More Examples of Hilbert-style proofs

I give you here a couple of Hilbert-style proofs for "visual practice". Of course, the *best practice* is when you prove things yourselves, not just reading other people's proofs. By the way, I use "□" to mark the end of a proof.

A.1 "Distributivity" (This is 8.15 in the GS text).

$$\vdash (\forall x)(A \Rightarrow B) \wedge (\forall x)(A \Rightarrow C) \equiv (\forall x)(A \Rightarrow B \wedge C) \tag{1}$$

In GS's notation—recall the translation: $(\forall x|A : B)$ stands for $(\forall x)(A \Rightarrow B)$—this is

$$\vdash (\forall x|A : B) \wedge (\forall x|A : C) \equiv (\forall x|A : B \wedge C)$$

Taking A (range) to be the formula *true* we have the special case mentioned in our "Toolbox", namely,

$$\vdash (\forall x)B \wedge (\forall x)C \equiv (\forall x)(B \wedge C) \tag{2}$$

Let us prove (1). We split $\equiv$ in two directions and use the DThm in each.

($\Rightarrow$ direction)

1.	$(\forall x)(A \Rightarrow B) \wedge (\forall x)(A \Rightarrow C)$	$\langle$assume$\rangle$
2.	$(\forall x)(A \Rightarrow B)$	$\langle$1. and taut. implication$\rangle$
3.	$(\forall x)(A \Rightarrow C)$	$\langle$1. and taut. implication$\rangle$
4.	$A \Rightarrow B$	$\langle$2. and *specialization*$\rangle$
5.	$A \Rightarrow C$	$\langle$3. and *specialization*$\rangle$
6.	$A \Rightarrow B \wedge C$	$\langle$4., 5. and taut. implication$\rangle$
7.	$(\forall x)(A \Rightarrow B \wedge C)$	$\langle$6. and generalization; OK: no free x in 1.$\rangle$

By the Deduction Theorem, we are done.

($\Leftarrow$) With amended "annotation", the above proof can be reversed (7.–1.) □

A.2 (8.16)–(8.18) in GS boil down to just (8.18) if "$*$" is "$\forall$". GS call (8.18) "Range split ". This is

$$\vdash (\forall x)(A \vee B \Rightarrow C) \equiv (\forall x)(A \Rightarrow C) \wedge (\forall x)(B \Rightarrow C)$$

To prove the above we again split $\equiv$ and use the DThm for each direction. Again we show only one direction as the other is entirely similar.

($\Rightarrow$)

1.	$(\forall x)(A \vee B \Rightarrow C)$	$\langle$assume$\rangle$
2.	$A \vee B \Rightarrow C$	$\langle$1. and *specialization*$\rangle$
3.	$A \Rightarrow C$	$\langle$2. and taut. implication$\rangle$
4.	$B \Rightarrow C$	$\langle$2. and taut. implication$\rangle$
5.	$(\forall x)(A \Rightarrow C)$	$\langle$3. and generalization; OK: no free x in 1.$\rangle$
6.	$(\forall x)(B \Rightarrow C)$	$\langle$4. and generalization; OK: no free x in 1.$\rangle$
7.	$(\forall x)(A \Rightarrow C) \wedge (\forall x)(B \Rightarrow C)$	$\langle$5., 6. and taut. implication$\rangle$

By the Deduction Theorem, we are done.

($\Leftarrow$) Reverse the above proof. □

A.3 The following is a famous result of Bertrand Russell's:

Let P be any predicate of **arity 2*** (this could be anything: E.g., $=, <, >, \leq, \in$)

Russell proved that the following is an *absolute theorem* (provable *without* any nonlogical assumptions—in particular, no axioms about P are needed)

$$\neg(\exists y)(\forall x)(P(x, y) \equiv \neg P(x, x)) \tag{3}$$

Now (3) is tautologically equivalent† to

$$(\exists y)(\forall x)(P(x, y) \equiv \neg P(x, x)) \equiv \mathit{false} \tag{4}$$

and since $\vdash \mathit{false} \Rightarrow A$ (Why?), to show (4) I only need to show

$$(\exists y)(\forall x)(P(x, y) \equiv \neg P(x, x)) \Rightarrow \mathit{false} \tag{5}$$

I prove (5) using the DThm:

1.	$(\exists y)(\forall x)(P(x, y) \equiv \neg P(x, x))$	$\langle$assume$\rangle$
2.	$(\forall x)(P(x, z) \equiv \neg P(x, x))$	$\langle$by 1, add new assumption with z *new*$\rangle$
3.	$P(z, z) \equiv \neg P(z, z)$	$\langle$2. and Axiom 2 (using z for "t")$\rangle$
4.	*false*	$\langle$3. and taut. implication$\rangle$

To sum up "in slow motion", the proof 1–4 establishes

$$1., 2. \vdash \mathit{false}$$

*Recall that "arity" is a word that mathematicians made up. It denotes the number of arguments that are syntactically appropriate for a function or predicate. It came from words such as "bin**ary**", "tern**ary**" (three argument slots), "n-ary".

†"A is tautologically equivalent to B" means $\models_{\text{taut}} A \equiv B$.

But z is in neither in 1. nor in *false*, thus, by the Auxiliary Variable Metatheorem, we have also $1. \vdash$ *false*. The DThm immediately gives (5). $\Box$

Why is (3) famous? Well, if you choose P to be specifically the "is a member of" predicate of set theory, "$\in$", then we have—in particular—proved that

$$(\exists y)(\forall x)(x \in y \equiv \neg x \in x) \tag{6}$$

is a contradiction; or as we say *refutable*‡.

But (6), in plain English, says "There *is* a set (y) whose members (x) are precisely those objects that *are not members of themselves*". Russell's result of the refutability of (6) means that no such set exists. (More on this when we do set theory).

‡The negation is provable.