Exercises 1

Disussion of solutions: Monday, Oct 2 in class

Let $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ be some hypothesis class over some domain $\mathcal{X}$.

1. **Empirical Risk Minimization**
   Recall that a learner $\mathcal{A}$ is an Empirical Risk Minimizer (ERM) for $\mathcal{H}$ if for all samples $S \in \bigcup_{i=1}^{\infty}(\mathcal{X} \times \{0,1\})^i$, it outputs a function from $\mathcal{H}$ of minimal empirical risk:
   $$\mathcal{A}(S) \in \text{argmin}_{h \in H} L_n(h).$$
   Describe two different learners that are ERM for the class $\mathcal{H}_{\text{rec}}$ of axis aligned rectangles over $\mathcal{X} = \mathbb{R}^d$, defined as
   $$\mathcal{H}_{\text{rec}} := \{h_b \ : \ \boldsymbol{b} \in \mathbb{R}^{2d}\}$$
   where $h_b(\boldsymbol{x}) = 1$ if and only if $x_i \in [b_i, b_{d+i}]$.

2. **Learnability**
   Show that the class of singletons $\mathcal{H}_{\text{sing}}$ is learnable in the realizable case. $\mathcal{H}_{\text{sing}}$ is defined as
   $$\mathcal{H}_{\text{sing}} = \{h \in \{0,1\}^{\mathcal{X}} \ : \ |\{x \in \mathcal{X} \ : \ h(x) = 1\}| \leq 1\}$$

3. **VC-dimension**

(a) Let
$$\mathcal{H}_{\leq \mathrm{k}} = \{h \in \{0,1\}^{\mathcal{X}} \ : \ |\{x \in \mathcal{X} \ : \ h(x) = 1\}| \leq k\}.$$
What is the VC-dimension of $\mathcal{H}_{\leq \mathrm{k}}$? How does it depend on the cardinality of the domain $\mathcal{X}$? Prove your claims.

(b) Let
$$\mathcal{H}_{\mathrm{k}} = \{h \in \{0,1\}^{\mathcal{X}} \ : \ |\{x \in \mathcal{X} \ : \ h(x) = 1\}| = k\}.$$
What is the VC-dimension of $\mathcal{H}_{\mathrm{k}}$? How does it depend on the cardinality of the domain $\mathcal{X}$? Prove your claims.

(c) Show that adding one function to a hypothesis class can increase the VC-dimension by at most one. That is, for any hypothesis class $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ and any $h \in \{0,1\}^{\mathcal{X}}$, we have
$$\mathrm{VC}(H \cup \{h\}) \leq \mathrm{VC}(H) + 1.$$

(d) Show that the inequality in the above question can be tight. That is, give an example of a class $\mathcal{H}$ over some domain and a function h with
$$\mathrm{VC}(H \cup \{h\}) = \mathrm{VC}(H) + 1.$$