Asymptotic Analysis of Algorithms

EECS2030 B: Advanced Object Oriented Programming
Fall 2018

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Algorithm and Data Structure

- A *data structure* is:
  - A systematic way to store and organize data in order to facilitate *access* and *modifications*
  - Never suitable for all purposes: it is important to know its *strengths* and *limitations*
- A *well-specified* *computational problem* precisely describes the desired *input/output relationship*.
  - **Input:** A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$
  - **Output:** A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \le a'_2 \le \ldots \le a'_n$
  - An *instance* of the problem: $\langle 3, 1, 2, 5, 4 \rangle$
- An *algorithm* is:
  - A solution to a well-specified *computational problem*
  - A *sequence of computational steps* that takes value(s) as *input* and produces value(s) as *output*
- Steps in an *algorithm* manipulate well-chosen *data structure(s)*.

Measuring "Goodness" of an Algorithm

1. *Correctness* :
   - Does the algorithm produce the expected output?
   - Use JUnit to ensure this.
2. Efficiency:
   - *Time Complexity*: processor time required to complete
   - *Space Complexity*: memory space required to store data

Correctness is always the priority.

How about efficiency? Is time or space more of a concern?

Measuring Efficiency of an Algorithm

- *Time* is more of a concern than is *storage*.
- Solutions that are meant to be run on a computer should run *as fast as possible*.
- Particularly, we are interested in how *running time* depends on two *input factors*:
  1. size
     e.g., sorting an array of 10 elements vs. 1m elements
  2. structure
     e.g., sorting an already-sorted array vs. a hardly-sorted array
- *How do you determine the running time of an algorithm?*
  1. Measure time via *experiments*
  2. Characterize time as a *mathematical function* of the input size

Measure Running Time via Experiments

- Once the algorithm is implemented in Java:
  - Execute the program on *test inputs* of various *sizes* and *structures*.
  - For each test, record the *elapsed time* of the execution.

```
long startTime = System.currentTimeMillis();
/* run the algorithm */
long endTime = System.currenctTimeMillis();
long elapsed = endTime - startTime;
```

  - *Visualize* the result of each test.
- To make *sound statistical claims* about the algorithm's *running time*, the set of input tests must be "reasonably" *complete*.

Example Experiment

- *Computational Problem*:
  - **Input**: A character *c* and an integer *n*
  - **Output**: A string consisting of *n* repetitions of character *c*
    e.g., Given input `'*'` and `15`, output `***************`.
- *Algorithm 1* using *String* Concatenations:

```
public static String repeat1(char c, int n) {
  String answer = "";
  for (int i = 0; i < n; i ++) { answer += c; }
  return answer; }
```

- *Algorithm 2* using *StringBuilder* `append`'s:

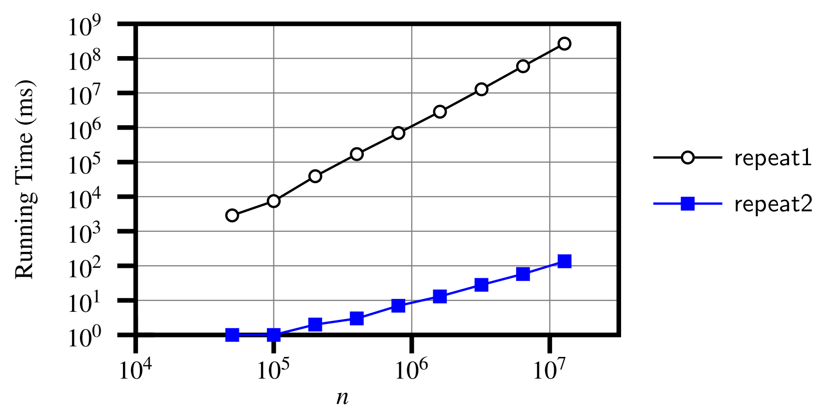
```
public static String repeat2(char c, int n) {
  StringBuilder sb = new StringBuilder();
  for (int i = 0; i < n; i ++) { sb.append(c); }
  return sb.toString(); }
```


Example Experiment: Detailed Statistics

n	`repeat1` (in ms)	`repeat2` (in ms)
50,000	2,884	1
100,000	7,437	1
200,000	39,158	2
400,000	170,173	3
800,000	690,836	7
1,600,000	2,847,968	13
3,200,000	12,809,631	28
6,400,000	59,594,275	58
12,800,000	265,696,421 (≈ 3 days)	135

- As *input size* is doubled, *rates of increase* for both algorithms are *linear*:
  - *Running time* of `repeat1` increases by ≈ 5 times.
  - *Running time* of `repeat2` increases by ≈ 2 times.

Example Experiment: Visualization



Experimental Analysis: Challenges

1. An algorithm must be *fully implemented* (i.e., translated into valid Java syntax) in order study its runtime behaviour *experimentally*.
   - What if our purpose is to *choose among alternative* data structures or algorithms to implement?
   - Can there be a *higher-level analysis* to determine that one algorithm or data structure is *superior* than others?
2. Comparison of multiple algorithms is only *meaningful* when experiments are conducted under the same environment of:
   - *Hardware*: CPU, running processes
   - *Software*: OS, JVM version
3. Experiments can be done only on *a limited set of test inputs*.
   - What if "*important*" inputs were not included in the experiments?

Moving Beyond Experimental Analysis

- A better approach to analyzing the *efficiency* (e.g., *running times*) of algorithms should be one that:
  - Allows us to calculate the *relative efficiency* (rather than absolute elapsed time) of algorithms in a ways that is *independent of* the hardware and software environment.
  - Can be applied using a *high-level description* of the algorithm (without fully implementing it).
  - Considers *all* possible inputs.
- We will learn a better approach that contains 3 ingredients:
  1. Counting *primitive operations*
  2. Approximating running time as *a function of input size*
  3. Focusing on the *worst-case* input (requiring the most running time)

Counting Primitive Operations

A *primitive operation* corresponds to a low-level instruction with a *constant execution time*.

- Assignment [e.g., `x = 5;`]
- Indexing into an array [e.g., `a[i]`]
- Arithmetic, relational, logical op. [e.g., `a + b`, `z > w`, `b1 && b2`]
- Accessing an attribute of an object [e.g., `acc.balance`]
- Returning from a method [e.g., `return result;`]

Q: Why is a method call in general *not* a primitive operation?
A: It may be a call to:
- a "*cheap*" method (e.g., printing `Hello World`), or
- an "*expensive*" method (e.g., sorting an array of integers)

Example: Counting Primitive Operations

```
1 findMax (int[] a, int n) {
2   currentMax = a[0];
3   for (int i = 1; i < n; ) {
4     if (a[i] > currentMax) {
5       currentMax = a[i]; }
6     i ++ }
7   return currentMax; }
```

\# of times `i < n` in **Line 3** is executed? [n]
\# of times the loop body (**Line 4** to **Line 6**) is executed? [$n-1$]

- **Line 2**: 2 [1 indexing + 1 assignment]
- **Line 3**: $n+1$ [1 assignment + n comparisons]
- **Line 4**: $(n-1)\cdot 2$ [1 indexing + 1 comparison]
- **Line 5**: $(n-1)\cdot 2$ [1 indexing + 1 assignment]
- **Line 6**: $(n-1)\cdot 2$ [1 addition + 1 assignment]
- **Line 7**: 1 [1 return]
- **Total # of Primitive Operations**: 7n - 2

From Absolute RT to Relative RT

- Each *primitive operation* (PO) takes approximately the same, constant amount of time to execute. [say t]
- The *number of primitive operations* required by an algorithm should be *proportional* to its *actual running time* on a specific environment.

  e.g., `findMax (int[] a, int n)` has $7n - 2$ POs

  $$RT = (7n - 2) \cdot t$$

  Say two algorithms with RT $(7n - 2) \cdot t$ and RT $(10n + 3) \cdot t$.

  ⇒ It suffices to compare their *relative* running time:

  $7n - 2$ vs. $10n + 3$.
- To determine the *time efficiency* of an algorithm, we only focus on their *number of POs*.

Example: Approx. # of Primitive Operations

- Given # of primitive operations counted precisely as $7n^1 - 2$, we view it as

  $$7 \cdot n - 2 \cdot n^0$$
- We say
  - n is the ***highest power***
  - 7 and 2 are the ***multiplicative constants***
  - 2 is the ***lower term***
- When approximating a function (considering that input size may be very large):
  - **Only** the ***highest power*** matters.
  - ***multiplicative constants*** and ***lower terms*** can be dropped.

  ⇒ $7n - 2$ is approximately n

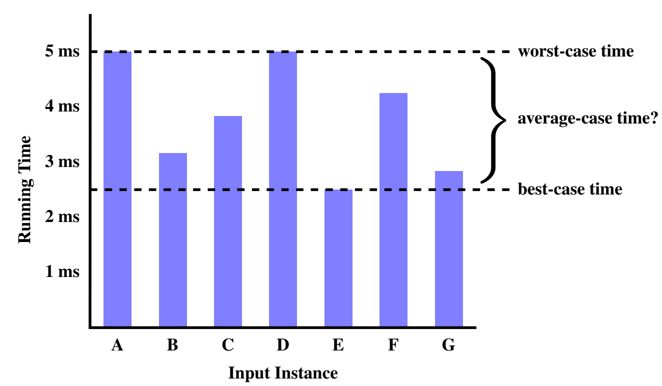
  Exercise: Consider $7n + 2n \cdot log\ n + 3n^2$:
  - ***highest power***? [n^2]
  - ***multiplicative constants***? [7, 2, 3]
  - ***lower terms***? [$7n + 2n \cdot log\ n$]

Approximating Running Time as a Function of Input Size

Given the *high-level description* of an algorithm, we associate it with a function f, such that $f(n)$ returns the *number of primitive operations* that are performed on an *input of size n*.

- $f(n) = 5$ [constant]
- $f(n) = log_2 n$ [logarithmic]
- $f(n) = 4 \cdot n$ [linear]
- $f(n) = n^2$ [quadratic]
- $f(n) = n^3$ [cubic]
- $f(n) = 2^n$ [exponential]

Focusing on the Worst-Case Input



- *Average-case* analysis calculates the *expected running times* based on the probability distribution of input values.
- *worst-case* analysis or *best-case* analysis?

What is Asymptotic Analysis?

Asymptotic analysis

- Is a method of describing *behaviour in the limit*:
  - How the *running time* of the algorithm under analysis changes as the *input size* changes without bound
  - e.g., contrast $RT_1(n) = n$ with $RT_2(n) = n^2$
- Allows us to compare the *relative* performance of alternative algorithms:
  - For large enough inputs, the *multiplicative constants* and *lower-order* terms of an exact running time can be disregarded.
  - e.g., $RT_1(n) = 3n^2 + 7n + 18$ and $RT_1(n) = 100n^2 + 3n - 100$ are considered **equally efficient**, *asymptotically*.
  - e.g., $RT_1(n) = n^3 + 7n + 18$ is considered **less efficient** than $RT_1(n) = 100n^2 + 100n + 2000$, *asymptotically*.

Three Notions of Asymptotic Bounds

We may consider three kinds of *asymptotic bounds* for the *running time* of an algorithm:

- Asymptotic *upper* bound [O]
- Asymptotic lower bound [Ω]
- Asymptotic tight bound [Θ]

Asymptotic Upper Bound: Definition

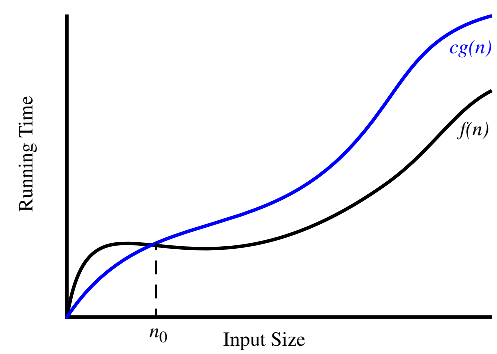
- Let $f(n)$ and $g(n)$ be functions mapping positive integers (input size) to positive real numbers (running time).
  - $f(n)$ characterizes the running time of some algorithm.
  - $O(g(n))$ denotes *a collection of* functions.
- $O(g(n))$ consists of *all* functions that can be upper bounded by $g(n)$, starting at some point, using some constant factor.
- $f(n) \in O(g(n))$ if there are:
  - A real *constant* $c > 0$
  - An integer *constant* $n_0 \geq 1$

  such that:

$$f(n) \leq c \cdot g(n) \quad \text{for } n \geq n_0$$

- For each member function $f(n)$ in $O(g(n))$, we say that:
  - $f(n) \in O(g(n))$ [f(n) is a member of "big-Oh of g(n)"]
  - $f(n)$ **is** $O(g(n))$ [f(n) is "big-Oh of g(n)"]
  - $f(n)$ **is order of** $g(n)$

Asymptotic Upper Bound: Visualization



From n_0, $f(n)$ is upper bounded by $c \cdot g(n)$, so $f(n)$ is $O(g(n))$.

Asymptotic Upper Bound: Example (1)

Prove: The function $8n + 5$ is $O(n)$.
Strategy: Choose a real constant $c > 0$ and an integer constant $n_0 \geq 1$, such that for every integer $n \geq n_0$:

$$8n + 5 \leq c \cdot n$$

Can we choose $c = 9$? What should the corresponding n_0 be?

n	8n + 5	9n
1	13	9
2	21	18
3	29	27
4	37	36
5	45	45
6	53	54
...		

Therefore, we prove it by choosing $c = 9$ and $n_0 = 5$.
We may also prove it by choosing $c = 13$ and $n_0 = 1$. Why?

Asymptotic Upper Bound: Example (2)

Prove: The function $f(n) = 5n^4 + 3n^3 + 2n^2 + 4n + 1$ is $O(n^4)$.

Strategy: Choose a real constant $c > 0$ and an integer constant $n_0 \geq 1$, such that for every integer $n \geq n_0$:

$$5n^4 + 3n^3 + 2n^2 + 4n + 1 \leq c \cdot n^4$$

$f(1) = 5 + 3 + 2 + 4 + 1 = 15$

Choose $c = 15$ and $n_0 = 1$!

Asymptotic Upper Bound: Proposition (1)

If $f(n)$ is a polynomial of degree d, i.e.,

$$f(n) = a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d$$

and $a_0, a_1, \ldots, a_d$ are integers (i.e., negative, zero, or positive), then $f(n)$ *is* $O(n^d)$.

- We prove by choosing

$$\begin{aligned} c &= |a_0| + |a_1| + \cdots + |a_d| \\ n_0 &= 1 \end{aligned}$$

- We know that for $n \geq 1$: $n^0 \leq n^1 \leq n^2 \leq \cdots \leq n^d$
- Upper-bound effect starts when $n_0 = 1$? $[f(1) \leq 1^d]$

$$a_0 \cdot 1^0 + a_1 \cdot 1^1 + \cdots + a_d \cdot 1^d \leq |a_0| \cdot 1^d + |a_1| \cdot 1^d + \cdots + |a_d| \cdot 1^d$$

- Upper-bound effect holds? $[f(n) \leq n^d]$

$$a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d \leq |a_0| \cdot n^d + |a_1| \cdot n^d + \cdots + |a_d| \cdot n^d$$

Asymptotic Upper Bound: Proposition (2)

$$O(n^0) \subset O(n^1) \subset O(n^2) \subset \ldots$$

If a function $f(n)$ is *upper bounded* by another function $g(n)$ of degree d, $d \geq 0$, then $f(n)$ is also upper bounded by all other functions of a *strictly higher degree* (i.e., $d + 1$, $d + 2$, *etc.*).

e.g., Family of $O(n)$ contains:
- $n^0, 2n^0, 3n^0, \ldots$ [functions with degree 0]
- $n, 2n, 3n, \ldots$ [functions with degree 1]

e.g., Family of $O(n^2)$ contains:
- $n^0, 2n^0, 3n^0, \ldots$ [functions with degree 0]
- $n, 2n, 3n, \ldots$ [functions with degree 1]
- $n^2, 2n^2, 3n^2, \ldots$ [functions with degree 2]

Asymptotic Upper Bound: More Examples

- $5n^2 + 3n \cdot logn + 2n + 5$ is $O(n^2)$ [$c = 15$, $n_0 = 1$]
- $20n^3 + 10n \cdot logn + 5$ is $O(n^3)$ [$c = 35$, $n_0 = 1$]
- $3 \cdot logn + 2$ is $O(logn)$ [$c = 5$, $n_0 = \boxed{2}$]
  - Why can't n_0 be 1?
  - Choosing $n_0 = 1$ means $\Rightarrow f(\boxed{1})$ ***is*** upper-bounded by $c \cdot log\boxed{1}$:
    - We have $f(\boxed{1}) = 3 \cdot log1 + 2$, which is 2.
    - We have $c \cdot log\boxed{1}$, which is 0.

    $\Rightarrow f(\boxed{1})$ ***is not*** upper-bounded by $c \cdot log\boxed{1}$ [Contradiction!]
- 2^{n+2} is $O(2^n)$ [$c = 4$, $n_0 = 1$]
- $2n + 100 \cdot logn$ is $O(n)$ [$c = 102$, $n_0 = 1$]

Using Asymptotic Upper Bound Accurately

- Use the big-Oh notation to characterize a function (of an algorithm's running time) *as closely as possible*.

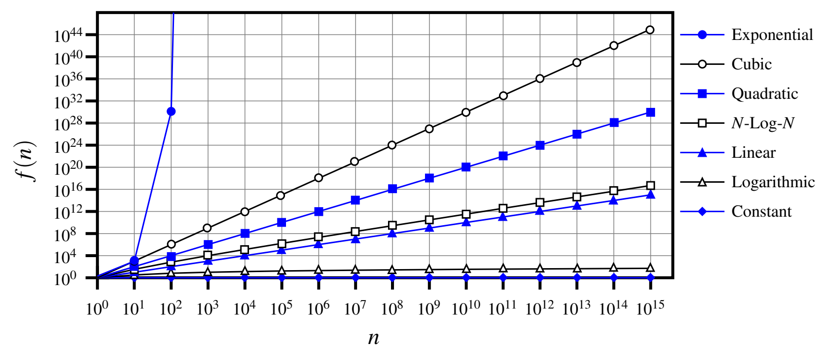
  For example, say $f(n) = 4n^3 + 3n^2 + 5$:
  - Recall: $O(n^3) \subset O(n^4) \subset O(n^5) \subset \dots$
  - It is the *most accurate* to say that $f(n)$ is $O(n^3)$.
  - It is ***true***, but not very useful, to say that $f(n)$ is $O(n^4)$ and that $f(n)$ is $O(n^5)$.
  - It is ***false*** to say that $f(n)$ is $O(n^2)$, $O(n)$, or $O(1)$.
- Do not include *constant factors* and *lower-order terms* in the big-Oh notation.

  For example, say $f(n) = 2n^2$ is $O(n^2)$, do not say $f(n)$ is $O(4n^2 + 6n + 9)$.

Classes of Functions

upper bound	class	cost
$O(1)$	constant	*cheapest*
$O(log(n))$	logarithmic	
$O(n)$	linear	
$O(n \cdot log(n))$	"n-log-n"	
$O(n^2)$	quadratic	
$O(n^3)$	cubic	
$O(n^k)$, $k \geq 1$	polynomial	
$O(a^n)$, $a > 1$	exponential	*most expensive*

Rates of Growth: Comparison



Upper Bound of Algorithm: Example (1)

```
maxOf (int x, int y) {
  int max = x;
  if (y > x) {
    max = y;
  }
  return max;
}
```

- # of primitive operations: 4
  2 assignments + 1 comparison + 1 return = 4
- Therefore, the running time is $O(1)$.
- That is, this is a *constant-time* algorithm.

Upper Bound of Algorithm: Example (2)

```
findMax (int[] a, int n) {
  currentMax = a[0];
  for (int i = 1; i < n; ) {
    if (a[i] > currentMax) {
      currentMax = a[i]; }
    i ++ }
  return currentMax; }
```

- From last lecture, we calculated that the # of primitive operations is $7n - 2$.
- Therefore, the running time is $O(n)$.
- That is, this is a *linear-time* algorithm.

Upper Bound of Algorithm: Example (3)

```
1 containsDuplicate (int[] a, int n) {
2   for (int i = 0; i < n; ) {
3     for (int j = 0; j < n; ) {
4       if (i != j && a[i] == a[j]) {
5         return true; }
6       j ++; }
7     i ++; }
8   return false; }
```

- Worst case is when we reach Line 8.
- # of primitive operations $\approx c_1 + n \cdot n \cdot c_2$, where c_1 and c_2 are some constants.
- Therefore, the running time is $O(n^2)$.
- That is, this is a *quadratic* algorithm.

Upper Bound of Algorithm: Example (4)

```
sumMaxAndCrossProducts (int[] a, int n) {
  int max = a[0];
  for(int i = 1; i < n;) {
    if (a[i] > max) { max = a[i]; }
  }
  int sum = max;
  for (int j = 0; j < n; j ++) {
    for (int k = 0; k < n; k ++) {
      sum += a[j] * a[k]; } }
  return sum; }
```

- # of primitive operations $\approx (c_1 \cdot n + c_2) + (c_3 \cdot n \cdot n + c_4)$, where c_1, c_2, c_3, and c_4 are some constants.
- Therefore, the running time is $O(n + n^2) = O(n^2)$.
- That is, this is a *quadratic* algorithm.

Upper Bound of Algorithm: Example (5)

```
triangularSum (int[] a, int n) {
  int sum = 0;
  for (int i = 0; i < n; i ++) {
    for (int j = i; j < n; j ++) {
      sum += a[j]; } }
  return sum; }
```

- # of primitive operations $\approx n + (n-1) + \cdots + 2 + 1 = \frac{n \cdot (n+1)}{2}$
- Therefore, the running time is $O(\frac{n^2+n}{2}) = O(n^2)$.
- That is, this is a *quadratic* algorithm.

Basic Data Structure: Arrays

- An array is a sequence of indexed elements.
- *Size* of an array is **fixed** at the time of its construction.
- Supported *operations* on an array:
  - *Accessing*: e.g., `int max = a[0];`
    Time Complexity: *O(1)* [constant operation]
  - *Updating*: e.g., `a[i] = a[i + 1];`
    Time Complexity: *O(1)* [constant operation]
  - *Inserting/Removing*:

```
String[] insertAt(String[] a, int n, String e, int i)
  String[] result = new String[n + 1];
  for(int j = 0; j <= i - 1; j ++){ result[j] = a[j]; }
  result[i] = e;
  for(int j = i + 1; j <= n - 1; j ++){ result[j] = a[j-1]; }
  return result;
```

Time Complexity: *O(n)* [linear operation]

Array Case Study: Comparing Two Sorting Strategies

- Problem:
  Input: An array a of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$
  Output: A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$
- We propose two *alternative implementation strategies* for solving this problem.
- At the end, we want to know which one to choose, based on *time complexity*.

Sorting: Strategy 1 – Selection Sort

- Maintain a (initially empty) *sorted portion* of array *a*.
- From left to right in array *a*, select and insert *the minimum element* to the end of this sorted portion, so it remains sorted.

```
1 selectionSort(int[] a, int n)
2   for (int i = 0; i <= (n - 2); i ++)
3     int minIndex = i;
4     for (int j = i; j <= (n - 1); j ++)
5       if (a[j] < a[minIndex]) { minIndex = j; }
6     int temp = a[i];
7     a[i] = a[minIndex];
8     a[minIndex] = temp;
```

- How many times does the body of *for loop* (Line 4) run?
- Running time? [$O(n^2)$]

$$\underbrace{n}_{\text{find } \{a[0],\ \ldots,\ a[n-1]\}} + \underbrace{(n-1)}_{\text{find } \{a[1],\ \ldots,\ a[n-1]\}} + \cdots + \underbrace{2}_{\text{find } \{a[n-2],\ a[a[n-1]]\}}$$

- So selection sort is a *quadratic-time algorithm*.

Sorting: Strategy 2 – Insertion Sort

- Maintain a (initially empty) *sorted portion* of array *a*.
- From left to right in array *a*, insert *one element at a time* into the "right" spot in this sorted portion, so it remains sorted.

```
1 insertionSort(int[] a, int n)
2   for (int i = 1; i < n; i ++)
3     int current = a[i];
4     int j = i;
5     while (j > 0 && a[j - 1] > current)
6       a[j] = a[j - 1];
7       j --;
8     a[j] = current;
```

- *while loop* (L5) exits when? `j <= 0` or `a[j - 1] <= current`
- Running time? [$O(n^2)$]

$$O(\underbrace{1}_{\text{insert into \{a[0]\}}} + \underbrace{2}_{\text{insert into \{a[0], a[1]\}}} + \cdots + \underbrace{(n-1)}_{\text{insert into \{a[0], ..., a[n-2]\}}})$$

- So insertion sort is a *quadratic-time algorithm*.

Sorting: Alternative Implementations?

- In the Java implementations for *selection* sort and *insertion* sort, we maintain the "sorted portion" from the *left* end.
  - For *selection* sort, we select the *minimum* element from the "unsorted portion" and insert it to the *end* in the "sorted portion".
- For *insertion* sort, we choose the *left-most* element from the "unsorted portion" and insert it at the "*right spot*" in the "sorted portion".
- **Question:** Can we modify the Java implementations, so that the "sorted portion" is maintained and grown from the *right* end instead?

Comparing Insertion & Selection Sorts

- *Asymptotically*, running times of selection sort and insertion sort are both $O(n^2)$.
- We will later see that there exist better algorithms that can perform better than quadratic: $O(n \cdot logn)$.

Index (1)

Index (2)

Index (3)