Exam Questions for Practice

1. Let $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ be some hypothesis class over some domain $\mathcal{X}$.
   (a) **Prove or refute** If $\mathcal{H}$ is learnable, then for all $h \in \{0,1\}^{\mathcal{X}}$, the class $\mathcal{H} \cup \{h\}$ is learnable.
   (b) **Prove or refute** If $\text{VC}(\mathcal{H}) = d$, then $\mathcal{H}$ shatters all domain subsets $C \subseteq \mathcal{X}$ with $|C| \leq \log d$.
   (c) Give a dataset in $\mathbb{R}^2$ on which the perceptron algorithm will not halt.
   (d) **Prove or refute** Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be convex functions and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha g + \beta f$ is also convex.

2. Let the domain $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^2 \ : \ x_1 \geq 0, x_2 \geq 0\}$ be the positive quadrant of the real plane. Consider the following two classes over $\mathcal{X}$. Let H_r be the class of rectangles whose lower left corner is the origin. That is,

$$H_r = \{h_{a,b} \ : \ a, b \in \mathbb{R}, \ a, b \geq 0\}$$

   with

$$h_{a,b}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } x_1 \leq a \text{ and } x_2 \leq b \\ 0 & \text{else.} \end{cases}$$

   Let H_{ds} be the class of positive decision stumps. That is,

$$H_{ds} = \{h_{c,d,i} \ : \ c, d \in \mathbb{R}, \ c, d \geq 0, \ i \in \{1,2\}\}$$

   with

$$h_{c,d,i}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } i = 1 \text{ and } x_1 \leq c \\ 1 & \text{if } i = 2 \text{ and } x_2 \leq d \\ 0 & \text{else.} \end{cases}$$

   (a) What is the VC-dimension of H_r?
   (b) What is the VC-dimension of H_{ds}?

3. Let $\mathcal{X}$ be some domain. Consider the learner *Memorize*. On a sample S, it will output a function that agrees with the label of the sample points, and is elsewhere (on points not in the sample) constant 0. If a point x appears multiple times in the sample, *Memorize* outputs the majority label with which it appears (and breaks ties to 0).

   (a) Let $\mathcal{H}_k$ be the class of functions that map at most k elements of the domain to 1. Show that *Memorize* (PAC)-learns (satisfies the definition of learnability for) $\mathcal{H}_k$ in the realizable case.

   (b) Give an example of a class, that is learnable, but not with *Memorize* (that is, for this class *Memorize* does not satisfy the definition of learnability).