Lecture # 21, Nov. 27, 2020

We saw in the lecture NOTES at link http://www.cs.yorku.ca/~gt/papers/lec15.pdf page 1, our first utilisation of Axioms 5 and 6 towards proving the ∀-version of the "one point rule".

Here are some more uses of these axioms.

0.0.1 Example. We prove that

$$\vdash x = y \to y = z \to x = z \tag{1}$$

The above is the transitivity of Equality since (by tautological implication back and forth) says the same thing as

$$\vdash x = y \land y = z \to x = z$$

We prove (1): In the application of **Ax6**

$$t = s \to (A[w := t] \equiv A[w := s])$$

in the proof below we took

- t to be x
- s to be y
- A to be $w = z$

1) $x = y \to (x = z \equiv y = z)$ ⟨**Ax6**⟩
2) $x = y \to (y = z \to x = z)$ ⟨1 + Post⟩

I did

$$p \to (q \equiv r) \models_{taut} p \to (r \to q)$$

Line 2 above is (1), as we see if we omit redundant brackets. □

0.0.2 Example. ("Replacing Equals by Equals") Here we prove

$$\vdash \mathbf{x} = \mathbf{y} \to f(\mathbf{x}) = f(\mathbf{y}), \text{ for any unary function symbols } f$$

which, awkwardly,* is captured by the quoted phrase above.

Here we use **Ax6**

$$t = s \to (A[\mathbf{w} := t] \equiv A[\mathbf{w} := s])]$$

in the special form below:

- t to be $\mathbf{x}$
- s to be $\mathbf{y}$
- A to be $f(\mathbf{w}) = f(\mathbf{y})$

1)	$\mathbf{x} = \mathbf{y} \to \Big(f(\mathbf{x}) = f(\mathbf{y}) \equiv f(\mathbf{y}) = f(\mathbf{y})\Big)$	$\langle$**Ax6**$\rangle$
2)	$(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$	$\langle$Partial gen. of **Ax5**$\rangle$
3)	$f(\mathbf{y}) = f(\mathbf{y})$	$\langle$2 + spec$\rangle$
4)	$\mathbf{x} = \mathbf{y} \to f(\mathbf{x}) = f(\mathbf{y})$	$\langle$(1, 3) + Post$\rangle$

I used the general version of *spec* in step 3 above with "$f(\mathbf{y})$" in place of the term "t" and "$\mathbf{x} = \mathbf{x}$" as "B":

$$(\forall \mathbf{x})B \vdash B[\mathbf{x} := t]$$

□

0.0.3 Exercise. Imitate the above proofs to prove commutativity of "=".

$$\vdash \mathbf{x} = \mathbf{y} \to \mathbf{y} = \mathbf{x}$$

□

*Nothing is "Equal" in the absence of other things.

0.1. Miscellaneous

Lecture # 22, Dec. 2, 2020

0.1.1 Example. Monotonicity of $\exists$, nickname $\exists$-MON or E-MON.

If $\Gamma \vdash A \to B$ and there is no free $\mathbf{x}$ in any wff of Γ then we have also

$$\Gamma \vdash (\exists \mathbf{x})A \to (\exists \mathbf{x})B \tag{1}$$

Here is a Hilbert proof.

1)	$A \to B$	$\langle \Gamma$-thm$\rangle$
2)	$\neg B \to \neg A$	$\langle 1$ + Post$\rangle$
3)	$(\forall \mathbf{x})\neg B \to (\forall \mathbf{x})\neg A$	$\langle 2$ + A-MON; conditions on Γ good!$\rangle$
4)	$\neg(\forall \mathbf{x})\neg A \to \neg(\forall \mathbf{x})\neg B$	$\langle 3$ + Post$\rangle$

The last line can be abbreviated as (1) □

0.1.2 Corollary. *If* $\vdash A \to B$*, then we have also*

$$\vdash (\exists \mathbf{x})A \to (\exists \mathbf{x})B \tag{2}$$

0.1.3 Example. We have seen via a countermodel in class/Text/NOTES that we CANNOT prove from the axioms

$$A \to (\forall \mathbf{x})A$$

if A contains free occurrences of $\mathbf{x}$.

How about

$$A[\mathbf{x} := c] \to (\forall \mathbf{x})A \tag{2}$$

where c is an unspecified (abstract) constant? *Can I prove it from the axioms?*

NO, here is a countermodel for the wff in (2), where I took A to be the atomic $x = y$ (not bold; actual variables).

So (2) becomes —after the simplification of A—

$$c = y \to (\forall x)x = y$$

Take $D = \mathbb{N}$, $y^{\mathfrak{D}} = 42$, $c^{\mathfrak{D}} = 42$. (2) translates as

$$c^{\mathfrak{D}} = y^{\mathfrak{D}} \to (\forall x \in \mathbb{N})x = y^{\mathfrak{D}}$$

or more explicitly

$$\overbrace{\underbrace{42 = 42}_{\mathbf{t}} \to \underbrace{(\forall x \in \mathbb{N})x = 42}_{\mathbf{f}}}^{\mathbf{f}}$$

□