Facts-List

Here I list some tools/facts from old M1090 classes I taught, which you can use in M2090 (and beyond) “off the shelf”, without proof.

The list is long, but hopefully useful. **The axioms of arithmetic and set theory are at the end.**

The following metatheorems are good for ***both*** Propositional (Ch. 3–4) and Predicate Calculus (Ch. 8–9):

1. *Redundant True.* $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv true$

   “Redundant true” is *very* convenient. Make a habit of using it. But do be careful! “*true*” is a “meaningless symbol”,[*] *not* the truth value **t** (also pronounced “true”) of the metatheory.

2. *Modus Ponens (MP).* $A, A \Rightarrow B \vdash B$
3. *Cut Rule.* $A \vee B, \neg A \vee C \vdash B \vee C$
4. *Deduction Theorem.* If $\Gamma, A \vdash B$, then $\Gamma \vdash A \Rightarrow B$
5. *Proof by contradiction.* $\Gamma, \neg A \vdash false$ iff $\Gamma \vdash A$
6. *Post’s Theorem.* (Also called “tautology theorem”, or even “completeness of Propositional Calculus theorem”)

   If $\models_{\text{taut}} A$, then $\vdash A$.

   Also: If $B_1, \ldots, B_n \models_{\text{taut}} A$, then $B_1, \ldots, B_n \vdash A$
7. *Proof by cases.* $A \Rightarrow B, C \Rightarrow D \vdash A \vee C \Rightarrow B \vee D$

   Also the special case: $A \Rightarrow B, C \Rightarrow B \vdash A \vee C \Rightarrow B$

Recall that if A is a formula and $x_1, \ldots x_n$, where $n \geq 0$, are *ANY* variables (occurring or not occurring free in A, we don’t care) then $(\forall x_1)(\forall x_2)\ldots(\forall x_n)A$ is called a “*partial generalisation*” of A. If $n = 0$, then the “prefix” $(\forall x_1)(\forall x_2)\ldots(\forall x_n)$ is empty.[†] Thus A is one of the partial generalizations of A. Example: Consider $x < y$ (where $<$ is some nonlogical predicate of arity 2). Here I list some partial generalizations of that formula:

$x < y$

[*] Yes, it is *one* (multiple character) symbol, just like the symbol “*else*” of a programming language like Pascal. As I mentioned in class, some people go out of their way to emphasise that *true* and *false* are just meaningless symbols—not “values”—and write instead $\top$ and $\bot$ respectively. We will not use $\top$ and $\bot$, but we must always remember that “$\Gamma \vdash A \equiv true$” is ***not*** pronounced “A ***is*** true” but “Γ ***proves*** that A is ***equivalent*** to the ***formula*** *true*”.

[†] This is a standard convention: A sequence $x_1, \ldots, x_n$, where $n \geq 0$, is “x_1” if $n = 1$, “x_1, x_2, x_3” if $n = 3$, etc. It is empty by convention, if $n = 0$, the thinking being that we have stopped listing the sequence before we started. So, nothing is listed.

$(\forall z)x < y$
$(\forall x)x < y$
$(\forall y)(\forall x)x < y$
$(\forall x)(\forall y)x < y$
$(\forall x)(\forall x)(\forall x)(\forall x)x < y.$

As far as we—i.e., this class—are concerned, the following are ***the*** axioms for Ch.9 and 8—***NOT*** the ones listed in G & S:

Any partial generalization of any formula in groups* Ax1–Ax6 *is an axiom for Predicate Calculus.

Groups **Ax1–Ax6** contain:

Ax1. All tautologies.

Ax2. For every formula A, $(\forall x)A \Rightarrow A[x := t]$, for any term t.

Ax3. For every formula A and variable x *not free in* A, the formula $A \Rightarrow (\forall x)A$.

Ax4. For every formulas A and B, $(\forall x)(A \Rightarrow B) \Rightarrow (\forall x)A \Rightarrow (\forall x)B$.

Ax5. For *each* object variable x, the formula $x = x$.

Ax6. (*Leibniz's characterisation of equality—1st order version. "3.83"*) For any formula A, any object variable x and any terms t, s, the formula $t = s \Rightarrow (A[x := t] \equiv A[x := s])$.

Primary rules of inference are **Equanimity** and **PSL** in ***both*** Ch.3 and Ch.9.

$$\frac{A \equiv B}{C[p := A] \equiv C[p := B]}, \text{ provided } p \text{ is } \textbf{not} \text{ in the scope of a quantifier.} \tag{PSL}$$

Translations

$(\exists x)A$ translates to $\neg(\forall x)\neg A$

$(\forall x|A : B)$ translates to $(\forall x)(A \Rightarrow B)$ (Range trading with $\forall$)

$(\exists x|A : B)$ translates to $(\exists x)(A \wedge B)$ (Range trading with $\exists$)

Useful facts from Predicate Calculus (proved in class—you may use them without proof):

We ***know*** that SLCS, WLUS (as well as GS-Leibniz "8.12(a)" and "8.12(b)") are **derived rules**. These are the following (I am using "GS"-notation for 8.12(a–b)):

$$\text{Same as PSL, without the condition: } A \equiv B \vdash C[p := A] \equiv C[p := B] \qquad (SLCS)$$

$$\text{if } \vdash A \equiv B, \text{ then } \vdash C[p \setminus A] \equiv C[p \setminus B] \qquad (WLUS)$$

$$\text{if } \vdash A \equiv B, \text{ then } \vdash \big(*\, x | C[p := A] : D\big) \equiv \big(*\, x | C[p := B] : D\big) \qquad (8.12(a))$$

$$\text{if } \vdash D \Rightarrow (A \equiv B), \text{ then } \vdash \big(*\, x | D : C[p := A]\big) \equiv \big(*\, x | D : C[p := B]\big) \qquad (8.12(b))$$

where in 8.12(a–b) "$*$" stands everywhere for the symbol "$\forall$", or the symbol "$\exists$".

▶ More "rules" and (meta)theorems. (Only the "$\forall$-versions" are listed. This should help you ***remember*** the "$\exists$-versions" that were also covered in the M1090 class.):

(i)

$$\vdash A \equiv (\forall x)A, \text{ **provided** } x \text{ is not free in } A$$

$$\vdash A \equiv (\exists x)A, \text{ **provided** } x \text{ is not free in } A$$

(ii) *Dummy renaming.*

If z does not occur in $(\forall x)A$ as either free or bound, then $\vdash (\forall x)A \equiv (\forall z)\big(A[x := z]\big)$

If z does not occur in $(\exists x)A$ as either free or bound, then $\vdash (\exists x)A \equiv (\exists z)\big(A[x := z]\big)$

(iii) *$\forall$ over $\circ$ distribution, where $\circ$ is "$\vee$" or "$\Rightarrow$".*

$$\vdash A \circ (\forall x)B \equiv (\forall x)(A \circ B), \text{ **provided** } x \text{ is not free in } A$$

$\exists$ over $\wedge$ distribution

$$\vdash A \wedge (\exists x)B \equiv (\exists x)(A \wedge B), \text{ **provided** } x \text{ is not free in } A$$

(iv) *$\forall$ over $\wedge$ distribution.*

$$\vdash (\forall x)A \wedge (\forall x)B \equiv (\forall x)(A \wedge B)$$

$\exists$ over $\vee$ distribution.

$$\vdash (\exists x)A \vee (\exists x)B \equiv (\exists x)(A \vee B)$$

(v) *$\forall$ commutativity (symmetry).*

$$\vdash (\forall x)(\forall y)A \equiv (\forall y)(\forall x)A$$

$\exists$ commutativity (symmetry).

$$\vdash (\exists x)(\exists y)A \equiv (\exists y)(\exists x)A$$

(vi) *Specialization. Follows from* **Ax2** *and MP.* $(\forall x)A \vdash A[x := t]$, for any term t.

(vii) *Generalization.* If $\Gamma \vdash A$ and if, moreover, the formulas in Γ have **no free** x **occurrences**, then also $\Gamma \vdash (\forall x)A$.

(viii) $\forall$ *Monotonicity.* If $\Gamma \vdash A \Rightarrow B$ so that the formulas in Γ have **no free** x **occurrences**, then we can infer

$$\Gamma \vdash (\forall x)A \Rightarrow (\forall x)B$$

$\exists$ *Monotonicity.* If $\Gamma \vdash A \Rightarrow B$ so that the formulas in Γ have **no free** x **occurrences**, then we can infer

$$\Gamma \vdash (\exists x)A \Rightarrow (\exists x)B$$

(ix) $\forall$ *Introduction; a special case of* $\forall$ *Monotonicity that uses (i) above.* If $\Gamma \vdash A \Rightarrow B$ so that neither the formulas in Γ nor A have **any free** x **occurrences**, then we can infer

$$\Gamma \vdash A \Rightarrow (\forall x)B$$

$\exists$ *Introduction; a special case of* $\exists$ *Monotonicity that uses (i) above.* If $\Gamma \vdash A \Rightarrow B$ so that neither the formulas in Γ nor B have **any free** x **occurrences**, then we can infer

$$\Gamma \vdash (\exists x)A \Rightarrow B$$

(x) *Super-WLUS or sWLUS.* If $\Gamma \vdash A \equiv B$ so that the formulas in Γ have **no free variables**, then we can infer

$$\Gamma \vdash C[p \setminus A] \equiv C[p \setminus B]$$

where C is any formula and p is a Boolean variable.

(xi) (*Equals-for-equals in terms*) For any terms t, s, t' and variable x,

$$\vdash t = t' \Rightarrow s[x := t] = s[x := t']$$

(xii) Finally, the *Auxiliary Variable ("witness") Metatheorem.* If $\Gamma \vdash (\exists x)A$, and if y is a variable that ***does not*** occur as either free or bound variable in any of A or B or the formulas of Γ, then

$$\Gamma, A[x := y] \vdash B \text{ implies } \Gamma \vdash B$$

Semantics facts

Propositional Calculus	Predicate Calculus
(Boolean Soundness) $\vdash A$ implies $\models_{\text{taut}} A$ (Post) $\models_{\text{taut}} A$ implies $\vdash A$	$\vdash A$ does **NOT** imply $\models_{\text{taut}} A$ However, (**Ax1**) $\models_{\text{taut}} A$ implies $\vdash A$ (Pred. Calc. Soundness) $\vdash A$ implies $\models A$ (Gödel Completeness) $\models A$ implies $\vdash A$

CAUTION! The above facts/tools are only a fraction of what one should have seen in M1090. They are ***very important and very useful***, and that is why I list them for your easy reference here.

You can still use *without proof* ***ALL*** the things one sees in M1090, such as "one-point-rule", "deMorgan's laws", etc. ***Subject to the following constraints:***

(1) AXIOMS for predicate calculus are those listed here, ***NOT*** those listed in Ch.8–9 in G & S. Happily, the latter are *theorems* for us, and *can* be used. But don't quote them as *axioms*!

(2) *Forget* all the "facts" in Ch.8 involving "$*$", ***UNLESS*** you interpret "$*$" *exclusively* as one of $\exists$ or $\forall$. NO OTHER INTERPRETATIONS (e.g., "$+, \cup, \cap$") of "$*$" lead to FACTS OF PURE LOGIC (because $+$, $\cup$, etc. are ***non logical symbols*** and one needs *nonlogical axioms* **before** discussing them!)

Peano Arithmetic Axioms

These are the **universal closures** of the following formulas ((Ind) is a schema):

(S)**1.** $\neg 0 = Sx$

(S)**2.** $Sx = Sy \Rightarrow x = y$ ("1-1ness of S")

$(+)$**1.** $x + 0 = x$

$(+)$**2.** $x + Sy = S(x + y)$

$(\cdot)$**1.** $x \cdot 0 = 0$

$(\cdot)$**2.** $x \cdot Sy = x \cdot y + x$

$(<)$**1.** $\neg x < 0$

$(<)$**2.** $x < Sy \equiv x < y \vee x = y$

$(<)$**3.** $x < y \vee x = y \vee y < x$

And the **Induction Schema**, one axiom for each formula A:

(Ind) $\quad A[x := 0] \wedge (\forall x)(A \Rightarrow A[x := Sx]) \Rightarrow A$

CVI (Course-of-Values Induction) is a derived schema, the following

(CVI) $\quad (\forall x)((\forall z < x)A[z] \Rightarrow A[x]) \Rightarrow (\forall x)A[x]$

(Ind) is applied by proving $A[0]$ (**Basis**) and then $A[Sx]$ (or informally written $A[x + 1]$—this is the **goto step**) by assuming $A[x]$ (the **I.H.**).

(CVI) is applied by assuming $(\forall z < x)A[z]$ (**I.H.**) and then proving $A[x]$ (**goto**). It is important to do the "boundary cases" (Basis cases) during this step. These are cases that are not helped by the I.H.

Set Theory

First off, for any set-type variable S formula A and typeless variable x (i.e., of type set or atom), $S = \{x|A[x]\}$ is short for $(\forall x)(x \in S \equiv A[x])$. From this we get the provable principle of $\in$-elimination:

$$ST \vdash t \in \{x|A\} \equiv A[x := t], \text{ for any term } t$$

Formally, "$\{x|A\}$ is a set" is captured by $(\exists y)y = \{x|A\}$ where y is of type set. Using the above convention we can eliminate "$\{\ldots\}$" and write

$$(\exists y)(\forall x)(x \in y \equiv A)$$

The axioms of set theory *that we covered* are the **universal closures** of the formulas that express the following statements.[‡] Note that, deliberately, I have *not* introduced three important axioms that I consider "non-elementary": The axiom of choice, the axiom of replacement, and the axiom of infinity. These three do not appear below.

Axiom1. Extensionality: For all variables S, T of type set, and all typeless variables x,

$$(\forall x)(x \in S \equiv x \in T) \Rightarrow S = T$$

Axiom2. Empty Set: $\{x|\mathit{false}\}$ is a set denoted by $\emptyset$

Axiom3. Atoms contain no members: If x is of type atom and y is typeless, then we have $\neg(\exists y)y \in x$

Axiom4. Subsets: Any subset of a set is a set.

Axiom5. Pair: For any sets or atoms x and y, $\{x, y\}$ is a set

Axiom6. Union: For any sets S and T their union is a set. For any family of sets F, its union $\bigcup F$ is a set

Axiom7. Foundation: It is impossible to have an infinite descending chain

$$\ldots \in a''' \in a'' \in a \in a$$

Axiom8. Power Set: For any set S, $\{x|x \subseteq S\}$ is a set

[‡]OK, some are already formulas