ***logic*** and ***set theory***.

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Specifying & Refining a Bridge Controller

MEB: Chapter 2

EECS3342 E: System Specification and Refinement
Fall 2024

Chen-Wei Wang

Learning Outcomes

This module is designed to help you understand:

- What a ***Requirement Document*** (***RD***) is
- What a ***refinement*** is
- Writing ***formal specifications***
  - (Static) contexts: constants, axioms, theorems
  - (Dynamic) machines: variables, invariants, events, guards, actions
- ***Proof Obligations*** (***POs***) associated with proving:
  - ***refinements***
  - system ***properties***
- Applying ***inference rules*** of the ***sequent calculus***

Recall: Correct by Construction

- Directly reasoning about **source code** (written in a programming language) is too complicated to be feasible.
- Instead, given a ***requirements document***, prior to **implementation**, we develop ***models*** through a series of ***refinement*** steps:
  - Each model formalizes an ***external observer***'s perception of the system.
  - Models are "sorted" with ***increasing levels of accuracy*** w.r.t. the system.
  - The ***first model***, though the most ***abstract***, can already be proved satisfying some ***requirements***.
  - Starting from the ***second model***, each model is analyzed and proved ***correct*** relative to two criteria:
    1. Some ***requirements*** (i.e., R-descriptions)
    2. ***Proof Obligations*** (***POs***) related to the ***preceding model*** being ***refined by*** the ***current model*** (via "extra" ***state*** variables and ***events***).
  - The ***last model*** (which is *correct by construction*) should be **sufficiently close** to be transformed into a **working program** (e.g., in C).

State Space of a Model

- A model's *state space* is the set of **all** configurations:
  - Each ***configuration*** assigns values to **constants** & **variables**, subject to:
    - ***axiom*** (e.g., typing constraints, assumptions)
    - ***invariant*** properties/theorems
  - Say an initial model of a bank system with two constants and a variable:

    $c \in \mathbb{N}1 \land L \in \mathbb{N}1 \land accounts \in String \rightarrowtail \mathbb{Z}$ /* typing constraint */

    $\forall id \bullet id \in \mathrm{dom}(accounts) \Rightarrow -c \leq accounts(id) \leq L$ /* desired property */

    Q. What is the ***state space*** of this initial model?

    A. All valid combinations of c, L, and $accounts$.
    - Configuration 1: $(c = 1{,}000, L = 500{,}000, b = \varnothing)$
    - Configuration 2: $(c = 2{,}375, L = 700{,}000, b = \{(\text{"}id1\text{"}, 500), (\text{"}id2\text{"}, 1{,}250)\})$

    ... [Challenge: ***Combinatorial Explosion***]
  - Model Concreteness ↑ ⇒ (State Space ↑ ∧ Verification Difficulty ↑)
- A model's ***complexity*** should be guided by those properties intended to be verified against that model.

  ⇒ ***Infeasible*** to prove **all** desired properties on a model.

  ⇒ ***Feasible*** to distribute desired properties over a list of ***refinements***.

Roadmap of this Module

- We will walk through the *development process* of constructing ***models*** of a control system regulating cars on a bridge.

  Such controllers exemplify a ***reactive system***.

  (with **sensors** and **actuators**)

- Always stay on top of the following roadmap:

  1. A ***Requirements Document*** (***RD***) of the bridge controller
  2. A brief overview of the ***refinement strategy***
  3. An initial, the most ***abstract*** model
  4. A subsequent ***model*** representing the ***1st refinement***
  5. A subsequent ***model*** representing the ***2nd refinement***
  6. A subsequent ***model*** representing the ***3rd refinement***

Requirements Document: Mainland, Island

Imagine you are asked to build a bridge (as an alternative to ferry) connecting the downtown and Toronto Island.

Page Source: https://soldbyshane.com/area/toronto-islands/

Requirements Document: E-Descriptions

Each ***E-Description*** is an **atomic** ***specification*** of a ***constraint*** or an ***assumption*** of the system's working environment.

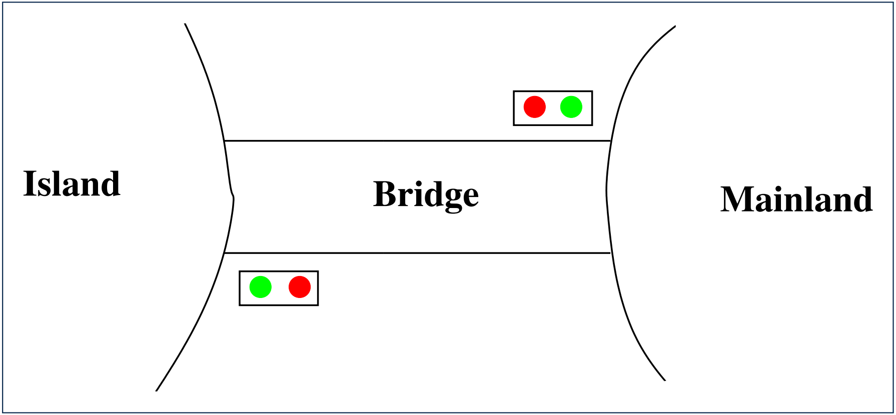
ENV1	The system is equipped with two traffic lights with two colors: green and red.
ENV2	The traffic lights control the entrance to the bridge at both ends of it.
ENV3	Cars are not supposed to pass on a red traffic light, only on a green one.
ENV4	The system is equipped with four sensors with two states: on or off.
ENV5	The sensors are used to detect the presence of a car entering or leaving the bridge: "on" means that a car is willing to enter the bridge or to leave it.

Requirements Document: R-Descriptions

Each ***R-Description*** is an **atomic** ***specification*** of an intended ***functionality*** or a desired ***property*** of the working system.

REQ1	The system is controlling cars on a bridge connecting the mainland to an island.
REQ2	The number of cars on bridge and island is limited.
REQ3	The bridge is one-way or the other, not both at the same time.

Requirements Document: Visual Summary of Equipment Pieces



Refinement Strategy

- Before diving into details of the ***models***, we first clarify the adopted ***design strategy of progressive refinements***.

  0. The ***initial model*** (m_0) will address the intended functionality of a limited number of cars on the island and bridge.

  [**REQ2**]

  1. A ***1st refinement*** (m_1 which ***refines*** m_0) will address the intended functionality of the *bridge being one-way*.

  [**REQ1**, **REQ3**]

  2. A ***2nd refinement*** (m_2 which ***refines*** m_1) will address the environment constraints imposed by *traffic lights*.

  [**ENV1, ENV2, ENV3**]

  3. A ***final, 3rd refinement*** (m_3 which ***refines*** m_2) will address the environment constraints imposed by *sensors* and the *architecture*: controller, environment, communication channels.

  [**ENV4, ENV5**]

- Recall *Correct by Construction*:

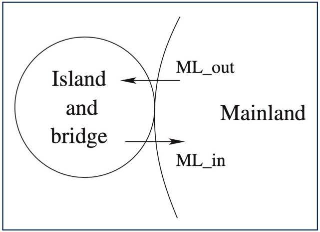
  From each ***model*** to its ***refinement***, only a manageable amount of details are added, making it ***feasible*** to conduct **analysis** and **proofs**.

Model m_0: Abstraction

- In this most ***abstract*** perception of the bridge controller, we do **not** even consider the bridge, traffic lights, and sensors!
- Instead, we focus on this single ***requirement***:

REQ2	The number of cars on bridge and island is limited.

- **Analogies**:
  - Observe the system from the sky: island and bridge appear only as a compound.



  - "***Zoom in***" on the system as ***refinements*** are introduced.

Model m_0: State Space

1. The ***static*** part is fixed and may be seen/imported.

   A ***constant*** d denotes the maximum number of cars allowed to be on the ***island-bridge compound*** at any time.

   (whereas cars on the mainland is unbounded)

   constants: d

   axioms:
   axm0_1 : $d \in \mathbb{N}$

   Remark. ***Axioms*** are assumed true and may be used to prove theorems.

2. The ***dynamic*** part changes as the system ***evolves***.

   A ***variable*** n denotes the actual number of cars, at a given moment, in the ***island-bridge compound***.

   variables: n

   invariants:
   inv0_1 : $n \in \mathbb{N}$
   inv0_2 : $n \leq d$

   Remark. ***Invariants*** should be (subject to **proofs**):
   - ***Established*** when the system is first initialized
   - ***Preserved***/***Maintained*** after any ***enabled event***'s actions take effect

Model m_0: State Transitions via Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)** : it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- At any given ***state*** (a valid ***configuration*** of constants/variables):
  - An event is said to be ***enabled*** if its guard evaluates to ***true***.
  - An event is said to be ***disabled*** if its guard evaluates to ***false***.
  - An ***enabled*** event makes a ***state transition*** if it occurs and its ***actions*** take effect.
- *1st* ***event***: A car exits mainland (and enters the island-bridge compound).

```
ML_out
  begin
    n := n + 1
  end
```

Correct Specification? Say $d = 2$.
Witness: ***Event Trace*** $\langle init, ML\_out, ML\_out, ML\_out \rangle$

- *2nd* ***event***: A car enters mainland (and exits the island-bridge compound).

```
ML_in
  begin
    n := n - 1
  end
```

Correct Specification? Say $d = 2$.
Witness: ***Event Trace*** $\langle init, ML\_in \rangle$

Model m_0: Actions vs. Before-After Predicates

- When an enabled event e occurs there are two notions of ***state***:
  - ***Before-/Pre-State***: Configuration just ***before*** e's actions take effect
  - ***After-/Post-State***: Configuration just ***after*** e's actions take effect

  Remark. When an enabled event occurs, its ***action(s)*** cause a **transition** from the ***pre-state*** to the ***post-state***.
- As examples, consider ***actions*** of m_0's two events:

Events	ML_out $n := n + 1$	ML_in $n := n - 1$
before–after predicates	$n' = n + 1$	$n' = n - 1$

  - An event ***action*** "$n := n + 1$" is not a variable assignment; instead, it is a ***specification***: "*n becomes n + 1 (when the state transition completes)*".
  - The ***before-after predicate*** (***BAP***) "$n' = n + 1$" expresses that n' (the ***post-state*** value of n) is one more than n (the ***pre-state*** value of n).
- When we express ***proof obligations*** (***POs***) associated with ***events***, we use ***BAP***.

Design of Events: Invariant Preservation

- Our design of the two events

  ML_out
  begin
  $n := n + 1$
  end

  ML_in
  begin
  $n := n - 1$
  end

  only specifies how the ***variable*** n should be updated.
- Remember, ***invariants*** are conditions that should never be ***violated***!

  invariants:
  inv0_1 : $n \in \mathbb{N}$
  inv0_2 : $n \leq d$

- By simulating the system as an ***ASM***, we discover ***witnesses*** (i.e., event traces) of the ***invariants*** not being preserved all the time.

$$\exists s \bullet s \in \textbf{STATE SPACE} \Rightarrow \neg \textit{invariants}(s)$$

- We formulate such a commitment to preserving ***invariants*** as a ***proof obligation (PO)*** rule (a.k.a. a ***verification condition (VC)*** rule).

Sequents: Syntax and Semantics

- We formulate each **PO/VC** rule as a (horizontal or vertical) ***sequent***:

$$\boxed{H \vdash G} \qquad \boxed{\begin{array}{c} H \\ \vdash \\ G \end{array}}$$

  - The symbol $\vdash$ is called the ***turnstile***.
  - H is a set of predicates forming the ***hypotheses***/***assumptions***.
    [assumed as ***true***]
  - G is a set of predicates forming the ***goal***/***conclusion***.
    [claimed to be ***provable*** from H]
- Informally:
  - $H \vdash G$ is ***true*** if G can be proved by assuming H.
    [i.e., We say "H ***entails*** G" or "H ***yields*** G"]
  - $H \vdash G$ is ***false*** if G cannot be proved by assuming H.
- Formally: $H \vdash G \iff (H \Rightarrow G)$

  Q. What does it mean when H is empty (i.e., no hypotheses)?

  A. $\boxed{\vdash G} \equiv \boxed{\textit{true} \vdash G}$ [Why not $\boxed{\vdash G} \equiv \boxed{\textit{false} \vdash G}$?]

PO of Invariant Preservation: Sketch

- Here is a sketch of the PO/VC rule for *invariant preservation*:

$$\begin{array}{|l|}\hline \text{Axioms} \\ \textit{Invariants} \text{ Satisfied at } \textit{Pre-State} \\ \text{Guards of the Event} \\ \vdash \\ \textit{Invariants} \text{ Satisfied at } \textit{Post-State} \\ \hline \end{array} \quad \underline{\text{INV}}$$

- Informally, this is what the above PO/VC *requires to prove*:

  Assuming **all** axioms, invariants, and the event's guards hold at the ***pre-state***,

  after the ***state transition*** is made by the event,

  all invariants hold at the ***post-state***.

PO of Invariant Preservation: Components

constants: d	**variables:** n	ML_out **begin** $n := n + 1$ **end**
axioms: **axm0_1** : $d \in \mathbb{N}$	**invariants:** **inv0_1** : $n \in \mathbb{N}$ **inv0_2** : $n \leq d$	ML_in **begin** $n := n - 1$ **end**

- c: list of ***constants*** $\langle d \rangle$
- $A(c)$: list of ***axioms*** $\langle \textbf{axm0\_1} \rangle$
- ***v*** and ***v'***: list of ***variables*** in ***pre***- and ***post***-states $\boldsymbol{v} \mathrel{\widehat{=}} \langle n \rangle$, $\boldsymbol{v'} \mathrel{\widehat{=}} \langle n' \rangle$
- $I(c, v)$: list of ***invariants*** $\langle \textbf{inv0\_1}, \textbf{inv0\_2} \rangle$
- $G(c, v)$: the ***event***'s list of guards

  $G(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \textbf{true} \rangle$, $G(\langle d \rangle, \langle n \rangle)$ of $ML\_in \mathrel{\widehat{=}} \langle \textbf{true} \rangle$
- $E(c, v)$: effect of the ***event***'s actions i.t.o. what variable values ***become***

  $E(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \boldsymbol{n} + 1 \rangle$, $E(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \boldsymbol{n} - 1 \rangle$
- $v' = E(c, v)$: ***before-after predicate*** formalizing E's actions

  BAP of ML_out: $\langle \boldsymbol{n'} \rangle = \langle \boldsymbol{n} + 1 \rangle$, BAP of ML_in: $\langle \boldsymbol{n'} \rangle = \langle \boldsymbol{n} - 1 \rangle$

Rule of Invariant Preservation: Sequents

- Based on the components $(c, A(c), v, I(c,v), E(c,v))$, we are able to formally state the ***PO/VC Rule of Invariant Preservation***:

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ G(c, \mathbf{v}) \\ \vdash \\ I_i(c, \mathbf{E(c, v)}) \end{array}} \quad \underline{\text{INV}}$$

where I_i denotes a single invariant condition

  - Accordingly, how many ***sequents*** to be proved? [# events × # invariants]
  - We have <u>two</u> ***sequents*** generated for ***event*** *ML_out* of model m_0:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \underline{\text{ML\_out}/\textbf{inv0\_1}/\text{INV}} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \underline{\text{ML\_out}/\textbf{inv0\_2}/\text{INV}}$$

<u>Exercise</u>. Write the ***POs of invariant preservation*** for event *ML_in*.

- Before claiming that a ***model*** is *correct*, outstanding ***sequents*** associated with <u>all</u> ***POs*** must be <u>proved/discharged</u>.

Inference Rules: Syntax and Semantics

- An *inference rule (IR)* has the following form:

$$\frac{A}{C}\ \mathbf{L}$$

Formally: $A \Rightarrow C$ is an axiom.

Informally: To prove C, it is sufficient to prove A instead.

Informally: C is the case, assuming that A is the case.

  - L is a name label for referencing the ***inference rule*** in proofs.
  - A is a **set** of sequents known as ***antecedents*** of rule L.
  - C is a **single** sequent known as ***consequent*** of rule L.
- Let's consider ***inference rules (IRs)*** with two different flavours:

$$\frac{H1 \vdash G}{H1, H2 \vdash G}\ \mathbf{MON} \qquad \frac{}{n \in \mathbb{N} \vdash n+1 \in \mathbb{N}}\ \mathbf{P2}$$

  - IR **MON**: To prove $H1, H2 \vdash G$, it suffices to prove $H1 \vdash G$ instead.
  - IR **P2**: $n \in \mathbb{N} \vdash n+1 \in \mathbb{N}$ is an ***axiom***.

    [proved automatically without further justifications]

Proof of Sequent: Steps and Structure

- To prove the following sequent (related to ***invariant preservation***):

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n+1 \in \mathbb{N} \end{array}} \quad \underline{\text{ML\_out/}\textbf{inv0\_1}\text{/INV}}$$

1. Apply a ***inference rule***, which ***transforms*** some "outstanding" **sequent** to <u>one</u> or <u>more</u> other **sequents** to be proved instead.
2. Keep applying ***inference rules*** until <u>**all**</u> ***transformed*** **sequents** are ***axioms*** that do <u>**not**</u> require any further justifications.

- Here is a *formal proof* of ML_out/**inv0_1**/INV, by applying IRs **MON** and **P2**:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n+1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n+1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

Example Inference Rules (1)

$$\frac{}{\vdash\ 0 \in \mathbb{N}}\ \textbf{P1}$$

1st Peano axiom: 0 is a natural number.

$$\frac{}{n \in \mathbb{N}\ \vdash\ n+1 \in \mathbb{N}}\ \textbf{P2}$$

2nd Peano axiom: $n+1$ is a natural number, assuming that n is a natural number.

$$\frac{}{0 < n\ \vdash\ n-1 \in \mathbb{N}}\ \textbf{P2'}$$

$n-1$ is a natural number, assuming that n is positive.

$$\frac{}{n \in \mathbb{N}\ \vdash\ 0 \leq n}\ \textbf{P3}$$

3rd Peano axiom: n is non-negative, assuming that n is a natural number.

Example Inference Rules (2)

$$\frac{}{n < m \vdash n + 1 \leq m} \quad \textbf{INC}$$

$n + 1$ is less than or equal to m, assuming that n is strictly less than m.

$$\frac{}{n \leq m \vdash n - 1 < m} \quad \textbf{DEC}$$

$n - 1$ is strictly less than m, assuming that n is less than or equal to m.

Example Inference Rules (3)

$$\frac{H1 \vdash G}{H1, H2 \vdash G} \quad \textbf{MON}$$

To prove a goal under certain hypotheses,
it suffices to prove it under less hypotheses.

$$\frac{H, P \vdash R \qquad H, Q \vdash R}{H, P \lor Q \vdash R} \quad \textbf{OR\_L}$$

Proof by Cases:
To prove a goal under a disjunctive assumption,
it suffices to prove **independently**
the same goal, twice, under each disjunct.

$$\frac{H \vdash P}{H \vdash P \lor Q} \quad \textbf{OR\_R1}$$

To prove a disjunction,
it suffices to prove the left disjunct.

$$\frac{H \vdash Q}{H \vdash P \lor Q} \quad \textbf{OR\_R2}$$

To prove a disjunction,
it suffices to prove the right disjunct.

Revisiting Design of Events: *ML_out*

- Recall that we already proved ***PO*** $\boxed{\text{ML\_out/}\textbf{inv0\_1}\text{/INV}}$:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \le d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

$\therefore$ ***ML_out/inv0_1/INV*** succeeds in being discharged.

- How about the other ***PO*** $\boxed{\text{ML\_out/}\textbf{inv0\_2}\text{/INV}}$ for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \le d \\ \vdash \\ n + 1 \le d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \le d \\ \vdash \\ n + 1 \le d \end{array}} \quad \textbf{?}$$

$\therefore$ ***ML_out/inv0_2/INV*** fails to be discharged.

Revisiting Design of Events: *ML_in*

- How about the ***PO*** $\boxed{\text{ML\_in/}\textbf{inv0\_1}\text{/INV}}$ for *ML_in*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{?}$$

∴ ***ML_in/inv0_1/INV*** fails to be discharged.

- How about the other ***PO*** $\boxed{\text{ML\_in/}\textbf{inv0\_2}\text{/INV}}$ for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n - 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \lor n - 1 = d \end{array}} \quad \textbf{OR\_1} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \end{array}} \quad \textbf{DEC}$$

∴ ***ML_in/inv0_2/INV*** succeeds in being discharged.

Fixing the Design of Events

- Proofs of ***ML_out/inv0_2/INV*** and ***ML_in/inv0_1/INV*** fail due to the two events being *enabled when they should not*.
- Having this feedback, we add proper ***guards*** to *ML_out* and *ML_in*:

```
ML_out
  when
    n < d
  then
    n := n + 1
  end
```

```
ML_in
  when
    n > 0
  then
    n := n − 1
  end
```

- Having changed both events, updated ***sequents*** will be generated for the PO/VC rule of ***invariant preservation***.
- All ***sequents*** ($\{ML\_out,\ ML\_in\} \times \{\mathbf{inv0\_1},\ \mathbf{inv0\_2}\}$) now ***provable***?

Revisiting Fixed Design of Events: *ML_out*

- How about the ***PO*** $\boxed{\text{ML\_out/}\textbf{inv0\_1}\text{/INV}}$ for *ML_out*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n < d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

∴ ***ML_out/inv0_1/INV*** still <u>succeeds</u> in being discharged!

- How about the other ***PO*** $\boxed{\text{ML\_out/}\textbf{inv0\_2}\text{/INV}}$ for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n < d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n < d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \textbf{INC}$$

∴ ***ML_out/inv0_2/INV*** now <u>succeeds</u> in being discharged!

Revisiting Fixed Design of Events: *ML_in*

- How about the ***PO*** $\boxed{\text{ML\_in/}\textbf{inv0\_1}\text{/INV}}$ for *ML_in*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n > 0 \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n > 0 \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{P2'}$$

$\therefore$ ***ML_in/inv0_1/INV*** now <u>succeeds</u> in being discharged!

- How about the other ***PO*** $\boxed{\text{ML\_in/}\textbf{inv0\_2}\text{/INV}}$ for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n > 0 \\ \vdash \\ n - 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \lor n - 1 = d \end{array}} \quad \textbf{OR\_1} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \end{array}} \quad \textbf{DEC}$$

$\therefore$ ***ML_in/inv0_2/INV*** still <u>succeeds</u> in being discharged!

Initializing the Abstract System m_0

- Discharging the four ***sequents*** proved that both ***invariant*** conditions are ***preserved*** between occurrences/interleavings of ***events*** `ML_out` and `ML_in`.
- But how are the ***invariants established*** in the first place?

  Analogy. Proving P via ***mathematical induction***, two cases to prove:
  - $P(1), P(2), \ldots$ [***base*** cases ≈ ***establishing*** inv.]
  - $P(n) \Rightarrow P(n+1)$ [***inductive*** cases ≈ ***preserving*** inv.]
- Therefore, we specify how the ***ASM***'s ***initial state*** looks like:

```
init
  begin
    n := 0
  end
```

✓ The IB compound, once ***initialized***, has no cars.

✓ Initialization always possible: guard is ***true***.

✓ There is no ***pre-state*** for *init*.

∴ The RHS of := must not involve variables.

∴ The RHS of := may only involve constants.

✓ There is only the ***post-state*** for *init*.

∴ Before-***After Predicate***: $n' = 0$

PO of Invariant Establishment

```
init
  begin
    n := 0
  end
```

- ✓ An ***reactive system***, once ***initialized***, should never terminate.
- ✓ Event *init* cannot "preserve" the ***invariants***.

  ∵ State before its occurrence (***pre-state***) does not exist.
- ✓ Event *init* only required to ***establish*** invariants for the first time

- A new formal component is needed:
  - $K(c)$: effect of ***init***'s actions i.t.o. what variable values ***become***

    e.g., $K(\langle d \rangle)$ of *init* $\mathrel{\widehat{=}} \langle 0 \rangle$
  - $v' = K(c)$: ***before-after predicate*** formalizing *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{n'} \rangle = \langle 0 \rangle$
- Accordingly, PO of ***invariant establishment*** is formulated as a ***sequent***:

Axioms $\vdash$ *Invariants* Satisfied at *Post-State*	INV	$A(c)$ $\vdash$ $I_i(c, \boldsymbol{K(c)})$	INV

Discharging PO of Invariant Establishment

- How many ***sequents*** to be proved? [# invariants]
- We have <u>two</u> ***sequents*** generated for ***event*** *init* of model m_0:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \in \mathbb{N} \end{array}} \quad \underline{\text{init/}\mathbf{inv0\_1}\text{/INV}} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \leq d \end{array}} \quad \underline{\text{init/}\mathbf{inv0\_2}\text{/INV}}$$

- Can we discharge the ***PO*** init/**inv0_1**/INV ?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} \vdash \\ 0 \in \mathbb{N} \end{array}} \quad \textbf{P1}$$

$\therefore$ ***init/inv0_1/INV*** <u>succeeds</u> in being discharged.

- Can we discharge the ***PO*** init/**inv0_2**/INV ?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \leq d \end{array}} \quad \textbf{P3}$$

$\therefore$ ***init/inv0_2/INV*** <u>succeeds</u> in being discharged.

System Property: Deadlock Freedom

- So far we have proved that our initial model m_0 is s.t. all ***invariant conditions*** are:
  - Established when system is first initialized via *init*
  - Preserved whenevner there is a ***state transition***

    (via an enabled event: *ML_out* or *ML_in*)
- However, whenever ***event occurrences*** are conditional (i.e., ***guards*** stronger than ***true***), there is a possibility of ***deadlock*** :
  - A state where ***guards*** of all events evaluate to ***false***
  - When a ***deadlock*** happens, none of the ***events*** is ***enabled***.

    ⇒ The system is blocked and not reactive anymore!
- We express this ***non-blocking*** property as a new requirement:

REQ4	Once started, the system should work for ever.

PO of Deadlock Freedom (1)

- Recall some of the formal components we discussed:
  - c: list of ***constants*** $\langle d \rangle$
  - $A(c)$: list of ***axioms*** $\langle \textbf{axm0\_1} \rangle$
  - $\mathbf{v}$ and $\mathbf{v'}$: list of ***variables*** in ***pre***- and ***post***-states $\mathbf{v} \mathrel{\widehat{=}} \langle n \rangle, \mathbf{v'} \mathrel{\widehat{=}} \langle n' \rangle$
  - $I(c, v)$: list of ***invariants*** $\langle \textbf{inv0\_1}, \textbf{inv0\_2} \rangle$
  - $G(c, v)$: the event's list of ***guards***

$$G(\langle d \rangle, \langle n \rangle) \text{ of } ML\_out \mathrel{\widehat{=}} \langle n < d \rangle,\ G(\langle d \rangle, \langle n \rangle) \text{ of } ML\_in \mathrel{\widehat{=}} \langle n > 0 \rangle$$

- A system is ***deadlock-free*** if at least one of its ***events*** is ***enabled***:

$$\boxed{\begin{array}{l} \text{Axioms} \\ \textit{\textbf{Invariants}} \text{ Satisfied at } \textit{\textbf{Pre-State}} \\ \vdash \\ \text{Disjunction of the guards satisfied at } \textit{\textbf{Pre-State}} \end{array}} \ \underline{\text{DLF}} \qquad \boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ \vdash \\ G_1(c, \mathbf{v}) \lor \cdots \lor G_m(c, \mathbf{v}) \end{array}} \ \underline{\text{DLF}}$$

To prove about deadlock freedom

- An event's effect of state transition is **not** relevant.
- Instead, the evaluation of all events' ***guards*** at the ***pre-state*** is relevant.

PO of Deadlock Freedom (2)

- ***Deadlock freedom*** is not necessarily a desired property.

  $\Rightarrow$ When it is (like m_0), then the generated ***sequents*** must be discharged.
- Applying the PO of ***deadlock freedom*** to the initial model m_0:

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ \vdash \\ G_1(c, \mathbf{v}) \lor \cdots \lor G_m(c, \mathbf{v}) \end{array}} \quad \underline{\text{DLF}} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \end{array}} \quad \underline{\text{DLF}}$$

  Our bridge controller being ***deadlock-free*** means that cars can ***always*** enter (via *ML_out*) or leave (via *ML_in*) the island-bridge compound.
- Can we formally discharge this ***PO*** for our ***initial model*** m_0?

Example Inference Rules (4)

$$\overline{H, P \vdash P} \quad \textbf{HYP}$$	A goal is proved if it can be assumed.
$$\overline{\bot \vdash P} \quad \textbf{FALSE\_L}$$	Assuming ***false*** ($\bot$), anything can be proved.
$$\overline{P \vdash \top} \quad \textbf{TRUE\_R}$$	***true*** ($\top$) is proved, regardless of the assumption.
$$\overline{P \vdash E = E} \quad \textbf{EQ}$$	An expression being equal to itself is proved, regardless of the assumption.

Example Inference Rules (5)

$$\frac{H(\mathbf{F}),\ \mathbf{E} = \mathbf{F} \ \vdash\ P(\mathbf{F})}{H(\mathbf{E}),\ \mathbf{E} = \mathbf{F} \ \vdash\ P(\mathbf{E})} \quad \textbf{EQ\_LR}$$

To prove a goal $P(\mathbf{E})$ assuming $H(\mathbf{E})$, where both P and H depend on expression $\mathbf{E}$, it suffices to prove $P(\mathbf{F})$ assuming $H(\mathbf{F})$, where both P and H depend on expresion $\mathbf{F}$, given that $\mathbf{E}$ is equal to $\mathbf{F}$.

$$\frac{H(\mathbf{E}),\ \mathbf{E} = \mathbf{F} \ \vdash\ P(\mathbf{E})}{H(\mathbf{F}),\ \mathbf{E} = \mathbf{F} \ \vdash\ P(\mathbf{F})} \quad \textbf{EQ\_RL}$$

To prove a goal $P(\mathbf{F})$ assuming $H(\mathbf{F})$, where both P and H depend on expression $\mathbf{F}$, it suffices to prove $P(\mathbf{E})$ assuming $H(\mathbf{E})$, where both P and H depend on expresion $\mathbf{E}$, given that $\mathbf{E}$ is equal to $\mathbf{F}$.

Discharging PO of DLF: Exercise

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ \vdash \\ G_1(c, \mathbf{v}) \lor \cdots \lor G_m(c, \mathbf{v}) \end{array}} \;\; \underline{\text{DLF}} \;\; \boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\; ??$$

Discharging PO of DLF: First Attempt

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \end{array}}$$

$\equiv$

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{MON}\; \boxed{\begin{array}{l} n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{OR\_L}\; \left\{ \begin{array}{l} \boxed{\begin{array}{l} n < d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{OR\_R1}\; \boxed{\begin{array}{l} n < d \\ \vdash \\ n < d \end{array}} \;\textbf{HYP} \\ \\ \boxed{\begin{array}{l} n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \;\textbf{EQ\_LR, MON}\; \boxed{\begin{array}{l} \\ \vdash \\ d < d \lor d > 0 \end{array}} \;\textbf{OR\_R2}\; \boxed{\begin{array}{l} \\ \vdash \\ d > 0 \end{array}} \;\textbf{?} \end{array} \right.$$

Why Did the DLF PO Fail to Discharge?

- In our first attempt, proof of the 2nd case failed: $\vdash\ d > 0$
- This ***unprovable*** sequent gave us a good hint:
  - For the model under consideration (m_0) to be ***deadlock-free***, it is required that $d > 0$. [≥ 1 car allowed in the IB compound]
  - But current ***specification*** of m_0 ***not*** strong enough to entail this:
    - $\neg(d > 0) \equiv d \leq 0$ is possible for the current model
    - Given **axm0_1** : $d \in \mathbb{N}$

  $\Rightarrow d = 0$ is allowed by m_0 which causes a ***deadlock***.
- Recall the *init* event and the two ***guarded*** events:

init	ML_out	ML_in
init **begin** $n := 0$ **end**	ML_out **when** $n < d$ **then** $n := n + 1$ **end**	ML_in **when** $n > 0$ **then** $n := n - 1$ **end**

When $d = 0$, the disjunction of guards evaluates to ***false***: $0 < 0 \vee 0 > 0$
$\Rightarrow$ As soon as the system is initialized, it ***deadlocks immediately*** as no car can either enter or leave the IR compound!!

Fixing the Context of Initial Model

- Having understood the failed proof, we add a proper ***axiom*** to m_0:

axioms:

axm0_2 : $d > 0$

- We have effectively elaborated on **REQ2**:

REQ2	The number of cars on bridge and island is limited but positive.

- Having changed the context, an updated ***sequent*** will be generated for the PO/VC rule of ***deadlock freedom***.
- Is this new sequent now ***provable***?

Discharging PO of DLF: Second Attempt

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array}$$

$\equiv$

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{MON}\; \begin{array}{|l|}\hline d > 0 \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{OR\_L}\; \begin{cases} \begin{array}{|l|}\hline d > 0 \\ n < d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{OR\_R1}\; \begin{array}{|l|}\hline d > 0 \\ n < d \\ \vdash \\ n < d \\ \hline \end{array} \;\textbf{HYP} \\ \begin{array}{|l|}\hline d > 0 \\ n = d \\ \vdash \\ n < d \lor n > 0 \\ \hline \end{array} \;\textbf{EQ\_LR, MON}\; \begin{array}{|l|}\hline d > 0 \\ \vdash \\ d < d \lor d > 0 \\ \hline \end{array} \;\textbf{OR\_R2}\; \begin{array}{|l|}\hline d > 0 \\ \vdash \\ d > 0 \\ \hline \end{array} \;\textbf{HYP} \end{cases}$$

Initial Model: Summary

- The final version of our ***initial model*** m_0 is ***provably correct*** w.r.t.:
  - Establishment of ***Invariants***
  - Preservation of ***Invariants***
  - ***Deadlock*** Freedom
- Here is the final ***specification*** of m_0:

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$
axm0_2 : $d > 0$

variables: n

invariants:
inv0_1 : $n \in \mathbb{N}$
inv0_2 : $n \leq d$

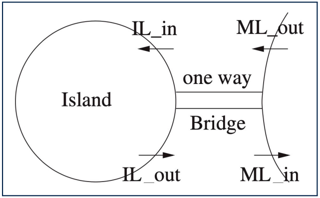
init
begin
$n := 0$
end

ML_out
when
$n < d$
then
$n := n + 1$
end

ML_in
when
$n > 0$
then
$n := n - 1$
end

Model m_1: "More Concrete" Abstraction

- First ***refinement*** has a more ***concrete*** perception of the bridge controller:
  - We "**zoom in**" by observing the system from **closer to the ground**, so that the island-bridge compound is split into:
    - the island
    - the (one-way) bridge



  - Nonetheless, traffic lights and sensors remain ***abstracted*** away!
- That is, we focus on these two ***requirement***:

REQ1	The system is controlling cars on a bridge connecting the mainland to an island.
REQ3	The bridge is one-way or the other, not both at the same time.

- We are ***obliged to prove*** this ***added concreteness*** is ***consistent*** with m_0.

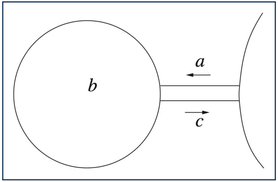
Model m_1: Refined State Space

1. The **static** part is the same as m_0's:

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$
axm0_2 : $d > 0$

2. The **dynamic** part of the ***concrete state*** consists of three ***variables***:



- ***a***: number of cars on the bridge, heading to the island
- ***b***: number of cars on the island
- ***c***: number of cars on the bridge, heading to the mainland

variables: a, b, c

invariants:
inv1_1 : $a \in \mathbb{N}$
inv1_2 : $b \in \mathbb{N}$
inv1_3 : $c \in \mathbb{N}$
inv1_4 : *??*
inv1_5 : *??*

✓ **inv1_1**, **inv1_2**, **inv1_3** are ***typing*** constraints.

✓ **inv1_4** ***links***/***glues*** the ***abstract*** and ***concrete*** states.

✓ **inv1_5** specifies that the bridge is one-way.

Model m_1: State Transitions via Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- We first consider the "old" ***events*** already existing in m_0.
- ***Concrete***/***Refined*** version of ***event*** *ML_out*:

```
ML_out
  when
    ??
  then
    a := a + 1
  end
```

  - Meaning of *ML_out* is ***refined***:
    a car exits mainland (getting on the bridge).
  - *ML_out* ***enabled*** only when:
    - the bridge's current traffic flows to the island
    - number of cars on both the bridge and the island is limited
- ***Concrete***/***Refined*** version of ***event*** *ML_in*:

```
ML_in
  when
    ??
  then
    c := c - 1
  end
```

  - Meaning of *ML_in* is ***refined***:
    a car enters mainland (getting off the bridge).
  - *ML_in* ***enabled*** only when:
    there is some car on the bridge heading to the mainland.

Model m_1: Actions vs. Before-After Predicates

- Consider the ***concrete***/***refined*** version of ***actions*** of m_0's two events:

Events	ML_in **when** $0 < c$ **then** $c := c - 1$ **end**	ML_out **when** $a + b < d$ $c = 0$ **then** $a := a + 1$ **end**
Before–after predicates	$a' = a \ \wedge\ b' = b \wedge$ $c' = c - 1$	$a' = a + 1 \ \wedge\ b' = b \wedge$ $c' = c$

  - An event's ***actions*** are a **specification**: "c becomes c - 1 after the transition".
  - The ***before-after predicate*** (***BAP***) "$c' = c - 1$" expresses that c' (the ***post-state*** value of c) is one less than c (the ***pre-state*** value of c).
  - Given that the ***concrete state*** consists of three variables:
    - An event's ***actions*** only specify those changing from ***pre***-state to ***post***-state. [e.g., $c' = c - 1$]
    - Other unmentioned variables have their ***post***-state values remain unchanged. [e.g., $a' = a \wedge b' = b$]

- When we express ***proof obligations*** (***POs***) associated with ***events***, we use ***BAP***.

States & Invariants: Abstract vs. Concrete

- m_1 refines m_0 by introducing more ***variables***:
  - ***Abstract*** State (of m_0 being refined):

    variables: n

  - ***Concrete*** State (of the refinement model m_1):

    variables: a, b, c

- Accordingly, ***invariants*** may involve different ***states***:
  - ***Abstract*** Invariants (involving the ***abstract*** state only):

    invariants:
    inv0_1 : $n \in \mathbb{N}$
    inv0_2 : $n \leq d$

  - ***Concrete*** Invariants (involving at least the ***concrete*** state):

    invariants:
    inv1_1 : $a \in \mathbb{N}$
    inv1_2 : $b \in \mathbb{N}$
    inv1_3 : $c \in \mathbb{N}$
    inv1_4 : $a + b + c = n$
    inv1_5 : $a = 0 \vee c = 0$

Events: Abstract vs. Concrete

- When an ***event*** exists in both models m_0 and m_1, there are two versions of it:
  - The ***abstract*** version modifies the ***abstract*** state.

(abstract_)ML_out	(abstract_)ML_in
when $n < d$ **then** $n := n + 1$ **end**	**when** $n > 0$ **then** $n := n - 1$ **end**

  - The ***concrete*** version modifies the ***concrete*** state.

(concrete_)ML_out	(concrete_)ML_in
when $a + b < d$, $c = 0$ **then** $a := a + 1$ **end**	**when** $c > 0$ **then** $c := c - 1$ **end**

- A ***new event*** may **only** exist in m_1 (the ***concrete*** model): we will deal with this kind of events later, separately from "redefined/overridden" events.

PO of Refinement: Components (1)

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$
axm0_2 : $d > 0$

variables: a, b, c

invariants:
inv1_1 : $a \in \mathbb{N}$
inv1_2 : $b \in \mathbb{N}$
inv1_3 : $c \in \mathbb{N}$
inv1_4 : $a + b + c = n$
inv1_5 : $a = 0 \lor c = 0$

```
ML_out
  when
    a + b < d
    c = 0
  then
    a := a + 1
  end
```

```
ML_in
  when
    c > 0
  then
    c := c - 1
  end
```

- c: list of ***constants*** $\langle d \rangle$
- $A(c)$: list of ***axioms*** $\langle$**axm0_1**$\rangle$
- v and v': ***abstract variables*** in pre- & post-states $v \mathrel{\widehat{=}} \langle n \rangle$, $v' \mathrel{\widehat{=}} \langle n \rangle$
- w and w': ***concrete variables*** in pre- & post-states $w \mathrel{\widehat{=}} \langle a, b, c \rangle$, $w' \mathrel{\widehat{=}} \langle a', b', c' \rangle$
- $I(c, v)$: list of ***abstract invariants*** $\langle$**inv0_1**, **inv0_2**$\rangle$
- $J(c, v, w)$: list of ***concrete invariants*** $\langle$**inv1_1**, **inv1_2**, **inv1_3**, **inv1_4**, **inv1_5**$\rangle$

PO of Refinement: Components (2)

constants: d

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$

variables: a, b, c

invariants:
- **inv1_1** : $a \in \mathbb{N}$
- **inv1_2** : $b \in \mathbb{N}$
- **inv1_3** : $c \in \mathbb{N}$
- **inv1_4** : $a + b + c = n$
- **inv1_5** : $a = 0 \lor c = 0$

ML_out
when
$a + b < d$
$c = 0$
then
$a := a + 1$
end

ML_in
when
$c > 0$
then
$c := c - 1$
end

- $G(c, v)$: list of guards of the ***abstract event***

$$G(\langle d \rangle, \langle n \rangle) \text{ of } ML\_out \ \widehat{=} \ \langle n < d \rangle, \ G(c, v) \text{ of } ML\_in \ \widehat{=} \ \langle n > 0 \rangle$$

- $H(c, w)$: list of guards of the ***concrete event***

$$H(\langle d \rangle, \langle a, b, c \rangle) \text{ of } ML\_out \ \widehat{=} \ \langle a + b < d, c = 0 \rangle, \ H(c, w) \text{ of } ML\_in \ \widehat{=} \ \langle c > 0 \rangle$$

PO of Refinement: Components (3)

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$
axm0_2 : $d > 0$

variables: a, b, c

invariants:
inv1_1 : $a \in \mathbb{N}$
inv1_2 : $b \in \mathbb{N}$
inv1_3 : $c \in \mathbb{N}$
inv1_4 : $a + b + c = n$
inv1_5 : $a = 0 \vee c = 0$

ML_out
when
$a + b < d$
$c = 0$
then
$a := a + 1$
end

ML_in
when
$c > 0$
then
$c := c - 1$
end

- $E(c, v)$: effect of the ***abstract event***'s actions i.t.o. what variable values **become**

$$E(\langle d \rangle, \langle n \rangle) \text{ of } ML\_out \mathrel{\widehat{=}} \langle \mathbf{n} + 1 \rangle,\ E(\langle d \rangle, \langle n \rangle) \text{ of } ML\_in \mathrel{\widehat{=}} \langle \mathbf{n} - 1 \rangle$$

- $F(c, w)$: effect of the ***concrete event***'s actions i.t.o. what variable values **become**

$$F(c, w) \text{ of } ML\_out \mathrel{\widehat{=}} \langle \mathbf{a} + 1, b, c \rangle,\ F(c, w) \text{ of } ML\_in \mathrel{\widehat{=}} \langle a, b, \mathbf{c} - 1 \rangle$$

Sketching PO of Refinement

The PO/VC rule for a *proper refinement* consists of two parts:

1. ***Guard Strengthening***

Axioms
Abstract Invariants Satisfied at Pre-State
Concrete Invariants Satisfied at Pre-State
Guards of the ***Concrete Event***
$\vdash$
Guards of the ***Abstract Event***

GRD

- A ***concrete*** transition always has an ***abstract*** counterpart.
- A ***concrete*** event is enabled *only if* ***abstract*** counterpart is enabled.

2. ***Invariant Preservation***

Axioms
Abstract Invariants Satisfied at Pre-State
Concrete Invariants Satisfied at Pre-State
Guards of the ***Concrete Event***
$\vdash$
Concrete Invariants Satisfied at **Post**-State

INV

- A ***concrete*** event performs a ***transition*** on ***concrete*** states.
- This ***concrete*** state ***transition*** must be consistent with how its ***abstract*** counterpart performs a corresponding ***abstract transition***.

Note. ***Guard strengthening*** and ***invariant preservation*** are only applicable to events that might be ***enabled*** after the system is launched.

The special, non-guarded `init` event will be discussed separately later.

Refinement Rule: Guard Strengthening

- Based on the components, we are able to formally state the ***PO/VC Rule of Guard Strengthening for Refinement***:

$$\boxed{\begin{array}{l} A(c) \\ I(c, v) \\ J(c, v, w) \\ H(c, w) \\ \vdash \\ G_i(c, v) \end{array}} \quad \underline{\text{GRD}}$$

where G_i denotes a single ***guard*** condition of the ***abstract*** event

  - How many ***sequents*** to be proved? [# *abstract* guards]
  - For ML_out, only one ***abstract*** guard, so one ***sequent*** is generated :

$$\boxed{\begin{array}{llllll} d \in \mathbb{N} & d > 0 & & & \\ n \in \mathbb{N} & n \leq d & & & \\ a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} & a + b + c = n & a = 0 \lor c = 0 \\ a + b < d & c = 0 & & & \\ \vdash & & & & \\ n < d & & & & \end{array}} \quad \underline{\text{ML\_out/GRD}}$$

- **Exercise**. Write ML_in's ***PO of Guard Strengthening for Refinement***.

PO Rule: Guard Strengthening of *ML_out*

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
Concrete guards of *ML_out*	$\left\{ \begin{array}{l} a + b < d \\ c = 0 \end{array} \right.$
	$\vdash$
Abstract guards of *ML_out*	$\{\ n < d$

ML_out/GRD

PO Rule: Guard Strengthening of *ML_in*

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
Concrete guards of *ML_in*	$\{\ c > 0$
	$\vdash$
Abstract guards of *ML_in*	$\{\ n > 0$

ML_in/GRD

Proving Refinement: ML_out/GRD

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a+b+c=n \\ a=0 \vee c=0 \\ a+b<d \\ c=0 \\ \vdash \\ n<d \\ \hline \end{array} \quad \textbf{MON} \quad \begin{array}{|l|}\hline a+b+c=n \\ a+b<d \\ c=0 \\ \vdash \\ n<d \\ \hline \end{array} \quad \textbf{EQ\_LR}, \textbf{MON} \quad \begin{array}{|l|}\hline a+b+0=n \\ a+b<d \\ \vdash \\ n<d \\ \hline \end{array} \quad \textbf{ARI} \quad \begin{array}{|l|}\hline a+b=n \\ a+b<d \\ \vdash \\ n<d \\ \hline \end{array} \quad \textbf{EQ\_LR}, \textbf{MON} \quad \begin{array}{|l|}\hline n<d \\ \vdash \\ n<d \\ \hline \end{array} \quad \textbf{HYP}$$

Proving Refinement: ML_in/GRD

$$
\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{MON}\
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{OR\_L}\
\left\{
\begin{array}{l}
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + c = n \\ a = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{EQ\_LR, MON}\
\boxed{\begin{array}{l} b \in \mathbb{N} \\ 0 + b + c = n \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{ARI}\
\boxed{\begin{array}{l} b \in \mathbb{N} \\ b + c = n \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{ARI}\
\boxed{\begin{array}{l} c \leq n \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{ARI}\
\boxed{\begin{array}{l} n > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{HYP}
\\[2ex]
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + c = n \\ c = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{EQ\_LR}\
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + 0 = n \\ c = 0 \\ 0 > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{MON}\
\boxed{\begin{array}{l} 0 > 0 \\ \vdash \\ n > 0 \end{array}}
\ \textbf{ARI}\
\boxed{\begin{array}{l} \bot \\ \vdash \\ n > 0 \end{array}}
\ \textbf{FALSE\_L}
\end{array}
\right.
$$

Refinement Rule: Invariant Preservation

- Based on the components, we are able to formally state the ***PO/VC Rule of Invariant Preservation for Refinement***:

$$\frac{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ J(c, \mathbf{v}, \mathbf{w}) \\ H(c, \mathbf{w}) \\ \vdash \\ J_i(c, E(c, \mathbf{v}), F(c, \mathbf{w})) \end{array}}{}\ \text{INV}$$

where J_i denotes a single ***concrete invariant***

  - \# ***sequents*** to be proved? [# *concrete*, **old** evts × # *concrete* invariants]
  - Here are two (of the ten) ***sequents*** generated:

$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a + b < d \\ c = 0 \\ \vdash \\ (a+1) + b + c = (n+1) \end{array} \quad \text{ML\_out/inv1\_4/INV}$$

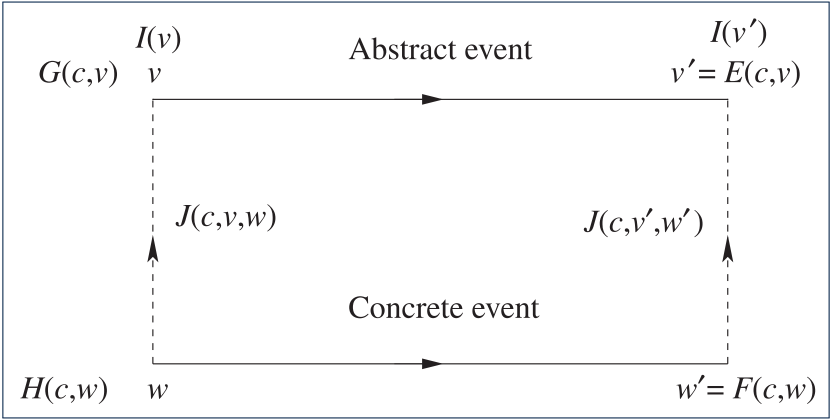
$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ c > 0 \\ \vdash \\ a = 0 \lor (c-1) = 0 \end{array} \quad \text{ML\_in/inv1\_5/INV}$$

- **Exercises**. Specify and prove other **eight** ***POs of Invariant Preservation***.

Visualizing Inv. Preservation in Refinement

Each ***concrete*** event (w to w') is ***simulated by*** an ***abstract*** event (v to v'):

- ***abstract*** & ***concrete*** pre-states related by ***concrete*** invariants $J(c, v, w)$
- ***abstract*** & ***concrete*** post-states related by ***concrete*** invariants $J(c, v', w')$



INV PO of m_1: ML_out/inv1_4/INV

axm0_1	$\{ \; d \in \mathbb{N}$
axm0_2	$\{ \; d > 0$
inv0_1	$\{ \; n \in \mathbb{N}$
inv0_2	$\{ \; n \leq d$
inv1_1	$\{ \; a \in \mathbb{N}$
inv1_2	$\{ \; b \in \mathbb{N}$
inv1_3	$\{ \; c \in \mathbb{N}$
inv1_4	$\{ \; a + b + c = n$
inv1_5	$\{ \; a = 0 \lor c = 0$
Concrete guards of *ML_out*	$\begin{cases} a + b < d \\ c = 0 \end{cases}$
	$\vdash$
Concrete invariant **inv1_4** with *ML_out*'s effect in the post-state	$\{ \; (a + 1) + b + c = (n + 1)$

ML_out/inv1_4/INV

INV PO of m_1: ML_in/inv1_5/INV

$$
\begin{array}{r l}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textit{Concrete}\text{ guards of } ML\_in & \{\ c > 0 \\
& \vdash \\
\textit{Concrete}\text{ invariant } \textbf{inv1\_5} & \\
\text{with } ML\_in\text{'s effect in the } \underline{\text{post}}\text{-state} & \{\ a = 0 \lor (c - 1) = 0
\end{array}
$$

ML_in/inv1_5/INV

Proving Refinement: ML_out/inv1_4/INV

$$
\begin{array}{|l|}
\hline
d \in \mathbb{N} \\
d > 0 \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
a + b < d \\
c = 0 \\
\vdash \\
(a+1) + b + c = (n+1) \\
\hline
\end{array}
\;\textbf{MON}\;
\begin{array}{|l|}
\hline
a + b + c = n \\
\vdash \\
(a+1) + b + c = (n+1) \\
\hline
\end{array}
\;\textbf{ARI}\;
\begin{array}{|l|}
\hline
a + b + c = n \\
\vdash \\
a + b + c + 1 = n + 1 \\
\hline
\end{array}
\;\textbf{EQ\_LR, MON}\;
\begin{array}{|l|}
\hline
\vdash \\
n + 1 = n + 1 \\
\hline
\end{array}
\;\textbf{EQ}
$$

Proving Refinement: ML_in/inv1_5/INV

$$
\boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
n \in \mathbb{N} \\
n \le d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
c > 0 \\
\vdash \\
a = 0 \lor (c - 1) = 0
\end{array}}
\quad \textbf{MON} \quad
\boxed{\begin{array}{l}
a = 0 \lor c = 0 \\
c > 0 \\
\vdash \\
a = 0 \lor (c - 1) = 0
\end{array}}
\quad \textbf{OR\_L} \quad
\left\{
\begin{array}{l}
\boxed{\begin{array}{l}
a = 0 \\
c > 0 \\
\vdash \\
a = 0 \lor (c - 1) = 0
\end{array}}
\quad \textbf{OR\_R1} \quad
\boxed{\begin{array}{l}
a = 0 \\
c > 0 \\
\vdash \\
a = 0
\end{array}}
\quad \textbf{HYP}
\\[2ex]
\boxed{\begin{array}{l}
c = 0 \\
c > 0 \\
\vdash \\
a = 0 \lor (c - 1) = 0
\end{array}}
\quad \textbf{EQ\_LR, MON} \quad
\boxed{\begin{array}{l}
0 > 0 \\
\vdash \\
a = 0 \lor (0 - 1) = 0
\end{array}}
\quad \textbf{ARI} \quad
\boxed{\begin{array}{l}
\bot \\
\vdash \\
a = 0 \lor -1 = 0
\end{array}}
\quad \textbf{FALSE\_L}
\end{array}
\right.
$$

Initializing the Refined System m_1

- Discharging the **twelve sequents** proved that:
  - ***concrete invariants*** preserved by `ML_out` & `ML_in`
  - ***concrete guards*** of *ML_out* & *ML_in* entail their ***abstract*** counterparts
- What's left is the specification of how the ***ASM*** 's ***initial state*** looks like:

```
init
  begin
    a := 0
    b := 0
    c := 0
  end
```

✓ No cars on bridge (heading either way) and island

✓ Initialization always possible: guard is ***true***.

✓ There is no ***pre-state*** for *init*.

∴ The RHS of := must not involve variables.

∴ The RHS of := may only involve constants.

✓ There is only the ***post-state*** for *init*.

∴ Before-***After Predicate***: $a' = 0 \land b' = 0 \land c' = 0$

PO of m_1 Concrete Invariant Establishment

- Some (new) formal components are needed:
  - $K(c)$: effect of ***abstract init***'s actions:

    e.g., $K(\langle d \rangle)$ of *init* $\widehat{=}$ $\langle 0 \rangle$
  - $v' = K(c)$: ***before-after predicate*** formalizing ***abstract*** *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{n'} \rangle = \langle 0 \rangle$
  - $L(c)$: effect of ***concrete init***'s actions:

    e.g., $K(\langle d \rangle)$ of *init* $\widehat{=}$ $\langle 0, 0, 0 \rangle$
  - $w' = L(c)$: ***before-after predicate*** formalizing ***concrete*** *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{a'}, \boldsymbol{b'}, \boldsymbol{c'} \rangle = \langle 0, 0, 0 \rangle$
- Accordingly, PO of ***invariant establisment*** is formulated as a **sequent**:

Axioms $\vdash$ *Concrete Invariants* Satisfied at Post-State	INV	$A(c)$ $\vdash$ $J_i(c, \boldsymbol{K(c)}, \boldsymbol{L(c)})$	INV

Discharging PO of m_1 Concrete Invariant Establishment

- How many ***sequents*** to be proved? [# ***concrete*** invariants]
- Two (of the five) sequents generated for ***concrete*** *init* of m_1:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 + 0 + 0 = 0 \end{array}} \ \text{init/}\textbf{inv1\_4}\text{/INV} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 = 0 \lor 0 = 0 \end{array}} \ \text{init/}\textbf{inv1\_5}\text{/INV}$$

- Can we discharge the ***PO*** $\boxed{\text{init/}\textbf{inv1\_4}\text{/INV}}$?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 + 0 + 0 = 0 \end{array}} \ \textbf{ARI, MON} \ \boxed{\vdash \ \top} \ \textbf{TRUE\_R}$$

∴ ***init/inv1_4/INV*** succeeds in being discharged.

- Can we discharge the ***PO*** $\boxed{\text{init/}\textbf{inv1\_5}\text{/INV}}$?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 = 0 \lor 0 = 0 \end{array}} \ \textbf{ARI, MON} \ \boxed{\vdash \ \top} \ \textbf{TRUE\_R}$$

∴ ***init/inv1_5/INV*** succeeds in being discharged.

Model m_1: New, Concrete Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- Considered ***concrete/refined events*** already existing in m_0: *ML_out* & *ML_in*
- ***New event*** *IL_in*:

```
IL_in
  when
    ??
  then
    ??
  end
```

  - *IL_in* denotes a car entering the island (getting off the bridge).
  - *IL_in* ***enabled*** only when:
    - The bridge's current traffic flows to the island.
      Q. Limited number of cars on the bridge and the island?
      A. Ensured when the earlier *ML_out* (of same car) occurred
- ***New event*** *IL_out*:

```
IL_out
  when
    ??
  then
    ??
  end
```

  - *IL_out* denotes a car exiting the island (getting on the bridge).
  - *IL_out* ***enabled*** only when:
    - There is some car on the island.
    - The bridge's current traffic flows to the mainland.

Model m_1: BA Predicates of Multiple Actions

Consider ***actions*** of m_1's two ***new*** events:

```
IL_in
  when
    a > 0
  then
    a := a − 1
    b := b + 1
  end
```

```
IL_out
  when
    b > 0
    a = 0
  then
    b := b − 1
    c := c + 1
  end
```

- What is the ***BAP*** of *ML_in*'s ***actions***?

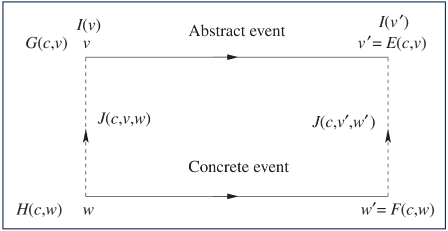
$$a' = a - 1 \land b' = b + 1 \land c' = c$$

- What is the ***BAP*** of *ML_in*'s ***actions***?

$$a' = a \land b' = b - 1 \land c' = c + 1$$

Visualizing Inv. Preservation in Refinement

- Recall how a ***concrete*** event is ***simulated*** by its ***abstract*** counterpart:



- For each ***new*** event:
  - Strictly speaking, it does <u>**not**</u> have an ***abstract*** counterpart.
  - It is ***simulated by*** a special ***abstract*** event (transforming v to v'):

```
skip
  begin

  end
```

- *skip* is a "dummy" event: <u>non</u>-guarded and does <u>nothing</u>
- <u>**Q**</u>. ***BAP*** of the skip event?

  <u>**A**</u>. $n' = n$

Refinement Rule: Invariant Preservation

- The new events *IL_in* and *IL_out* do not exist in $\mathbf{m_0}$, but:
  - They **exist** in $\mathbf{m_1}$ and may impact upon the ***concrete*** state space.
  - They ***preserve*** the ***concrete invariants***, just as *ML_out* & *ML_in* do.
- Recall the ***PO/VC Rule of Invariant Preservation for Refinement***:

$$\frac{\begin{array}{l} A(c) \\ I(c, v) \\ J(c, v, w) \\ H(c, w) \\ \vdash \\ J_i(c, E(c, v), F(c, w)) \end{array}}{} \quad \text{INV}$$

where J_i denotes a single ***concrete invariant***

  - How many ***sequents*** to be proved? [# *new* evts × # *concrete* invariants]
  - Here are two (of the ten) ***sequents*** generated:

$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a > 0 \\ \vdash \\ (a - 1) + (b + 1) + c = n \end{array} \quad \text{IL\_in/inv1\_4/INV}$$

$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a > 0 \\ \vdash \\ (a - 1) = 0 \lor c = 0 \end{array} \quad \text{IL\_in/inv1\_5/INV}$$

- **Exercises**. Specify and prove other **eight** ***POs of Invariant Preservation***.

INV PO of m_1: IL_in/inv1_4/INV

axm0_1	{	$d \in \mathbb{N}$
axm0_2	{	$d > 0$
inv0_1	{	$n \in \mathbb{N}$
inv0_2	{	$n \leq d$
inv1_1	{	$a \in \mathbb{N}$
inv1_2	{	$b \in \mathbb{N}$
inv1_3	{	$c \in \mathbb{N}$
inv1_4	{	$a + b + c = n$
inv1_5	{	$a = 0 \vee c = 0$
Guards of *IL_in*	{	$a > 0$
	$\vdash$	
Concrete invariant **inv1_4** with *IL_in*'s effect in the post-state	{	$(a - 1) + (b + 1) + c = n$

IL_in/inv1_4/INV

INV PO of m_1: IL_in/inv1_5/INV

$$
\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textit{Guards} \text{ of } \textit{IL\_in} & \{\ a > 0 \\
 & \vdash \\
\begin{array}{r}\textit{Concrete} \text{ invariant } \textbf{inv1\_5} \\ \text{with } \textit{IL\_in}\text{'s effect in the } \underline{\text{post}}\text{-state}\end{array} & \{\ (a - 1) = 0 \lor c = 0
\end{array}
$$

IL_in/inv1_5/INV

Proving Refinement: IL_in/inv1_4/INV

$$\begin{array}{|l|}\hline d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \le d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a > 0 \\ \vdash \\ (a-1) + (b+1) + c = n \\ \hline \end{array} \ \textbf{MON} \ \begin{array}{|l|}\hline a + b + c = n \\ \vdash \\ (a-1) + (b+1) + c = n \\ \hline \end{array} \ \textbf{ARI} \ \begin{array}{|l|}\hline a + b + c = n \\ \vdash \\ a + b + c = n \\ \hline \end{array} \ \textbf{HYP}$$

Proving Refinement: IL_in/inv1_5/INV

$$
\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a > 0 \\ \vdash \\ (a-1) = 0 \lor c = 0 \end{array}}
\;\textbf{MON}\;
\boxed{\begin{array}{l} a = 0 \lor c = 0 \\ a > 0 \\ \vdash \\ (a-1) = 0 \lor c = 0 \end{array}}
\;\textbf{OR\_L}\;
\left\{ \begin{array}{l}
\boxed{\begin{array}{l} a = 0 \\ a > 0 \\ \vdash \\ (a-1) = 0 \lor c = 0 \end{array}}
\;\textbf{EQ\_LR, MON}\;
\boxed{\begin{array}{l} 0 > 0 \\ \vdash \\ (0-1) = 0 \lor c = 0 \end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l} \bot \\ \vdash \\ -1 = 0 \lor c = 0 \end{array}}
\;\textbf{FALSE\_L} \\[2ex]
\boxed{\begin{array}{l} c = 0 \\ a > 0 \\ \vdash \\ (a-1) = 0 \lor c = 0 \end{array}}
\;\textbf{OR\_R2}\;
\boxed{\begin{array}{l} c = 0 \\ a > 0 \\ \vdash \\ c = 0 \end{array}}
\;\textbf{HYP}
\end{array} \right.
$$

Livelock Caused by New Events Diverging

- An alternative m_1 (with **inv1_4**, **inv1_5**, and **guards** of new events removed):

constants: d	**axioms:** **axm0_1** : $d \in \mathbb{N}$ **axm0_2** : $d > 0$	**variables:** a, b, c	**invariants:** **inv1_1** : $a \in \mathbb{Z}$ **inv1_2** : $b \in \mathbb{Z}$ **inv1_3** : $c \in \mathbb{Z}$
ML_out **when** $a + b < d$ $c = 0$ **then** $a := a + 1$ **end**	ML_in **when** $c > 0$ **then** $c := c - 1$ **end**	IL_in **begin** $a := a - 1$ $b := b + 1$ **end**	IL_out **begin** $b := b - 1$ $c := c + 1$ **end**

Concrete invariants are **under-specified**: only typing constraints.

Exercises: Show that **Invariant Preservation** is provable, but **Guard Strengthening** is not.

- Say this alternative m_1 is implemented as is:
  IL_in and *IL_out* **always** ***enabled*** and may occur ***indefinitely***, preventing other "old" events (*ML_out* and *ML_in*) from ever happening:

$$\langle init, ML\_out, IL\_in, IL\_out, IL\_in, IL\_out, \dots \rangle$$

  Q: What are the corresponding ***abstract*** transitions?

  A: $\langle init, ML\_out, skip, skip, skip, skip, \dots \rangle$ [≈ executing `while(true);`]

- We say that these two ***new*** events *diverge*, creating a *livelock*:
  - Different from a ***deadlock*** ∵ **always** an event occurring (*IL_in* or *IL_out*).
  - But their ***indefinite*** occurrences contribute **nothing** useful.

PO of Convergence of New Events

The PO/VC rule for *non-divergence/livelock freedom* consists of two parts:

- Interleaving of ***new*** events characterized as an integer expr.: ***variant***.
- A variant $V(c, w)$ may refer to constants and/or ***concrete*** variables.
- In the original m_1, let's try $\boxed{\textbf{variants} : 2 \cdot a + b}$

1. ***Variant Stays Non-Negative***

$$\frac{A(c) \quad I(c,v) \quad J(c,v,w) \quad H(c,w) \quad \vdash \quad V(c,w) \in \mathbb{N}}{}\ \text{NAT}$$

- Variant $V(c, w)$ measures how many more times the ***new*** events can occur.
- If a ***new*** event is ***enabled***, then $V(c, w) > 0$.
- When $V(c, w)$ reaches 0, some "old" events must happen s.t. $V(c, w)$ goes back above 0.

2. ***A New Event Occurrence Decreases Variant***

$$\frac{A(c) \quad I(c,v) \quad J(c,v,w) \quad H(c,w) \quad \vdash \quad V(c,F(c,w)) < V(c,w)}{}\ \text{VAR}$$

- If a ***new*** event is ***enabled*** and occurs, the value of $V(c, w) \downarrow$.

PO of Convergence of New Events: NAT

- Recall: PO related to ***Variant Stays Non-Negative***:

$$\begin{array}{l} A(c) \\ I(c,v) \\ J(c,v,w) \\ H(c,w) \\ \vdash \\ V(c,w) \in \mathbb{N} \end{array} \quad \text{NAT}$$

How many ***sequents*** to be proved?

[# ***new*** events]

- For the ***new*** event IL_in:

$$\begin{array}{lll} d \in \mathbb{N} & d > 0 & \\ n \in \mathbb{N} & n \leq d & \\ a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\ a + b + c = n & a = 0 \lor c = 0 & \\ a > 0 & & \\ \vdash & & \\ 2 \cdot a + b \in \mathbb{N} & & \end{array} \quad \text{IL\_in/NAT}$$

Exercises: Prove **IL_in/NAT** and Formulate/Prove **IL_out/NAT**.

PO of Convergence of New Events: VAR

- Recall: PO related to ***A New Event Occurrence Decreases Variant***

$$\frac{\begin{array}{l} A(c) \\ I(c,v) \\ J(c,v,w) \\ H(c,w) \\ \vdash \\ V(c,F(c,w)) < V(c,w) \end{array}}{}\ \text{VAR}$$

How many ***sequents*** to be proved?

[# ***new*** events]

- For the ***new*** event *IL_in*:

$$\begin{array}{l} \begin{array}{lll} d \in \mathbb{N} & d > 0 & \\ n \in \mathbb{N} & n \leq d & \\ a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\ a+b+c=n & a=0 \lor c=0 & \\ a > 0 & & \end{array} \\ \vdash \\ 2 \cdot (a-1) + (b+1) < 2 \cdot a + b \end{array} \quad \text{IL\_in/VAR}$$

Exercises: Prove **IL_in/VAR** and Formulate/Prove **IL_out/VAR**.

Convergence of New Events: Exercise

Given the original $\mathbf{m_1}$, what if the following ***variant*** expression is used:

$$\boxed{\textbf{variants} : a + b}$$

Are the formulated sequents still ***provable***?

PO of Refinement: Deadlock Freedom

- Recall:
  - We proved that the initial model m_0 is deadlock free (see **DLF**).
  - We proved, according to ***guard strengthening***, that if a ***concrete*** event is enabled, then its ***abstract*** counterpart is enabled.
- PO of ***relative deadlock freedom*** for a ***refinement*** model:

$$\boxed{\begin{array}{l} A(c) \\ I(c,v) \\ J(c,v,w) \\ G_1(c,v) \vee \cdots \vee G_m(c,v) \\ \vdash \\ H_1(c,w) \vee \cdots \vee H_n(c,w) \end{array}} \quad \underline{\text{DLF}}$$

If an ***abstract*** state does not *deadlock* (i.e., $G_1(c,v) \vee \cdots \vee G_m(c,v)$), then its ***concrete*** counterpart does not *deadlock* (i.e., $H_1(c,w) \vee \cdots \vee H_n(c,w)$).

- Another way to think of the above PO:

  The ***refinement*** does **not** introduce, in the ***concrete***, any "new" *deadlock* scenarios **not** existing in the ***abstract*** state.

PO Rule: Relative Deadlock Freedom m_1

$$
\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\text{Disjunction of } \textit{abstract} \text{ guards} & \left\{ \begin{array}{rrl} & n < d & \} \ \textbf{guards of } \textit{ML\_out} \textbf{ in } m_0 \\ \lor & n > 0 & \} \ \textbf{guards of } \textit{ML\_in} \textbf{ in } m_0 \end{array} \right. \\
& \vdash \\
\text{Disjunction of } \textit{concrete} \text{ guards} & \left\{ \begin{array}{rrl} & a + b < d \land c = 0 & \} \ \textbf{guards of } \textit{ML\_out} \textbf{ in } m_1 \\ \lor & c > 0 & \} \ \textbf{guards of } \textit{ML\_in} \textbf{ in } m_1 \\ \lor & a > 0 & \} \ \textbf{guards of } \textit{IL\_in} \textbf{ in } m_1 \\ \lor & b > 0 \land a = 0 & \} \ \textbf{guards of } \textit{IL\_out} \textbf{ in } m_1 \end{array} \right.
\end{array}
\qquad \textbf{DLF}
$$

Example Inference Rules (6)

$$\frac{H, \neg P \vdash Q}{H \vdash P \vee Q} \quad \textbf{OR\_R}$$

To prove a ***disjunctive goal***,
it suffices to prove one of the disjuncts,
with the the negation of the the other disjunct
serving as an additional hypothesis.

$$\frac{H, P, Q \vdash R}{H, P \wedge Q \vdash R} \quad \textbf{AND\_L}$$

To prove a goal with a ***conjunctive hypothesis***,
it suffices to prove the same goal,
with the the two conjuncts
serving as two separate hypotheses.

$$\frac{H \vdash P \qquad H \vdash Q}{H \vdash P \wedge Q} \quad \textbf{AND\_R}$$

To prove a goal with a ***conjunctive goal***,
it suffices to prove each conjunct
as a separate goal.

Proving Refinement: DLF of m_1

$$
\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
n < d \lor n > 0 \\
\vdash \\
\quad\ \ a + b < d \land c = 0 \\
\lor \ \ c > 0 \\
\lor \ \ a > 0 \\
\lor \ \ b > 0 \land a = 0
\end{array}
$$

MON

$$
\begin{array}{l}
d > 0 \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
\vdash \\
\quad\ \ a + b < d \land c = 0 \\
\lor \ \ c > 0 \\
\lor \ \ a > 0 \\
\lor \ \ b > 0 \land a = 0
\end{array}
$$

OR_R, ARI

$$
\begin{array}{l}
d > 0 \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c = 0 \\
\vdash \\
\quad\ \ a + b < d \land c = 0 \\
\lor \ \ c > 0 \\
\lor \ \ a > 0 \\
\lor \ \ b > 0 \land a = 0
\end{array}
$$

EQ_LR, MON

$$
\begin{array}{l}
d > 0 \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
\vdash \\
\quad\ \ a + b < d \land 0 = 0 \\
\lor \ \ 0 > 0 \\
\lor \ \ a > 0 \\
\lor \ \ b > 0 \land a = 0
\end{array}
$$

OR_R, ARI

$$
\begin{array}{l}
d > 0 \\
a = 0 \\
b \in \mathbb{N} \\
\vdash \\
\quad\ \ a + b < d \land 0 = 0 \\
\lor \ \ b > 0 \land a = 0
\end{array}
$$

EQ_LR, MON

$$
\begin{array}{l}
d > 0 \\
b \in \mathbb{N} \\
\vdash \\
\quad\ \ 0 + b < d \land 0 = 0 \\
\lor \ \ b > 0 \land 0 = 0
\end{array}
$$

ARI

$$
\begin{array}{l}
d > 0 \\
b = 0 \lor b > 0 \\
\vdash \\
\quad\ \ b < d \land 0 = 0 \\
\lor \ \ b > 0 \land 0 = 0
\end{array}
$$

...

Proving Refinement: DLF of m_1 (continued)

$$\begin{array}{l} d > 0 \\ b = 0 \lor b > 0 \\ \vdash \\ \quad b < d \land 0 = 0 \\ \lor \quad b > 0 \land 0 = 0 \end{array}$$

OR_L

$$\begin{array}{l} d > 0 \\ b = 0 \\ \vdash \\ \quad b < d \land 0 = 0 \\ \lor \quad b > 0 \land 0 = 0 \end{array}$$

OR_R1

$$\begin{array}{l} d > 0 \\ b = 0 \\ \vdash \\ b < d \land 0 = 0 \end{array}$$

OR_R1, MON

$$\begin{array}{l} d > 0 \\ \vdash \\ 0 < d \land 0 = 0 \end{array}$$

AND_R

$$\begin{array}{l} d > 0 \\ \vdash \\ 0 < d \end{array}$$

ARI, HYP

$$\begin{array}{l} d > 0 \\ \vdash \\ 0 = 0 \end{array}$$

EQ

$$\begin{array}{l} d > 0 \\ b > 0 \\ \vdash \\ \quad b < d \land 0 = 0 \\ \lor \quad b > 0 \land 0 = 0 \end{array}$$

OR_R2

$$\begin{array}{l} d > 0 \\ b > 0 \\ \vdash \\ b > 0 \land 0 = 0 \end{array}$$

AND_R

$$\begin{array}{l} d > 0 \\ b > 0 \\ \vdash \\ b > 0 \end{array}$$

HYP

$$\begin{array}{l} d > 0 \\ b > 0 \\ \vdash \\ 0 = 0 \end{array}$$

EQ

First Refinement: Summary

- The final version of our ***first refinement*** m_1 is ***provably correct*** w.r.t.:
  - Establishment of ***Concrete Invariants*** [*init*]
  - Preservation of ***Concrete Invariants*** [old & new events]
  - Strengthening of ***guards*** [old events]
  - ***Convergence*** (a.k.a. livelock freedom, non-divergence) [new events]
  - Relative ***Deadlock*** Freedom
- Here is the final specification of m_1:

constants: d

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$

variables: a, b, c

invariants:
- **inv1_1** : $a \in \mathbb{N}$
- **inv1_2** : $b \in \mathbb{N}$
- **inv1_3** : $c \in \mathbb{N}$
- **inv1_4** : $a + b + c = n$
- **inv1_5** : $a = 0 \vee c = 0$

variants:
$2 \cdot a + b$

```
init
  begin
    a := 0
    b := 0
    c := 0
  end
```

```
ML_out
  when
    a + b < d
    c = 0
  then
    a := a + 1
  end
```

```
ML_in
  when
    c > 0
  then
    c := c - 1
  end
```

```
IL_in
  when
    a > 0
  then
    a := a - 1
    b := b + 1
  end
```

```
IL_out
  when
    b > 0
    a = 0
  then
    b := b - 1
    c := c + 1
  end
```

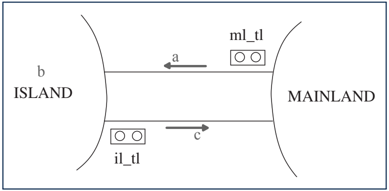
Model m_2: "More Concrete" Abstraction

- 2nd ***refinement*** has even more ***concrete*** perception of the bridge controller:
  - We "**zoom in**" by observing the system from **even closer to the ground**, so that the one-way traffic of the bridge is controlled via:

ml_tl: a traffic light for exiting the ML

il_tl: a traffic light for exiting the IL

abstract variables ***a***, ***b***, ***c*** from m_1 still used (instead of being replaced)



  - Nonetheless, sensors remain ***abstracted*** away!
- That is, we focus on these three ***environment constraints***:

ENV1	The system is equipped with two traffic lights with two colors: green and red.
ENV2	The traffic lights control the entrance to the bridge at both ends of it.
ENV3	Cars are not supposed to pass on a red traffic light, only on a green one.

- We are ***obliged to prove*** this ***added concreteness*** is ***consistent*** with m_1.

Model m_2: Refined, Concrete State Space

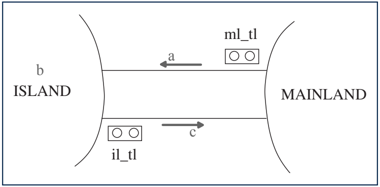
1. The **static** part introduces the notion of traffic light colours:

sets: $COLOR$

constants: $red, green$

axioms:
axm2_1 : $COLOR = \{green, red\}$
axm2_2 : $green \neq red$

2. The **dynamic** part shows the ***superposition refinement*** scheme:



- ***Abstract*** variables ***a***, ***b***, ***c*** from m_1 are still in use in ***m_2***.
- Two new, ***concrete*** variables are introduced: ***ml_tl*** and ***il_tl***
- **Constrast**: In m_1, ***abstract*** variable n is replaced by ***concrete*** variables a, b, c.

variables:
a, b, c
ml_tl
il_tl

invariants:
inv2_1 : $ml\_tl \in COLOUR$
inv2_2 : $il\_tl \in COLOUR$
inv2_3 : ??
inv2_4 : ??

- ◇ **inv2_1** & **inv2_2**: typing constraints
- ◇ **inv2_3**: being allowed to exit ML **means** cars within limit and no opposite traffic
- ◇ **inv2_4**: being allowed to exit IL **means** some car in IL and no opposite traffic

Model m_2: Refining Old, Abstract Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- ***Concrete/Refined*** version of ***event*** *ML_out*:

```
ML_out
  when
    ??
  then
    a := a + 1
  end
```

  - Recall the ***abstract*** guard of *ML_out* in m_1: $(c = 0) \land (a + b < d)$

    $\Rightarrow$ Unrealistic as drivers should **not** know about a, b, c!
  - *ML_out* is ***refined***: a car exits the ML (to the bridge) only when:
    - the traffic light *ml_tl* allows
- ***Concrete/Refined*** version of ***event*** *IL_out*:

```
IL_out
  when
    ??
  then
    b := b − 1
    c := c + 1
  end
```

  - Recall the ***abstract*** guard of *IL_out* in m_1: $(a = 0) \land (b > 0)$

    $\Rightarrow$ Unrealistic as drivers should **not** know about a, b, c!
  - *IL_out* is ***refined***: a car exits the IL (to the bridge) only when:
    - the traffic light *il_tl* allows

Q1. How about the other two "old" ***events*** *IL_in* and *ML_in*?
A1. No need to ***refine*** as already ***guarded*** by *ML_out* and *IL_out*.
Q2. What if the driver disobeys *ml_tl* or *il_tl*? [**A2**. **ENV3**]

Model m_2: New, Concrete Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)** : it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- Considered ***events*** already existing in m_1:
  - *ML_out* & *IL_out* [REFINED]
  - *IL_in* & *ML_in* [UNCHANGED]
- ***New event*** *ML_tl_green*:

```
ML_tl_green
  when
    ??
  then
    ml_tl := green
  end
```

  - *ML_tl_green* denotes the traffic light *ml_tl* turning green.
  - *ML_tl_green* ***enabled*** only when:
    - the traffic light not already green
    - limited number of cars on the bridge and the island
    - No opposite traffic

  [⇒ *ML_out*'s ***abstract*** guard in m_1]

- ***New event*** *IL_tl_green*:

```
IL_tl_green
  when
    ??
  then
    il_tl := green
  end
```

  - *IL_tl_green* denotes the traffic light *il_tl* turning green.
  - *IL_tl_green* ***enabled*** only when:
    - the traffic light not already green
    - some cars on the island (i.e., island not empty)
    - No opposite traffic

  [⇒ *IL_out*'s ***abstract*** guard in m_1]

Invariant Preservation in Refinement m_2

constants: d

sets: $COLOR$

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$
- **axm2_1** : $COLOR = \{green, red\}$
- **axm2_2** : $green \neq red$

variables: a, b, c, ml_tl, il_tl

invariants:
- **inv2_1** : $ml\_tl \in COLOUR$
- **inv2_2** : $il\_tl \in COLOUR$
- **inv2_3** : $ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
- **inv2_4** : $il\_tl = green \Rightarrow b > 0 \wedge a = 0$

ML_tl_green
when $ml\_tl = red$, $a + b < d$, $c = 0$
then $ml\_tl := green$
end

ML_out
when $ml\_tl = green$
then $a := a + 1$
end

IL_in
when $a > 0$
then $a := a - 1$, $b := b + 1$
end

IL_tl_green
when $il\_tl = red$, $b > 0$, $a = 0$
then $il\_tl := green$
end

IL_out
when $il\_tl = green$
then $b := b - 1$, $c := c + 1$
end

ML_in
when $c > 0$
then $c := c - 1$
end

Recall the ***PO/VC Rule of Invariant Preservation for Refinement***:

$$\frac{A(c) \quad I(c, v) \quad J(c, v, w) \quad H(c, w) \quad \vdash \quad J_i(c, E(c, v), F(c, w))}{} \;\text{INV}$$

where J_i denotes a single ***concrete invariant***

- How many ***sequents*** to be proved? [# *concrete* evts × # *concrete* invariants = 6 × 4]
- We discuss two sequents: *ML_out*/**inv2_4**/*INV* and *IL_out*/**inv2_3**/*INV*

Exercises. Specify and prove (some of) other twenty-two ***POs of Invariant Preservation***.

INV PO of m_2: ML_out/inv2_4/INV

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
axm2_1	$\{\ COLOUR = \{green, red\}$
axm2_2	$\{\ green \neq red$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
inv2_1	$\{\ ml\_tl \in COLOUR$
inv2_2	$\{\ il\_tl \in COLOUR$
inv2_3	$\{\ ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$\{\ il\_tl = green \Rightarrow b > 0 \land a = 0$
Concrete guards of *ML_out*	$\{\ ml\_tl = green$
	$\vdash$
Concrete invariant **inv2_4** with *ML_out*'s effect in the post-state	$\{\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0$

ML_out/inv2_4/INV

INV PO of m_2: IL_out/inv2_3/INV

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
axm2_1	$\{\ COLOUR = \{green, red\}$
axm2_2	$\{\ green \neq red$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
inv2_1	$\{\ ml\_tl \in COLOUR$
inv2_2	$\{\ il\_tl \in COLOUR$
inv2_3	$\{\ ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$\{\ il\_tl = green \Rightarrow b > 0 \land a = 0$
Concrete guards of *IL_out*	$\{\ il\_tl = green$
	$\vdash$
Concrete invariant **inv2_3** with *ML_out*'s effect in the post-state	$\{\ ml\_tl = green \Rightarrow a + (b - 1) < d \land (c + 1) = 0$

IL_out/inv2_3/INV

Example Inference Rules (7)

$$\frac{H, P, Q \vdash R}{H, P, P \Rightarrow Q \vdash R} \quad \textbf{IMP\_L}$$

If a hypothesis ***P*** matches the assumption of another ***implicative hypothesis P ⇒ Q***, then the conclusion ***Q*** of the ***implicative hypothesis*** can be used as a new hypothesis for the sequent.

$$\frac{H, P \vdash Q}{H \vdash P \Rightarrow Q} \quad \textbf{IMP\_R}$$

To prove an ***implicative goal P ⇒ Q***, it suffices to prove its conclusion ***Q***, with its assumption ***P*** serving as a new hypotheses.

$$\frac{H, \neg Q \vdash P}{H, \neg P \vdash Q} \quad \textbf{NOT\_L}$$

To prove a goal ***Q*** with a ***negative hypothesis ¬ P***, it suffices to prove the negated hypothesis $\neg(\neg P) \equiv$ ***P*** with the negated original goal ¬ ***Q*** serving as a new hypothesis.

Proving ML_out/inv2_4/INV: First Attempt

$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ COLOUR = \{green, red\} \\ green \neq red \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ ml\_tl \in COLOUR \\ il\_tl \in COLOUR \\ ml\_tl = green \Rightarrow a + b < d \land c = 0 \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ \vdash \\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0 \end{array}$$

MON

$$\begin{array}{l} green \neq red \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ \vdash \\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0 \end{array}$$

IMP_R

$$\begin{array}{l} green \neq red \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

IMP_L

$$\begin{array}{l} green \neq red \\ b > 0 \land a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

AND_L

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

AND_R

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \end{array}$$

HYP

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ (a + 1) = 0 \end{array}$$

EQ_LR, MON

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ (0 + 1) = 0 \end{array}$$

ARI

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ 1 = 0 \end{array}$$

??

Proving IL_out/inv2_3/INV: First Attempt

$$
\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
COLOUR = \{green, red\} \\
green \neq red \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
ml\_tl \in COLOUR \\
il\_tl \in COLOUR \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
il\_tl = green \Rightarrow b > 0 \land a = 0 \\
il\_tl = green \\
\vdash \\
ml\_tl = green \Rightarrow a + (b - 1) < d \land (c + 1) = 0
\end{array}
$$

MON

$$
\begin{array}{l}
green \neq red \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
il\_tl = green \\
\vdash \\
ml\_tl = green \Rightarrow a + (b - 1) < d \land (c + 1) = 0
\end{array}
$$

IMP_R

$$
\begin{array}{l}
green \neq red \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
a + (b - 1) < d \land (c + 1) = 0
\end{array}
$$

IMP_L

$$
\begin{array}{l}
green \neq red \\
a + b < d \land c = 0 \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
a + (b - 1) < d \land (c + 1) = 0
\end{array}
$$

AND_L

$$
\begin{array}{l}
green \neq red \\
a + b < d \\
c = 0 \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
a + (b - 1) < d \land (c + 1) = 0
\end{array}
$$

AND_R

$$
\begin{array}{l}
green \neq red \\
a + b < d \\
c = 0 \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
a + (b - 1) < d
\end{array}
$$

MON

$$
\begin{array}{l}
a + b < d \\
\vdash \\
a + (b - 1) < d
\end{array}
$$

ARI

$$
\begin{array}{l}
green \neq red \\
a + b < d \\
c = 0 \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
(c + 1) = 0
\end{array}
$$

EQ_LR, MON

$$
\begin{array}{l}
green \neq red \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
(0 + 1) = 0
\end{array}
$$

ARI

$$
\begin{array}{l}
green \neq red \\
il\_tl = green \\
ml\_tl = green \\
\vdash \\
1 = 0
\end{array}
$$

??

Failed: ML_out/inv2_4/INV, IL_out/inv2_3/INV

- Our first attempts of proving ***ML_out/inv2_4/INV*** and ***IL_out/inv2_3/INV*** both failed the 2nd case (resulted from applying IR **AND_R**):

$$\boxed{green \neq red \land il\_tl = green \land ml\_tl = green \;\vdash\; 1 = 0}$$

- This ***unprovable*** sequent gave us a good hint:
  - Goal $\boxed{1 = 0 \equiv \textbf{false}}$ suggests that the ***safety requirements*** $a = 0$ (for **inv2_4**) and $c = 0$ (for **inv2_3**) ***contradict*** with the current m_2.
  - Hyp. $\boxed{il\_tl = green = ml\_tl}$ suggests a ***possible, dangerous state*** of m_2, where two cars heading different directions are on the one-way bridge:

⟨ *init*	, *ML_tl_green*	, *ML_out*	, *IL_in*	, *IL_tl_green*	, *IL_out*	, *ML_out* ⟩
$d = 2$	$d = 2$	$d = 2$	$d = 2$	$d = 2$	$d = 2$	$d = 2$
$a' = 0$	$a' = 0$	**a' = 1**	**a' = 0**	$a' = 0$	$a' = 0$	**a' = 1**
$b' = 0$	$b' = 0$	$b' = 0$	**b' = 1**	$b' = 1$	**b' = 0**	$b' = 0$
$c' = 0$	$c' = 0$	$c' = 0$	$c' = 0$	$c' = 0$	**c' = 1**	$c' = 1$
$ml\_tl' = red$	**ml_tl' = green**	$ml\_tl' = green$	$ml\_tl' = green$	$ml\_tl' = green$	$ml\_tl' = green$	$ml\_tl' = green$
$il\_tl' = red$	$il\_tl' = red$	$il\_tl' = red$	$il\_tl' = red$	**il_tl' = green**	$il\_tl' = green$	$il\_tl' = green$

Fixing m_2: Adding an Invariant

- Having understood the failed proofs, we add a proper ***invariant*** to m_2:

invariants:

$\ldots$

$\textbf{inv2\_5} : ml\_tl = red \lor il\_tl = red$

- We have effectively resulted in an improved m_2 more faithful w.r.t. **REQ3**:

REQ3	The bridge is one-way or the other, not both at the same time.

- Having added this new invariant ***inv2_5***:
  - Original 6×4 generated sequents to be updated: **inv2_5** a new hypothesis e.g., Are ***ML_out/inv2_4/INV*** and ***IL_out/inv2_3/INV*** now ***provable***?
  - Additional 6×1 sequents to be generated due to this new invariant e.g., Are ***ML_tl_green/inv2_5/INV*** and ***IL_tl_green/inv2_5/INV*** ***provable***?

INV PO of m_2: ML_out/inv2_4/INV – Updated

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
axm2_1	$\{\ COLOUR = \{green, red\}$
axm2_2	$\{\ green \neq red$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
inv2_1	$\{\ ml\_tl \in COLOUR$
inv2_2	$\{\ il\_tl \in COLOUR$
inv2_3	$\{\ ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$\{\ il\_tl = green \Rightarrow b > 0 \land a = 0$
inv2_5	$\{\ ml\_tl = red \lor il\_tl = red$
Concrete guards of *ML_out*	$\{\ ml\_tl = green$
	$\vdash$
Concrete invariant **inv2_4** with *ML_out*'s effect in the post-state	$\{\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0$

ML_out/inv2_4/INV

INV PO of m_2: IL_out/inv2_3/INV – Updated

axm0_1	$\{ d \in \mathbb{N}$
axm0_2	$\{ d > 0$
axm2_1	$\{ COLOUR = \{green, red\}$
axm2_2	$\{ green \neq red$
inv0_1	$\{ n \in \mathbb{N}$
inv0_2	$\{ n \leq d$
inv1_1	$\{ a \in \mathbb{N}$
inv1_2	$\{ b \in \mathbb{N}$
inv1_3	$\{ c \in \mathbb{N}$
inv1_4	$\{ a + b + c = n$
inv1_5	$\{ a = 0 \lor c = 0$
inv2_1	$\{ ml\_tl \in COLOUR$
inv2_2	$\{ il\_tl \in COLOUR$
inv2_3	$\{ ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$\{ il\_tl = green \Rightarrow b > 0 \land a = 0$
inv2_5	$\{ ml\_tl = red \lor il\_tl = red$
Concrete guards of *IL_out*	$\{ il\_tl = green$
	$\vdash$
Concrete invariant **inv2_3** with *ML_out*'s effect in the post-state	$\{ ml\_tl = green \Rightarrow a + (b-1) < d \land (c+1) = 0$

IL_out/inv2_3/INV

Proving ML_out/inv2_4/INV: Second Attempt

$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ COLOUR = \{green, red\} \\ green \neq red \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ ml\_tl \in COLOUR \\ il\_tl \in COLOUR \\ ml\_tl = green \Rightarrow a + b < d \land c = 0 \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = red \lor il\_tl = red \\ ml\_tl = green \\ \vdash \\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0 \end{array}$$

MON

$$\begin{array}{l} green \neq red \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = red \lor il\_tl = red \\ ml\_tl = green \\ \vdash \\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0 \end{array}$$

IMP_R

$$\begin{array}{l} green \neq red \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

IMP_L

$$\begin{array}{l} green \neq red \\ b > 0 \land a = 0 \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

AND_L

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

AND_R

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ b > 0 \end{array}$$

HYP

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ (a + 1) = 0 \end{array}$$

EQ_LR, MON

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ (0 + 1) = 0 \end{array}$$

ARI

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ ml\_tl = red \lor il\_tl = red \\ il\_tl = green \\ \vdash \\ 1 = 0 \end{array}$$

OR_L

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ ml\_tl = red \\ il\_tl = green \\ \vdash \\ 1 = 0 \end{array}$$

EQ_LR, MON

$$\begin{array}{l} green \neq red \\ green = red \\ il\_tl = green \\ \vdash \\ 1 = 0 \end{array}$$

NOT_L

$$\begin{array}{l} green = red \\ il\_tl = green \\ 1 \neq 0 \\ \vdash \\ green = red \end{array}$$

HYP

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ il\_tl = red \\ il\_tl = green \\ \vdash \\ 1 = 0 \end{array}$$

EQ_LR, MON

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ red = green \\ \vdash \\ 1 = 0 \end{array}$$

NOT_L

$$\begin{array}{l} ml\_tl = green \\ red = green \\ 1 \neq 0 \\ \vdash \\ green = red \end{array}$$

HYP

Proving IL_out/inv2_3/INV: Second Attempt

$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow a + (b-1) < d \land (c+1) = 0$

MON

$green \neq red$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow a + (b-1) < d \land (c+1) = 0$

IMP_R

$green \neq red$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b-1) < d \land (c+1) = 0$

IMP_L

$green \neq red$
$a + b < d \land c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b-1) < d \land (c+1) = 0$

AND_L

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b-1) < d \land (c+1) = 0$

AND_R

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b-1) < d$

MON

$a + b < d$
$\vdash$
$a + (b-1) < d$

ARI

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$(c+1) = 0$

EQ_LR, MON

$green \neq red$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$(0 + 1) = 0$

ARI

$green \neq red$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

OR_L

$green \neq red$
$il\_tl = green$
$ml\_tl = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

EQ_LR, MON

$green \neq red$
$il\_tl = green$
$red = green$
$\vdash$
$1 = 0$

NOT_L

$il\_tl = green$
$red = green$
$1 \neq 0$
$\vdash$
$green = red$

HYP

$green \neq red$
$il\_tl = green$
$il\_tl = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

EQ_LR, MON

$green \neq red$
$green = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

NOT_L

$green = red$
$ml\_tl = green$
$1 \neq 0$
$\vdash$
$green = red$

HYP

Fixing m_2: Adding Actions

- Recall that an ***invariant*** was added to m_2:

> **invariants:**
> **inv2_5** : $ml\_tl = red \vee il\_tl = red$

- Additional 6×1 sequents to be generated due to this new invariant:
  - e.g., ***ML_tl_green*/inv2_5/INV** [for ***ML_tl_green*** to preserve **inv2_5**]
  - e.g., ***IL_tl_green*/inv2_5/INV** [for ***IL_tl_green*** to preserve **inv2_5**]
- For the above ***sequents*** to be ***provable***, we need to revise the two events:

```
ML_tl_green
  when
    ml_tl = red
    a + b < d
    c = 0
  then
    ml_tl := green
    il_tl := red
  end
```

```
IL_tl_green
  when
    il_tl = red
    b > 0
    a = 0
  then
    il_tl := green
    ml_tl := red
  end
```

Exercise: Specify and prove ***ML_tl_green*/inv2_5/INV** & ***IL_tl_green*/inv2_5/INV**.

INV PO of m_2: ML_out/inv2_3/INV

$$
\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{axm2\_1} & \{\ COLOUR = \{green, red\} \\
\textbf{axm2\_2} & \{\ green \neq red \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textbf{inv2\_1} & \{\ ml\_tl \in COLOUR \\
\textbf{inv2\_2} & \{\ il\_tl \in COLOUR \\
\textbf{inv2\_3} & \{\ ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
\textbf{inv2\_4} & \{\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\
\textbf{inv2\_5} & \{\ ml\_tl = red \lor il\_tl = red \\
\textit{Concrete}\text{ guards of } ML\_out & \{\ ml\_tl = green \\
 & \vdash \\
\textit{Concrete}\text{ invariant } \textbf{inv2\_3} \text{ with } ML\_out\text{'s effect in the post-state} & \{\ ml\_tl = green \Rightarrow (a + 1) + b < d \land c = 0
\end{array}
$$

ML_out/inv2_3/INV

Proving ML_out/inv2_3/INV: First Attempt

$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow (a + 1) + b < d \land c = 0$

MON

$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$\vdash$
$ml\_tl = green \Rightarrow (a + 1) + b < d \land c = 0$

IMP_R

$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d \land c = 0$

IMP_R

$a + b < d \land c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d \land c = 0$

AND_L

$a + b < d$
$c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d \land c = 0$

AND_R

$a + b < d$
$c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d$

??

$a + b < d$
$c = 0$
$ml\_tl = green$
$\vdash$
$c = 0$

HYP

Failed: ML_out/inv2_3/INV

- Our first attempt of proving ***ML_out/inv2_3/INV*** failed the 1st case (resulted from applying IR **AND_R**):

$$\boxed{a + b < d \land c = 0 \land ml\_tl = green \vdash (a+1) + b < d}$$

- This ***unprovable*** sequent gave us a good hint:
  - Goal $\underbrace{(a+1)}_{a'} + \underbrace{b}_{b'} < d$ specifies the ***capacity requirement***.
  - Hypothesis $\boxed{c = 0 \land ml\_tl = green}$ assumes that it's safe to exit the ML.
  - Hypothesis $\boxed{a + b < d}$ is **not** strong enough to entail $(a+1) + b < d$.

    e.g., $d = 3$, $b = 0$, $a = 0$ [$(a+1) + b < d$ evaluates to ***true***]

    e.g., $d = 3$, $b = 1$, $a = 0$ [$(a+1) + b < d$ evaluates to ***true***]

    e.g., $d = 3$, $b = 0$, $a = 1$ [$(a+1) + b < d$ evaluates to ***true***]

    e.g., $d = 3$, $b = 0$, $a = 2$ [$(a+1) + b < d$ evaluates to ***false***]

    e.g., $d = 3$, $b = 1$, $a = 1$ [$(a+1) + b < d$ evaluates to ***false***]

    e.g., $d = 3$, $b = 2$, $a = 0$ [$(a+1) + b < d$ evaluates to ***false***]
  - Therefore, $a + b < d$ (allowing one more car to exit ML) should be split:

    $a + b + 1 \neq d$ [more later cars may exit ML, ***ml_tl*** remains ***green***]

    $a + b + 1 = d$ [no more later cars may exit ML, ***ml_tl*** turns ***red***]

Fixing m_2: Splitting *ML_out* and *IL_out*

- Recall that ***ML_out/inv2_3/INV*** failed $\because$ two cases not handled separately:

  $a + b + 1 \neq d$ [more later cars may exit ML, ***ml_tl*** remains ***green***]

  $a + b + 1 = d$ [no more later cars may exit ML, ***ml_tl*** turns ***red***]

- Similarly, ***IL_out/inv2_4/INV*** would fail $\because$ two cases not handled separately:

  $b - 1 \neq 0$ [more later cars may exit IL, ***il_tl*** remains ***green***]

  $b - 1 = 0$ [no more later cars may exit IL, ***il_tl*** turns ***red***]

- Accordingly, we split ***ML_out*** and ***IL_out*** into two with corresponding guards.

```
ML_out_1
  when
    ml_tl = green
    a + b + 1 ≠ d
  then
    a := a + 1
  end
```

```
ML_out_2
  when
    ml_tl = green
    a + b + 1 = d
  then
    a := a + 1
    ml_tl := red
  end
```

```
IL_out_1
  when
    il_tl = green
    b ≠ 1
  then
    b := b − 1
    c := c + 1
  end
```

```
IL_out_2
  when
    il_tl = green
    b = 1
  then
    b := b − 1
    c := c + 1
    il_tl := red
  end
```

Exercise: Given the latest m_2, how many sequents to prove for ***invariant preservation***?
Exercise: Specify and prove ***ML_out_i*/inv2_3/INV** & ***IL_out_i*/inv2_4/INV** (where $i \in 1 .. 2$).
Exercise: Each split event (e.g., *ML_out_1*) refines its ***abstract*** counterpart (e.g., *ML_out*)?

m_2 Livelocks: New Events Diverging

- Recall that a system may ***livelock*** if the new events diverge.
- Current m_2's two new events ***ML_tl_green*** and ***IL_tl_green*** may *diverge*:

ML_tl_green	IL_tl_green
ML_tl_green **when** $ml\_tl = red$ $a + b < d$ $c = 0$ **then** $ml\_tl := green$ $il\_tl := red$ **end**	IL_tl_green **when** $il\_tl = red$ $b > 0$ $a = 0$ **then** $il\_tl := green$ $ml\_tl := red$ **end**

- *ML_tl_green* and *IL_tl_green* both ***enabled*** and may occur ***indefinitely***, preventing other "old" events (e.g., *ML_out*) from ever happening:

$$\langle \underbrace{init}_{\begin{array}{c} d=2 \\ a'=0 \\ b'=0 \\ c'=0 \\ ml\_tl = red \\ il\_tl = red \end{array}},\ \underbrace{ML\_tl\_green}_{\begin{array}{c} d=2 \\ a'=0 \\ b'=0 \\ c'=0 \\ ml\_tl' = green \\ il\_tl' = red \end{array}},\ \underbrace{ML\_out\_1}_{\begin{array}{c} d=2 \\ a'=1 \\ b'=0 \\ c'=0 \\ ml\_tl' = green \\ il\_tl' = red \end{array}},\ \underbrace{IL\_in}_{\begin{array}{c} d=2 \\ a'=0 \\ b'=1 \\ c'=0 \\ ml\_tl' = green \\ il\_tl' = red \end{array}},\ \underbrace{IL\_tl\_green}_{\begin{array}{c} d=2 \\ a'=0 \\ b'=1 \\ c'=0 \\ ml\_tl' = red \\ il\_tl' = green \end{array}},\ \underbrace{ML\_tl\_green}_{\begin{array}{c} d=2 \\ a'=0 \\ b'=1 \\ c'=0 \\ ml\_tl' = green \\ il\_tl' = red \end{array}},\ \underbrace{IL\_tl\_green}_{\begin{array}{c} d=2 \\ a'=0 \\ b'=1 \\ c'=0 \\ ml\_tl' = red \\ il\_tl' = green \end{array}},\ \dots \rangle$$

$\Rightarrow$ Two traffic lights keep changing colors so rapidly that **no** drivers can ever pass!

- **Solution**: Allow color changes between traffic lights in a disciplined way.

Fixing m_2: Regulating Traffic Light Changes

We introduce two variables/flags for regulating traffic light changes:

- ***ml_pass*** is ***1*** **if**, since ***ml_tl*** was last turned ***green***, at least one car exited the ML onto the bridge. Otherwise, ***ml_pass*** is ***0***.
- ***il_pass*** is ***1*** **if**, since ***il_tl*** was last turned ***green***, at least one car exited the IL onto the bridge. Otherwise, ***il_pass*** is ***0***.

variables: ml_pass, il_pass

invariants:
inv2_6 : $ml\_pass \in \{0, 1\}$
inv2_7 : $il\_pass \in \{0, 1\}$
inv2_8 : $ml\_tl = red \Rightarrow ml\_pass = 1$
inv2_9 : $il\_tl = red \Rightarrow il\_pass = 1$

```
ML_out_1
  when
    ml_tl = green
    a + b + 1 ≠ d
  then
    a := a + 1
    ml_pass := 1
  end
```

```
ML_out_2
  when
    ml_tl = green
    a + b + 1 = d
  then
    a := a + 1
    ml_tl := red
    ml_pass := 1
  end
```

```
IL_out_1
  when
    il_tl = green
    b ≠ 1
  then
    b := b − 1
    c := c + 1
    il_pass := 1
  end
```

```
IL_out_2
  when
    il_tl = green
    b = 1
  then
    b := b − 1
    c := c + 1
    il_tl := red
    il_pass := 1
  end
```

```
ML_tl_green
  when
    ml_tl = red
    a + b < d
    c = 0
    il_pass = 1
  then
    ml_tl := green
    il_tl := red
    ml_pass := 0
  end
```

```
IL_tl_green
  when
    il_tl = red
    b > 0
    a = 0
    ml_pass = 1
  then
    il_tl := green
    ml_tl := red
    il_pass := 0
  end
```

Fixing m_2: Measuring Traffic Light Changes

- Recall:
  - Interleaving of ***new*** events charactered as an integer expression: ***variant***.
  - A variant $V(c, w)$ may refer to constants and/or ***concrete*** variables.
  - In the latest m_2, let's try $\boxed{\textbf{variants} : ml\_pass + il\_pass}$
- Accordingly, for the ***new*** event *ML_tl_green*:

$$
\boxed{
\begin{array}{lll}
d \in \mathbb{N} & d > 0 & \\
COLOUR = \{green, red\} & green \neq red & \\
n \in \mathbb{N} & n \leq d & \\
a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\
a + b + c = n & a = 0 \lor c = 0 & \\
ml\_tl \in COLOUR & il\_tl \in COLOUR & \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 & il\_tl = green \Rightarrow b > 0 \land a = 0 & \\
ml\_tl = red \lor il\_tl = red & & \\
ml\_pass \in \{0, 1\} & il\_pass \in \{0, 1\} & \\
ml\_tl = red \Rightarrow ml\_pass = 1 & il\_tl = red \Rightarrow il\_pass = 1 & \\
ml\_tl = red & a + b < d & c = 0 \\
il\_pass = 1 & & \\
\vdash & & \\
0 + il\_pass < ml\_pass + il\_pass & &
\end{array}
}
$$

ML_tl_green/VAR

Exercises: Prove **ML_tl_green/VAR** and Formulate/Prove **IL_tl_green/NAT**.

PO Rule: Relative Deadlock Freedom of m_2

$$
\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{axm2\_1} & \{\ COLOUR = \{green, red\} \\
\textbf{axm2\_2} & \{\ green \neq red \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textbf{inv2\_1} & \{\ ml\_tl \in COLOUR \\
\textbf{inv2\_2} & \{\ il\_tl \in COLOUR \\
\textbf{inv2\_3} & \{\ ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
\textbf{inv2\_4} & \{\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\
\textbf{inv2\_5} & \{\ ml\_tl = red \lor il\_tl = red \\
\textbf{inv2\_6} & \{\ ml\_pass \in \{0, 1\} \\
\textbf{inv2\_7} & \{\ il\_pass \in \{0, 1\} \\
\textbf{inv2\_8} & \{\ ml\_tl = red \Rightarrow ml\_pass = 1 \\
\textbf{inv2\_9} & \{\ il\_tl = red \Rightarrow il\_pass = 1 \\
\text{Disjunction of } \textit{abstract} \text{ guards} & \left\{\begin{array}{rrl}
 & a + b < d \land c = 0 & \} \ \textbf{guards of } ML\_out \textbf{ in } m_1 \\
\lor & c > 0 & \} \ \textbf{guards of } ML\_in \textbf{ in } m_1 \\
\lor & a > 0 & \} \ \textbf{guards of } IL\_in \textbf{ in } m_1 \\
\lor & b > 0 \land a = 0 & \} \ \textbf{guards of } IL\_out \textbf{ in } m_1
\end{array}\right. \\
 & \vdash \\
\text{Disjunction of } \textit{concrete} \text{ guards} & \left\{\begin{array}{rrl}
 & ml\_tl = red \land a + b < d \land c = 0 \land il\_pass = 1 & \} \ \textbf{guards of } ML\_tl\_green \textbf{ in } m_2 \\
\lor & il\_tl = red \land b > 0 \land a = 0 \land ml\_pass = 1 & \} \ \textbf{guards of } IL\_tl\_green \textbf{ in } m_2 \\
\lor & ml\_tl = green \land a + b + 1 \neq d & \} \ \textbf{guards of } ML\_out\_1 \textbf{ in } m_2 \\
\lor & ml\_tl = green \land a + b + 1 = d & \} \ \textbf{guards of } ML\_out\_2 \textbf{ in } m_2 \\
\lor & il\_tl = green \land b \neq 1 & \} \ \textbf{guards of } IL\_out\_1 \textbf{ in } m_2 \\
\lor & il\_tl = green \land b = 1 & \} \ \textbf{guards of } IL\_out\_2 \textbf{ in } m_2 \\
\lor & a > 0 & \} \ \textbf{guards of } ML\_in \textbf{ in } m_2 \\
\lor & c > 0 & \} \ \textbf{guards of } IL\_in \textbf{ in } m_2
\end{array}\right.
\end{array}
$$

DLF

Proving Refinement: DLF of m_2

$$
\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
COLOUR = \{green, red\} \\
green \neq red \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
ml\_tl \in COLOUR \\
il\_tl \in COLOUR \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
il\_tl = green \Rightarrow b > 0 \land a = 0 \\
ml\_tl = red \lor il\_tl = red \\
ml\_pass \in \{0, 1\} \\
il\_pass \in \{0, 1\} \\
ml\_tl = red \Rightarrow ml\_pass = 1 \\
il\_tl = red \Rightarrow il\_pass = 1 \\
\quad\;\; a + b < d \land c = 0 \\
\lor \;\; c > 0 \\
\lor \;\; a > 0 \\
\lor \;\; b > 0 \land a = 0 \\
\vdash \\
\quad\;\; ml\_tl = red \land a + b < d \land c = 0 \land il\_pass = 1 \\
\lor \;\; il\_tl = red \land b > 0 \land a = 0 \land ml\_pass = 1 \\
\lor \;\; ml\_tl = green \\
\lor \;\; il\_tl = green \\
\lor \;\; a > 0 \\
\lor \;\; c > 0
\end{array}
$$

$\vdots$

$$
\boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
b \in \mathbb{N} \\
ml\_tl = red \\
il\_tl = red \\
ml\_tl = red \Rightarrow ml\_pass = 1 \\
il\_tl = red \Rightarrow il\_pass = 1 \\
\vdash \\
\quad\;\; b < d \land ml\_pass = 1 \land il\_pass = 1 \\
\lor \;\; b > 0 \land ml\_pass = 1 \land il\_pass = 1
\end{array}}
\;\ldots\;
\boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
b \in \mathbb{N} \\
ml\_tl = red \\
il\_tl = red \\
ml\_pass = 1 \\
il\_pass = 1 \\
\vdash \\
\quad\;\; b < d \land ml\_pass = 1 \land il\_pass = 1 \\
\lor \;\; b > 0 \land ml\_pass = 1 \land il\_pass = 1
\end{array}}
\;\ldots\;
\boxed{\begin{array}{l}
d > 0 \\
b \in \mathbb{N} \\
\vdash \\
b < d \lor b > 0
\end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l}
d > 0 \\
b > 0 \lor b = 0 \\
\vdash \\
b < d \lor b > 0
\end{array}}
\;\textbf{OR\_L}
\begin{cases}
\boxed{\begin{array}{l}
d > 0 \\
b > 0 \\
\vdash \\
b < d \lor b > 0
\end{array}}
\;\textbf{OR\_R2}\;
\boxed{\begin{array}{l}
d > 0 \\
b > 0 \\
\vdash \\
b > 0
\end{array}}
\;\textbf{HYP} \\
\boxed{\begin{array}{l}
d > 0 \\
b = 0 \\
\vdash \\
b < d \lor b > 0
\end{array}}
\;\textbf{EQ\_LR, MON}\;
\boxed{\begin{array}{l}
d > 0 \\
\\
\vdash \\
0 < d \lor 0 > 0
\end{array}}
\;\textbf{OR\_R1}\;
\boxed{\begin{array}{l}
d > 0 \\
\vdash \\
0 < d
\end{array}}
\;\textbf{HYP}
\end{cases}
$$

Second Refinement: Summary

- The final version of our ***second refinement*** m_2 is ***provably correct*** w.r.t.:
  - Establishment of ***Concrete Invariants*** [*init*]
  - Preservation of ***Concrete Invariants*** [old & new events]
  - Strengthening of ***guards*** [old events]
  - ***Convergence*** (a.k.a. livelock freedom, non-divergence) [new events]
  - Relative ***Deadlock*** Freedom
- Here is the final specification of m_2:

constants: d

sets: $COLOR$

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$
- **axm2_1** : $COLOR = \{green, red\}$
- **axm2_2** : $green \neq red$

variables: a, b, c, ml_tl, il_tl, ml_pass, il_pass

invariants:
- **inv2_1** : $ml\_tl \in COLOUR$
- **inv2_2** : $il\_tl \in COLOUR$
- **inv2_3** : $ml\_tl = green \Rightarrow a + b < d \land c = 0$
- **inv2_4** : $il\_tl = green \Rightarrow b > 0 \land a = 0$
- **inv2_5** : $ml\_tl = red \lor il\_tl = red$
- **inv2_6** : $ml\_pass \in \{0, 1\}$
- **inv2_7** : $il\_pass \in \{0, 1\}$
- **inv2_8** : $ml\_tl = red \Rightarrow ml\_pass = 1$
- **inv2_9** : $il\_tl = red \Rightarrow il\_pass = 1$

variants: $ml\_pass + il\_pass$

ML_tl_green
when $ml\_tl = red$, $a + b < d$, $c = 0$, $il\_pass = 1$
then $ml\_tl := green$, $il\_tl := red$, $ml\_pass := 0$
end

IL_tl_green
when $il\_tl = red$, $b > 0$, $a = 0$, $ml\_pass = 1$
then $il\_tl := green$, $ml\_tl := red$, $il\_pass := 0$
end

ML_out_1
when $ml\_tl = green$, $a + b + 1 \neq d$
then $a := a + 1$, $ml\_pass := 1$
end

ML_out_2
when $ml\_tl = green$, $a + b + 1 = d$
then $a := a + 1$, $ml\_tl := red$, $ml\_pass := 1$
end

IL_out_1
when $il\_tl = green$, $b \neq 1$
then $b := b - 1$, $c := c + 1$, $il\_pass := 1$
end

IL_out_2
when $il\_tl = green$, $b = 1$
then $b := b - 1$, $c := c + 1$, $il\_tl := red$, $il\_pass := 1$
end

ML_in
when $c > 0$
then $c := c - 1$
end

IL_in
when $a > 0$
then $a := a - 1$, $b := b + 1$
end

Index (1)

Index (2)

Index (3)

Index (4)

Index (5)

Index (6)

Index (7)

Index (8)

Index (9)

Index (10)

Index (11)