

Recursion



EECS2030 E&F: Advanced
Object Oriented Programming
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Learning Outcomes

This module is designed to help you learn about:

1. How to solve problems *recursively*
2. Example *recursions* on string and arrays
3. Some more advanced example (if time permitted)

Beyond this lecture ...

- Fantastic resources for sharpening your recursive skills for the exam:

`http://codingbat.com/java/Recursion-1`

`http://codingbat.com/java/Recursion-2`

- The **best** approach to learning about recursion is via a functional programming language:

Haskell Tutorial: `https://www.haskell.org/tutorial/`

Recursion: Principle

- **Recursion** is useful in expressing solutions to problems that can be **recursively** defined:
 - **Base Cases:** Small problem instances immediately solvable.
 - **Recursive Cases:**
 - Large problem instances *not immediately solvable*.
 - Solve by reusing *solution(s) to strictly smaller problem instances*.
- Similar idea learnt in high school: [**mathematical induction**]
- Recursion can be easily expressed programmatically in Java:

```
m(i) {  
    if(i == ...) { /* base case: do something directly */ }  
    else {  
        m(j); /* recursive call with strictly smaller value */  
    }  
}
```

- In the body of a method m , there might be *a call or calls to m itself*.
- Each such self-call is said to be a **recursive call**.
- Inside the execution of $m(i)$, a recursive call $m(j)$ must be that $j < i$.

Tracing Method Calls via a Stack

- When a method is called, it is **activated** (and becomes **active**) and **pushed** onto the stack.
- When the body of a method makes a (helper) method call, that (helper) method is **activated** (and becomes **active**) and **pushed** onto the stack.
 - ⇒ The stack contains activation records of all **active** methods.
 - **Top** of stack denotes the current point of execution.
 - Remaining parts of stack are (temporarily) **suspended**.
- When entire body of a method is executed, stack is **popped**.
 - ⇒ The current point of execution is returned to the new **top** of stack (which was **suspended** and just became **active**).
- Execution terminates when the stack becomes **empty**.

Recursion: Factorial (1)

- Recall the formal definition of calculating the n factorial:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1 & \text{if } n \geq 1 \end{cases}$$

- How do you define the same problem *recursively*?

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

- To solve $n!$, we combine n and the solution to $(n-1)!$.

```
int factorial(int n) {  
    int result;  
    if(n == 0) { /* base case */ result = 1; }  
    else { /* recursive case */  
        result = n * factorial(n - 1);  
    }  
    return result;  
}
```

Common Errors of Recursive Methods

- Missing Base Case(s).

```
int factorial(int n) {  
    return n * factorial(n - 1);  
}
```

Base case(s) are meant as points of stopping growing the runtime stack.

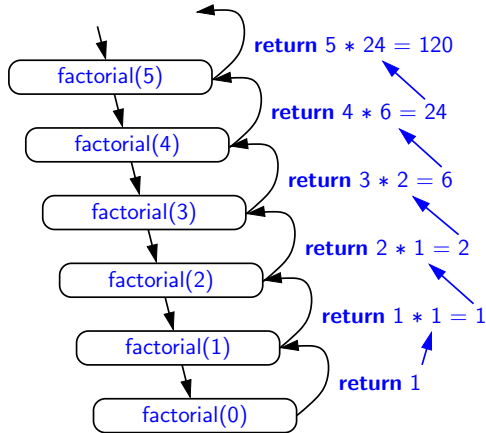
- Recursive Calls on Non-Smaller Problem Instances.

```
int factorial(int n) {  
    if(n == 0) { /* base case */ return 1; }  
    else { /* recursive case */ return n * factorial(n); }  
}
```

Recursive calls on **strictly smaller** problem instances are meant for moving gradually towards the base case(s).

- In both cases, a `StackOverflowException` will be thrown.

Recursion: Factorial (2)



Recursion: Factorial (3)

- When running *factorial(5)*, a *recursive call factorial(4)* is made. Call to *factorial(5)* suspended until *factorial(4)* returns a value.
- When running *factorial(4)*, a *recursive call factorial(3)* is made. Call to *factorial(4)* suspended until *factorial(3)* returns a value.
- ...
- *factorial(0)* returns 1 back to *suspended call factorial(1)*.
- *factorial(1)* receives 1 from *factorial(0)*, multiplies 1 to it, and returns 1 back to the *suspended call factorial(2)*.
- *factorial(2)* receives 1 from *factorial(1)*, multiplies 2 to it, and returns 2 back to the *suspended call factorial(3)*.
- *factorial(3)* receives 2 from *factorial(1)*, multiplies 3 to it, and returns 6 back to the *suspended call factorial(4)*.
- *factorial(4)* receives 6 from *factorial(3)*, multiplies 4 to it, and returns 24 back to the *suspended call factorial(5)*.
- *factorial(5)* receives 24 from *factorial(4)*, multiplies 5 to it, and returns 120 as the result.

Recursion: Factorial (4)

- When the execution of a method (e.g., *factorial(5)*) leads to a nested method call (e.g., *factorial(4)*):
 - The execution of the current method (i.e., *factorial(5)*) is *suspended*, and a structure known as an *activation record* or *activation frame* is created to store information about the progress of that method (e.g., values of parameters and local variables).
 - The nested methods (e.g., *factorial(4)*) may call other nested methods (*factorial(3)*).
 - When all nested methods complete, the activation frame of the *latest suspended* method is re-activated, then continue its execution.
- What kind of data structure does this activation-suspension process correspond to? [LIFO Stack]

Recursion: Fibonacci Sequence (1)

- Can you identify the pattern of a Fibonacci sequence?

$$F = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

- Here is the formal, **recursive** definition of calculating the n_{th} number in a Fibonacci sequence (denoted as F_n):

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{if } n > 2 \end{cases}$$

```
int fib(int n) {  
    int result;  
    if(n == 1) { /* base case */ result = 1; }  
    else if(n == 2) { /* base case */ result = 1; }  
    else { /* recursive case */  
        result = fib(n - 1) + fib(n - 2);  
    }  
    return result;  
}
```

Recursion: Fibonacci Sequence (2)

```

fib(5)
= { fib(5) = fib(4) + fib(3); push(fib(5)); suspended: {fib(5)}; active: fib(4) }
  fib(4) + fib(3)
= { fib(4) = fib(3) + fib(2); suspended: {fib(4), fib(5)}; active: fib(3) }
  ( fib(3) + fib(2) ) + fib(3)
= { fib(3) = fib(2) + fib(1); suspended: {fib(3), fib(4), fib(5)}; active: fib(2) }
  (( fib(2) + fib(1) ) + fib(2)) + fib(3)
= { fib(2) returns 1; suspended: {fib(3), fib(4), fib(5)}; active: fib(1) }
  (( 1 + fib(1) ) + fib(2)) + fib(3)
= { fib(1) returns 1; suspended: {fib(3), fib(4), fib(5)}; active: fib(3) }
  (( 1 + 1 ) + fib(2)) + fib(3)
= { fib(3) returns 1 + 1; pop(); suspended: {fib(4), fib(5)}; active: fib(2) }
  (2 + fib(2)) + fib(3)
= { fib(2) returns 1; suspended: {fib(4), fib(5)}; active: fib(4) }
  (2 + 1) + fib(3)
= { fib(4) returns 2 + 1; pop(); suspended: {fib(5)}; active: fib(3) }
  3 + fib(3)
= { fib(3) = fib(2) + fib(1); suspended: {fib(3), fib(5)}; active: fib(2) }
  3 + ( fib(2) + fib(1) )
= { fib(2) returns 1; suspended: {fib(3), fib(5)}; active: fib(1) }
  3 + (1 + fib(1) )
= { fib(1) returns 1; suspended: {fib(3), fib(5)}; active: fib(3) }
  3 + (1 + 1)
= { fib(3) returns 1 + 1; pop(); suspended: {fib(5)}; active: fib(5) }
  3 + 2
  { fib(5) returns 3 + 2; suspended: {} }

```

Java Library: String

```
public class StringTester {  
    public static void main(String[] args) {  
        String s = "abcd";  
        System.out.println(s.isEmpty()); /* false */  
        /* Characters in index range [0, 0) */  
        String t0 = s.substring(0, 0);  
        System.out.println(t0); /* "" */  
        /* Characters in index range [0, 4) */  
        String t1 = s.substring(0, 4);  
        System.out.println(t1); /* "abcd" */  
        /* Characters in index range [1, 3) */  
        String t2 = s.substring(1, 3);  
        System.out.println(t2); /* "bc" */  
        String t3 = s.substring(0, 2) + s.substring(2, 4);  
        System.out.println(s.equals(t3)); /* true */  
        for(int i = 0; i < s.length(); i++) {  
            System.out.print(s.charAt(i));  
        }  
        System.out.println();  
    }  
}
```

Recursion: Palindrome (1)

Problem: A palindrome is a word that reads the same forwards and backwards. Write a method that takes a string and determines whether or not it is a palindrome.

```
System.out.println(isPalindrome("")); true
System.out.println(isPalindrome("a")); true
System.out.println(isPalindrome("madam")); true
System.out.println(isPalindrome("racecar")); true
System.out.println(isPalindrome("man")); false
```

Base Case 1: Empty string \rightarrow Return *true* immediately.

Base Case 2: String of length 1 \rightarrow Return *true* immediately.

Recursive Case: String of length $\geq 2 \rightarrow$

- 1st and last characters match, **and**
- *the rest (i.e., middle) of the string* is a palindrome .

Recursion: Palindrome (2)

```
boolean isPalindrome (String word) {  
    if (word.length() == 0 || word.length() == 1) {  
        /* base case */  
        return true;  
    }  
    else {  
        /* recursive case */  
        char firstChar = word.charAt(0);  
        char lastChar = word.charAt(word.length() - 1);  
        String middle = word.substring(1, word.length() - 1);  
        return  
            firstChar == lastChar  
            /* See the API of java.lang.String.substring. */  
            && isPalindrome (middle);  
    }  
}
```

Recursion: Reverse of String (1)

Problem: The reverse of a string is written backwards. Write a method that takes a string and returns its reverse.

```
System.out.println(reverseOf("")); /* "" */  
System.out.println(reverseOf("a"));  "a"  
System.out.println(reverseOf("ab"));  "ba"  
System.out.println(reverseOf("abc"));  "cba"  
System.out.println(reverseOf("abcd"));  "dcba"
```

Base Case 1: Empty string \rightarrow Return *empty string*.

Base Case 2: String of length 1 \rightarrow Return *that string*.

Recursive Case: String of length $\geq 2 \rightarrow$

- 1) Head of string (i.e., first character)
- 2) Reverse of the tail of string (i.e., all but the first character)

Return the concatenation of **2)** and **1)**.

Recursion: Reverse of a String (2)

```
String reverseOf (String s) {  
    if(s.isEmpty()) { /* base case 1 */  
        return "";  
    }  
    else if(s.length() == 1) { /* base case 2 */  
        return s;  
    }  
    else { /* recursive case */  
        String tail = s.substring(1, s.length());  
        String reverseOfTail = reverseOf(tail);  
        char head = s.charAt(0);  
        return reverseOfTail + head;  
    }  
}
```

Recursion: Number of Occurrences (1)

Problem: Write a method that takes a string s and a character c , then count the number of occurrences of c in s .

```
System.out.println(occurrencesOf("", 'a')); /* 0 */  
System.out.println(occurrencesOf("a", 'a')); /* 1 */  
System.out.println(occurrencesOf("b", 'a')); /* 0 */  
System.out.println(occurrencesOf("baaba", 'a')); /* 3 */  
System.out.println(occurrencesOf("baaba", 'b')); /* 2 */  
System.out.println(occurrencesOf("baaba", 'c')); /* 0 */
```

Base Case: Empty string \rightarrow Return 0 .

Recursive Case: String of length $\geq 1 \rightarrow$

- 1) Head of s (i.e., first character)
- 2) Number of occurrences of c in the tail of s (i.e., all but the first character)

If head is equal to c , return $1 + 2$).

If head is not equal to c , return $0 + 2$).

Recursion: Number of Occurrences (2)

```
int occurrencesOf (String s, char c) {  
    if (s.isEmpty()) {  
        /* Base Case */  
        return 0;  
    }  
    else {  
        /* Recursive Case */  
        char head = s.charAt(0);  
        String tail = s.substring(1, s.length());  
        if (head == c) {  
            return 1 + occurrencesOf (tail, c);  
        }  
        else {  
            return 0 + occurrencesOf (tail, c);  
        }  
    }  
}
```

Making Recursive Calls on an Array

- Recursive calls denote solutions to *smaller* sub-problems.
- Naively*, explicitly create a new, smaller array:

```
void m(int[] a) {
    if(a.length == 0) { /* base case */ }
    else if(a.length == 1) { /* base case */ }
    else {
        int[] sub = new int[a.length - 1];
        for(int i = 1; i < a.length; i++) { sub[i - 1] = a[i]; }
        m(sub) } }
```

- For *efficiency*, we pass the *reference* of the same array and specify the *range of indices* to be considered:

```
void m(int[] a, int from, int to) {
    if(from > to) { /* base case */ }
    else if(from == to) { /* base case */ }
    else { m(a, from + 1, to) } }
```

- $m(a, 0, a.length - 1)$ [Initial call; entire array]
- $m(a, 1, a.length - 1)$ [1st r.c. on array of size $a.length - 1$]
- $m(a, a.length - 1, a.length - 1)$ [Last r.c. on array of size 1]

Recursion: All Positive (1)

Problem: Determine if an array of integers are all positive.

```
System.out.println(allPositive({})); /* true */  
System.out.println(allPositive({1, 2, 3, 4, 5})); /* true */  
System.out.println(allPositive({1, 2, -3, 4, 5})); /* false */
```

Base Case: Empty array \rightarrow Return *true* immediately.

The base case is *true* \because we can *not* find a counter-example (i.e., a number *not* positive) from an empty array.

Recursive Case: Non-Empty array \rightarrow

- 1st element positive, **and**
- *the rest of the array* is all positive.

Exercise: Write a method `boolean somePositive(int[]`

a) which *recursively* returns *true* if there is some positive number in `a`, and *false* if there are no positive numbers in `a`.

Hint: What to return in the base case of an empty array? [*false*]

\because No witness (i.e., a positive number) from an empty array

Recursion: All Positive (2)

```
boolean allPositive(int[] a) {  
    return allPositiveHelper(a, 0, a.length - 1);  
}  
  
boolean allPositiveHelper(int[] a, int from, int to) {  
    if (from > to) { /* base case 1: empty range */  
        return true;  
    }  
    else if (from == to) { /* base case 2: range of one element */  
        return a[from] > 0;  
    }  
    else { /* recursive case */  
        return a[from] > 0 && allPositiveHelper(a, from + 1, to);  
    }  
}
```

Recursion: Is an Array Sorted? (1)

Problem: Determine if an array of integers are sorted in a non-descending order.

```
System.out.println(isSorted({}));    true  
  
System.out.println(isSorted({1, 2, 2, 3, 4}));    true  
  
System.out.println(isSorted({1, 2, 2, 1, 3}));    false
```

Base Case: Empty array → Return *true* immediately.

The base case is *true* ∵ we can *not* find a counter-example (i.e., a pair of adjacent numbers that are *not* sorted in a non-descending order) from an empty array.

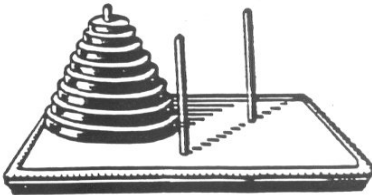
Recursive Case: Non-Empty array →

- 1st and 2nd elements are sorted in a non-descending order, **and**
- *the rest of the array*, starting from the 2nd element, **are sorted in a non-descending order**.

Recursion: Is an Array Sorted? (2)

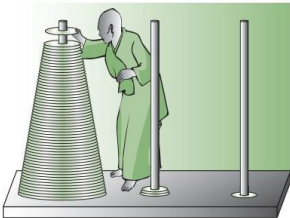
```
boolean isSorted(int[] a) {  
    return isSortedHelper(a, 0, a.length - 1);  
}  
  
boolean isSortedHelper(int[] a, int from, int to) {  
    if (from > to) { /* base case 1: empty range */  
        return true;  
    }  
    else if (from == to) { /* base case 2: range of one element */  
        return true;  
    }  
    else {  
        return a[from] <= a[from + 1]  
            && isSortedHelper(a, from + 1, to);  
    }  
}
```


Tower of Hanoi: Specification



- **Given:** A tower of 8 disks, initially stacked in decreasing size on one of 3 pegs
- **Rules:**
 - Move only one disk at a time.
 - Never move a larger disk onto a smaller one.
- **Problem:** Transfer the entire tower to one of the other pegs.

Tower of Hanoi: Llegend



*Brahmins at a temple in Benares, India have been carrying out movement of “Sacred Tower of Brahma”, consisting of **sixty-four** golden disks, according to the same rules as in the Tower of Hanoi game, and that the completion of the tower would lead to the end of the world.*

Tower of Hanoi: A Recursive Solution

The general, a recursive solution requires 3 steps:

1. Transfer the $n - 1$ smallest disks to a *second* peg.
2. Move the largest peg to the *third* peg (free of disks).
3. Transfer the $n - 1$ smallest disks back onto the largest disk.

Tower of Hanoi in Java (1)

```
void towerOfHanoi(String[] disks) {  
    toHelper(disks, 0, disks.length - 1, 1, 3);  
}  
void toHelper(String[] disks, int from, int to, int ori, int des){  
    if(from > to) { }  
    else if(from == to) {  
        print("move " + disks[to] + " from " + ori + " to " + des);  
    }  
    else {  
        int intermediate = 6 - ori - des;  
        toHelper(disks, from, to - 1, ori, intermediate);  
        print("move " + disks[to] + " from " + ori + " to " + des);  
        toHelper(disks, from, to - 1, intermediate, des);  
    }  
}
```

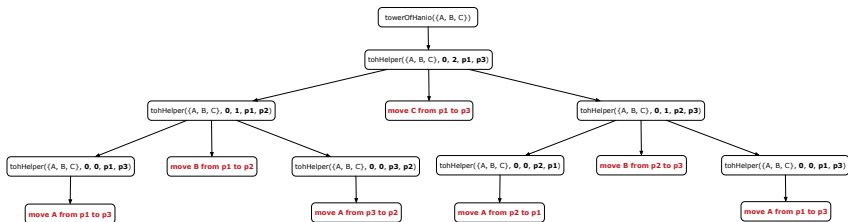
- `toHelper(disks, from, to, ori, des)` moves disks $\{disks[from], disks[from + 1], \dots, disks[to]\}$ from peg *ori* to peg *des*.
- Peg id's are 1, 2, and 3 \Rightarrow The intermediate one is $6 - ori - des$.

Tower of Hanoi in Java (2)

Say ds (disks) is $\{A, B, C\}$, where $A < B < C$.

$$\begin{aligned}
 & \text{tohH}(ds, \underbrace{0, 2}_{\{A, B, C\}}, p1, p3) = \left\{ \begin{array}{l} \text{Move C: } p1 \text{ to } p3 \\ \text{tohH}(ds, \underbrace{0, 1}_{\{A, B\}}, p1, p2) = \left\{ \begin{array}{l} \text{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p1, p3) = \left\{ \begin{array}{l} \text{Move A: } p1 \text{ to } p3 \end{array} \right. \\ \text{Move B: } p1 \text{ to } p2 \\ \text{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p3, p2) = \left\{ \begin{array}{l} \text{Move A: } p3 \text{ to } p2 \end{array} \right. \end{array} \right. \\ \text{tohH}(ds, \underbrace{0, 1}_{\{A, B\}}, p2, p3) = \left\{ \begin{array}{l} \text{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p2, p1) = \left\{ \begin{array}{l} \text{Move A: } p2 \text{ to } p1 \end{array} \right. \\ \text{Move B: } p2 \text{ to } p3 \\ \text{tohH}(ds, \underbrace{0, 0}_{\{A\}}, p1, p3) = \left\{ \begin{array}{l} \text{Move A: } p1 \text{ to } p3 \end{array} \right. \end{array} \right. \end{array} \right.
 \end{aligned}$$

Tower of Hanoi in Java (3)



Running Time: Tower of Hanoi (1)

- Generalize the problem by considering **n** disks.
- Let **$T(n)$** denote the number of moves required to transfer **n** disks from one to another under the rules.
- Recall the general solution pattern:
 1. Transfer the **n - 1** **smallest** disks to a **second** peg.
 2. Move the **largest** peg to the **third** peg (free of disks).
 3. Transfer the **n - 1** **smallest** disks back onto the **largest** disk.
- We end up with the following recurrence relation that allows us to compute **$T(n)$** for any **n** we like:

$$\begin{cases} T(1) &= 1 \\ T(n) &= 2 \cdot T(n-1) + 1 \quad \text{where } n > 0 \end{cases}$$

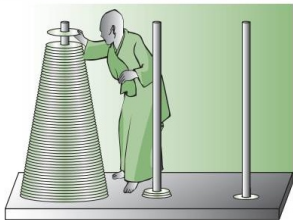
- To solve this recurrence relation, we study the pattern of **$T(n)$** and observe how it reaches the **base case(s)**.

Running Time: Tower of Hanoi (2)

$$\begin{aligned}
 T(n) &= \underbrace{2}_{1 \text{ term}} \times T(n-1) + \underbrace{1}_{1 \text{ term}} \\
 &= \underbrace{2 \times (2 \times T(n-2) + 1)}_{2 \text{ terms}} + \underbrace{1}_{2 \text{ terms}} \\
 &= \underbrace{2 \times (2 \times (2 \times T(n-3) + 1) + 1)}_{3 \text{ terms}} + \underbrace{1}_{3 \text{ terms}} \\
 &= \dots \\
 &= \underbrace{2 \times (2 \times (2 \times (\dots \times (2 \times T(n - (n-1))) + 1) + \dots) + 1) + 1)}_{n-1 \text{ terms}} \\
 &= 2^{n-1} + (n-1)
 \end{aligned}$$

$\therefore T(n)$ is $O(2^n)$

Tower of Hanoi: Legend



*Brahmins at a temple in Benares, India have been carrying out movement of “Sacred Tower of Brahma”, consisting of **sixty-four** golden disks, according to the same rules as in the Tower of Hanoi game, and that the completion of the tower would lead to the end of the world.*

Say one disk can be moved in one second.

Q. How long does it take to finish moving 64 disks ($n = 64$)?

A. 2^{64} seconds \approx 585 billion years ($>>$ 5 billion centuries)!

Beyond this lecture ...

- Recursions on Arrays: Lab Exercise from EECS2030-F19
- Notes on Recursion:
http://www.eecs.yorku.ca/~jackie/teaching/lectures/2024/F/EECS2030/notes/EECS2030_F24_Notes_Recursion.pdf
- API for String:
<https://docs.oracle.com/javase/8/docs/api/java/lang/String.html>
- Fantastic resources for sharpening your recursive skills for the exam:
<http://codingbat.com/java/Recursion-1>
<http://codingbat.com/java/Recursion-2>
- The **best** approach to learning about recursion is via a functional programming language:
Haskell Tutorial: <https://www.haskell.org/tutorial/>

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