Lecture #7 (Sept. 30)

0.0.1 Corollary. *If $\Gamma \vdash A$ and also $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash B$.*

In words, the conclusion says that A drops out as a hypothesis and we get $\Gamma \vdash B$.

That is, a THEOREM A can be <u>invoked</u> just like an axiom in a proof!

Proof. We have *two* proofs:

$$\boxed{\overbrace{\ldots A}^{\text{from } \Gamma}}$$

and

$$\boxed{\overbrace{\ldots A \ldots B}^{\text{from } \Gamma \cup \{A\}}}$$

When the second box is *standalone*, the justification for A is “hyp”.

Now concatenate the two proofs above in the order

$$\boxed{\overbrace{\ldots A}^{\text{from } \Gamma}}\boxed{\overbrace{\ldots A \ldots B}^{\text{from } \Gamma \cup \{A\}}}$$

Now change all the justifications for that A in the right box from “hyp” to the same exact reason you gave to the A in box one.

Thus, the status of A as “hyp” is removed and B is proved from Γ alone. □

0.0.2 Corollary. If $\Gamma \cup \{A\} \vdash B$ and $\vdash A$, then $\Gamma \vdash B$.

Proof. By hyp strengthening, I have $\Gamma \vdash A$. Now apply the previous theorem. $\square$

0.0.3 Theorem. $A \equiv B \vdash B \equiv A$

Proof.

$$\begin{array}{lll} (1) & A \equiv B & \langle \text{hyp} \rangle \\ (2) & (A \equiv B) \equiv (B \equiv A) & \langle \text{axiom} \rangle \\ (3) & B \equiv A & \langle \text{ (1,2) + Eqn} \rangle \end{array}$$

0.0.4 Theorem. $\vdash (A \equiv (B \equiv C)) \equiv ((A \equiv B) \equiv C)$

NOTE. This is the mirror image of Axiom (1).

Proof.

$$
\begin{array}{ll}
(1) & ((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C)) \ \langle \text{axiom} \rangle \\
(2) & (A \equiv (B \equiv C)) \equiv ((A \equiv B) \equiv C) \ \langle (1)\text{+}0.0.3 \rangle
\end{array}
$$
□

0.0.5 Remark. Thus, *in a chain of two "≡" we can shift brackets from left to right (axiom) but also right to left (above theorem).*

So it does not matter how brackets are inserted in such chain.

An induction proof on chain length (see course URL) extends this remark to any chain of "≡", of any length. □

0.0.6 Theorem. (The other (Eqn)) $B, A \equiv B \vdash A$

Proof.

$$\begin{array}{lll} (1) & B & \langle \text{hyp} \rangle \\ (2) & A \equiv B & \langle \text{hyp} \rangle \\ (3) & B \equiv A & \langle (2) + 0.0.3 \rangle \\ (4) & A & \langle (1, 3) + (Eqn) \rangle \end{array}$$

□

Lecture #8 (Oct. 2)

0.0.7 Corollary. $\vdash \top$

Proof.

$$\begin{array}{lll} (1) & \top \equiv \bot \equiv \bot & \langle \text{axiom} \rangle \\ (2) & \bot \equiv \bot & \langle \text{theorem} \rangle \\ (3) & \top & \langle (1, 2) + (Eqn) \rangle \end{array}$$

□

0.0.8 Theorem. $\vdash A \equiv A \equiv B \equiv B$

$$
\begin{array}{ll}
(1) & (A \equiv B \equiv B) \equiv A \ \langle \text{axiom} \rangle \\
(2) & A \equiv (A \equiv B \equiv B) \ \langle (1) + 0.0.3 \rangle
\end{array}
$$
□

0.0.9 Corollary. $\vdash \bot \equiv \bot \equiv B \equiv B$ *and* $\vdash A \equiv A \equiv \bot \equiv \bot$

NOTE *absence of brackets in theorem AND corollary!*

0.0.10 Corollary. (Redundant $\top$) *$\vdash \top \equiv A \equiv A$ and $\vdash A \equiv A \equiv \top$.*

Proof.

$$
\begin{array}{lll}
(1) & \top \equiv \bot \equiv \bot & \langle \text{axiom} \rangle \\
(2) & \bot \equiv \bot \equiv A \equiv A & \langle \text{absolute theorem} \rangle \\
(3) & \top \equiv A \equiv A & \langle (Trans) + (1,\, 2) \rangle
\end{array}
$$

0.0.11 Metatheorem. *For any Γ and A, we have $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$.*

Proof. Say $\Gamma \vdash A$.

Thus

$$\begin{array}{lll} & \Gamma & \\ & \vdots & \\ (1) & A & \langle \Gamma\text{-theorem} \rangle \\ (2) & A \equiv A \equiv \top & \langle \text{theorem} \rangle \\ (3) & A \equiv \top & \langle (1, 2) + \text{Eqn} \rangle \end{array}$$

The other direction is similar. □

EQUATIONAL PROOFS

Example from high school trigonometry.

Prove that $1 + (\tan x)^2 = (\sec x)^2$ given the identities

$$\tan x = \frac{\sin x}{\cos x} \tag{i}$$

$$\sec x = \frac{1}{\cos x} \tag{ii}$$

$$(\sin x)^2 + (\cos x)^2 = 1 \text{ (Pythagoras' Theorem)} \tag{iii}$$

Equational proof with annotation

$$\begin{aligned}
& 1 + (\tan x)^2 \\
= & \langle \text{by } (i) \rangle \\
& 1 + (\sin x / \cos x)^2 \\
= & \langle \text{arithmetic} \rangle \\
& \frac{(\sin x)^2 + (\cos x)^2}{(\cos x)^2} \\
= & \langle \text{by } (iii) \rangle \\
& \frac{1}{(\cos x)^2} \\
= & \langle \text{by } (ii) \rangle \\
& (\sec x)^2
\end{aligned} \tag{E}$$

An equational proof looks like:

$$\overbrace{A_1 \equiv A_2}^{\text{reason}}, \overbrace{A_2 \equiv A_3}^{\text{reason}}, \ldots, \overbrace{A_n \equiv A_{n+1}}^{\text{reason}} \tag{1}$$

0.0.12 Metatheorem.

$$A_1 \equiv A_2, A_2 \equiv A_3, \ldots, A_{n-1} \equiv A_n, A_n \equiv A_{n+1} \vdash A_1 \equiv A_{n+1} \tag{2}$$

Proof. *By repeated application of (derived) rule (Trans).*

For example to show the "special case"

$$A \equiv B, B \equiv C, C \equiv D, D \equiv E \vdash A \equiv E \tag{3}$$

the proof is

(1)	$A \equiv B$	$\langle$hyp$\rangle$
(2)	$B \equiv C$	$\langle$hyp$\rangle$
(3)	$C \equiv D$	$\langle$hyp$\rangle$
(4)	$D \equiv E$	$\langle$hyp$\rangle$
(5)	$A \equiv C$	$\langle 1 + 2 +$ Trans$\rangle$
(6)	$A \equiv D$	$\langle 3 + 5 +$ Trans$\rangle$
(7)	$A \equiv E$	$\langle 4 + 6 +$ Trans$\rangle$

For the "general case (2)" do induction on n with Basis at $n = 1$ (see text; better still do it without looking!) □

0.0.13 Corollary. *In an Equational proof (from Γ) like the one in (1) of p.10 we have $\Gamma \vdash A_1 \equiv A_{n+1}$.*

Proof. So we have n Γ-proofs, for $i = 1, \ldots, n$,

$$\boxed{\ldots A_i \equiv A_{i+1}}$$

Concatenate them all to get ONE Γ-proof

$$\boxed{\ldots A_1 \equiv A_2} \ldots \boxed{\ldots A_i \equiv A_{i+1}} \ldots \boxed{\ldots A_n \equiv A_{n+1}}$$

By the DERIVED RULE 0.0.12 the following is a Γ-proof of $A_1 \equiv A_{n+1}$

$$\boxed{\ldots A_1 \equiv A_2} \ldots \boxed{\ldots A_i \equiv A_{i+1}} \ldots \boxed{\ldots A_n \equiv A_{n+1}} \quad A_1 \equiv A_{n+1}$$

□

0.0.14 Corollary. *In an Equational proof (from Γ) like the one in (1) of p.10 we have $\Gamma \vdash A_1$ iff $\Gamma \vdash A_{n+1}$.*

Proof. *From the above Corollary we have*

$$\Gamma \vdash A_1 \equiv A_{n+1} \qquad (\dagger)$$

Now split the "iff" in two directions:

- IF: So we have

$$\Gamma \vdash A_{n+1}$$

  This plus (†) plus Eqn yield $\Gamma \vdash A_1$.

- ONLY IF: So we have

$$\Gamma \vdash A_1$$

  This plus (†) plus Eqn yield $\Gamma \vdash A_{n+1}$.

$\Gamma \vdash A_1 \equiv A_{n+1}$. Telei'wnoume m'esw tou (*Eqn*). □

Equational Proof Layout

Successive equivalences like "$A_i \equiv A_{i+1}$ and $A_{i+1} \equiv A_{i+2}$" we write vertically, without repeating the shared formula A_{i+1}.

WITH annotation in $\langle \ldots \rangle$ brackets

$$\begin{array}{ll} & A_1 \\ \equiv & \langle \text{annotation} \rangle \\ & A_2 \\ \equiv & \langle \text{annotation} \rangle \\ & \vdots \\ & A_{n-1} \\ \equiv & \langle \text{annotation} \rangle \\ & A_n \\ \equiv & \langle \text{annotation} \rangle \\ & A_{n+1} \end{array} \qquad (ii)$$

EXCEPT FOR ONE THING!

(ii) is just ONE FORMULA, namely

$$A_1 \equiv A_2 \equiv \ldots \equiv A_n \equiv A_{n+1}$$

which is NOT the same as (1) of p.10.

For example, "$\top \equiv \bot \equiv \bot$" is NOT the same as "$\top \equiv \bot$ AND $\bot \equiv \bot$"

The former (blue) is true but the latter (red) is false.

What do we do?

We introduce a metasymbol for an equivalence that acts ONLY on two formulas!

Cannot be chained to form ONE formula.

The symbol is "$\Leftrightarrow$" and thus

"$A \Leftrightarrow B \Leftrightarrow C$" MEANS "$A \Leftrightarrow B$ AND $B \Leftrightarrow C$".

We say that "$\Leftrightarrow$" is CONJUNCTIONAL while "$\equiv$" is associative.

So the final layout is:

$$
\begin{array}{ll}
 & A_1 \\
\Leftrightarrow & \langle \text{annotation} \rangle \\
 & A_2 \\
\Leftrightarrow & \langle \text{annotation} \rangle \\
 & \vdots \\
 & A_{n-1} \\
\Leftrightarrow & \langle \text{annotation} \rangle \\
 & A_n \\
\Leftrightarrow & \langle \text{annotation} \rangle \\
 & A_{n+1}
\end{array}
\qquad \text{(The Architecture of an Equational Proof)}
$$

Examples.

0.0.15 Theorem. $\vdash \neg(A \equiv B) \equiv \neg A \equiv B$

Proof. (Equational)

$$
\begin{aligned}
& \neg(A \equiv B) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \equiv B \equiv \bot \\
\Leftrightarrow\ & \langle (\textit{Leib}) + \text{axiom: } B \equiv \bot \equiv \bot \equiv B\text{; Denom: } A \equiv \mathbf{p}\text{; } \mathbf{p} \text{ fresh} \rangle \\
& A \equiv \bot \equiv B \\
\Leftrightarrow\ & \langle (\textit{Leib}) + \text{axiom: } A \equiv \bot \equiv \neg A\text{; Denom: } \mathbf{q} \equiv B\text{; } \mathbf{q} \text{ fresh} \rangle \\
& \neg A \equiv B
\end{aligned}
$$

□

0.0.16 Corollary. $\vdash \neg(A \equiv B) \equiv A \equiv \neg B$

Proof. (Equational)

$$
\begin{aligned}
& \neg(A \equiv B) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \equiv B \equiv \bot \\
\Leftrightarrow\ & \langle (\textit{Leib}) + \text{axiom: } B \equiv \bot \equiv \neg B\text{; Denom: } A \equiv \mathbf{p}\text{; } \mathbf{p} \text{ fresh} \rangle \\
& A \equiv \neg B
\end{aligned}
$$

□

Lecture #9, Oct. 7

0.0.17 Theorem. (Double Negation) $\vdash \neg\neg A \equiv A$

Proof. (Equational)

$$
\begin{aligned}
& \neg\neg A \\
\Leftrightarrow\ & \langle \text{axiom "}\neg X \equiv X \equiv \bot\text{"}\rangle \\
& \neg A \equiv \bot \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } \neg A \equiv A \equiv \bot\text{; Denom: } \mathbf{p} \equiv \bot\ \rangle \\
& A \equiv \bot \equiv \bot \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } \top \equiv \bot \equiv \bot\text{; Denom: } A \equiv \mathbf{q}\rangle \\
& A \equiv \top \\
\Leftrightarrow\ & \langle \text{red. } \top\rangle \\
& A
\end{aligned}
$$

□