You already know that removing a leading $\forall$ "uncovers" (in general[†]) "Boolean structure" which is amenable to proofs "by Post".

It would be a shame if we did not have techiques to remove a leading $\exists$.

We DO have such a technique! Read on.

0.0.1 Metatheorem. (Aux. Hypothesis Metatheorem) *Suppose that* $\Gamma \vdash (\exists \mathbf{x})A$.

Moreover, suppose that we know that $\Gamma, A[\mathbf{x} := \mathbf{z}] \vdash B$, *where* $\mathbf{z}$ *is fresh for ALL of* Γ, $(\exists \mathbf{x})A$, *and* B.

Then we have $\Gamma \vdash B$.

[†] Clearly, removing $\forall$ from $(\forall x)x = y$ uncovers $x = y$. But that has no Boolean structure —no glue. Hence I said "in general".

In our annotation we call $A[\mathbf{x} := \mathbf{z}]$ an "**auxiliary hypothesis associated with** $(\exists \mathbf{x})A$". $\mathbf{z}$ is called the auxiliary variable that we chose.

Essentially the fact that we proved $(\exists \mathbf{x})A$ allows us to adopt $A[\mathbf{x} := \mathbf{z}]$ as a *NEW AUXILIARY HYPOTHESIS* to help in the proof of B.

▶ How does it help? (1) I have a new hypothesis to work with; (2) $A[\mathbf{x} := \mathbf{z}]$ has *NO LEADING QUANTIFIER*.

(2), in general, results in uncovering the Boolean structure of $A[\mathbf{x} := \mathbf{z}]$ to enable proof by "Post"!

Halt-and-Take-Notice-Important! $A[\mathbf{x} := \mathbf{z}]$ is an *ADDED HYPOTHESIS!*

▶ It is *NOT TRUE* that either $(\exists \mathbf{x})A \vdash A[\mathbf{x} := \mathbf{z}]$ or that $\Gamma \vdash A[\mathbf{x} := \mathbf{z}]$.◀

WE WILL PROVE LATER IN THE COURSE THAT SUCH A THING IS NOT TRUE!

Proof. of the Metatheorem.

By the DThm, the metatheorem assumption yields

$$\Gamma \vdash A[\mathbf{x} := \mathbf{z}] \to B$$

Thus, ∃-Intro —*Corollary 0.1.5 on p.7* of the lecture Notes

http://www.cs.yorku.ca/~gt/papers/lec15.pdf

we get

$$\Gamma \vdash (\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to B \qquad (1)$$

We now can prove $\Gamma \vdash B$ as follows:

1)	$(\exists \mathbf{x})A$	$\langle \Gamma - \mathit{thm} \rangle$
2)	$(\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to B$	$\langle \Gamma - \mathit{thm}; (1)\ \mathit{above} \rangle$
3)	$(\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \equiv (\exists \mathbf{x})A$	⟨Bound var. renaming since $\mathbf{z}$ fresh⟩
4)	$(\exists \mathbf{x})A \to B$	⟨(2, 3) + Post⟩
5)	B	⟨(1, 4) + MP⟩

□

The most frequent form encountered in using Metatheorem 0.0.1 is the following corollary.

0.0.2 Corollary. *To prove* $(\exists \mathbf{x})A \vdash B$ IT SUFFICES *to pick a* $\mathbf{z}$ *that is* FRESH *for* $(\exists \mathbf{x})A$ *and* B *and*

PROVE INSTEAD $(\exists \mathbf{x})A, A[\mathbf{x} := \mathbf{z}] \vdash B$.

Proof. Take $\Gamma = \{(\exists x)A\}$ and invoke Metatheorem 0.0.1. $\square$

Some folks believe that the most important thing in logic is to know that the following is provable but the converse is not.

True, it is important.

But so are so many other things in logic, like Metatheorem 0.0.1, *precisely and correctly formulated* AND proved in our earlier pages.

0.0.3 Example. $\vdash (\exists \mathbf{x})(\forall \mathbf{y})A \to (\forall \mathbf{y})(\exists \mathbf{x})A$.

Let us share two proofs!

First *Proof.* By DThm it suffices to prove instead:

$(\exists \mathbf{x})(\forall \mathbf{y})A \vdash (\forall \mathbf{y})(\exists \mathbf{x})A$

(1)	$(\exists \mathbf{x})(\forall \mathbf{y})A$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{y})A[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$(2) + spec$\rangle$
(4)	$(\exists \mathbf{x})A$	$\langle$(3) + Dual spec$\rangle$
(5)	$(\forall \mathbf{y})(\exists \mathbf{x})A$	$\langle$(4) + gen; OK, *all hyp lines*, (1,2), have no free $\mathbf{y}\rangle$

We used the Corollary 0.0.2 of Metatheorem 0.0.1.

Second *Proof.* $\vdash A \to (\exists \mathbf{x})A$ (that is, the Dual of Ax2) we get $\vdash (\forall \mathbf{y})A \to (\forall \mathbf{y})(\exists \mathbf{x})A$ by $\forall$-mon.

Applying $\exists$-intro (Cor. 0.1.5 in the previous lecture Notes PDF, referred to also on p.3 of the present Notes) we get

$$\vdash (\exists \mathbf{x})(\forall \mathbf{y})A \to (\forall \mathbf{y})(\exists \mathbf{x})A \qquad \square$$

0.0.4 Example. We prove $(\exists\mathbf{x})(A \to B), (\forall\mathbf{x})A \vdash (\exists\mathbf{x})B$.

(1)	$(\exists\mathbf{x})(A \to B)$	$\langle$hyp$\rangle$
(2)	$(\forall\mathbf{x})A$	$\langle$hyp$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}] \to B[\mathbf{x} := \mathbf{z}]$	$\big\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\big\rangle$
(4)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$(2) + spec$\rangle$
(5)	$B[\mathbf{x} := \mathbf{z}]$	$\langle(3, 4) + MP\rangle$
(6)	$(\exists\mathbf{x})B$	$\langle$(5) + Dual spec$\rangle$

Remark. The above proves the conclusion using 0.0.1 and $\Gamma = \{(\exists\mathbf{x})(A \to B), (\forall\mathbf{x})A\}$. Of course, this Γ proves $(\exists\mathbf{x})(A \to B)$. □

0.0.5 Example. We prove $(\forall \mathbf{x})(A \to B), (\exists \mathbf{x})A \vdash (\exists \mathbf{x})B$.

(1)	$(\forall \mathbf{x})(A \to B)$	$\langle$hyp$\rangle$
(2)	$(\exists \mathbf{x})A$	$\langle$hyp$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (2); $\mathbf{z}$ fresh$\rangle$
(4)	$A[\mathbf{x} := \mathbf{z}] \to B[\mathbf{x} := \mathbf{z}]$	$\langle$(1) + spec$\rangle$
(5)	$B[\mathbf{x} := \mathbf{z}]$	$\langle$(3, 4) + $MP\rangle$
(6)	$(\exists \mathbf{x})B$	$\langle$(5) + Dual spec$\rangle$ $\square$

0.0.6 Example. Here is a common mistake people make when arguing informally.

Let us prove the following informally.

$\vdash (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \rightarrow (\exists \mathbf{x})(A \wedge B)$.

So let $(\exists \mathbf{x})A(\mathbf{x})$ *and* $(\exists \mathbf{x})B(\mathbf{x})$ *be true.*[†]

Thus, for some value c *of* $\mathbf{x}$ *we have that* $A(c)$ *and* $B(c)$ *are true.*

But then so is $A(c) \wedge B(c)$.

The latter implies the truth of $(\exists \mathbf{x})\big(A(\mathbf{x}) \wedge B(\mathbf{x})\big)$.

Nice, crisp and short.

And very, very wrong as we will see once we have **1st-order Soundness** in hand. Namely, we will show in the near future that $(\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \rightarrow (\exists \mathbf{x})(A \wedge B)$ *is NOT* a theorem schema. It is NOT provable.

[†]The experienced mathematician considers self-evident and unworthy of mention at least two things:

(1) The deduction theorem, and

(2) The Split Hypothesis metatheorem.

What went wrong above?

We said

"Thus, for some value c of $\mathbf{x}$ we have that $A(c)$ and $B(c)$ are true".

The blunder was to assume that THE SAME c verified BOTH $A(x)$ and $B(x)$.

Let us see that formalism protects even the inexperienced from such blunders.

Here are the first few steps of a(n attempted) FORMAL proof via the Deduction theorem:

(1)	$(\exists \mathbf{x})A \wedge (\exists \mathbf{x})B$	$\langle$hyp$\rangle$
(2)	$(\exists \mathbf{x})A$	$\langle(1) + \text{Post}\rangle$
(3)	$(\exists \mathbf{x})B$	$\langle(1) + \text{Post}\rangle$
(4)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (2); $\mathbf{z}$ fresh$\rangle$
(5)	$B[\mathbf{x} := \mathbf{w}]$	$\langle$aux. hyp for (3); $\mathbf{w}$ fresh$\rangle$

The requirement of freshness makes $\mathbf{w}$ DIFFERENT from $\mathbf{z}$. These variables play the role of two distinct c and c'. Thus the proof cannot continued. Saved by freshness! $\square$

0.0.7 Example. The last Example in this section makes clear that the Russell Paradox was the result of applying bad Logic, not just bad Set Theory!

I will prove that for any binary predicate ϕ we have

$$\vdash \neg(\exists \mathbf{y})(\forall \mathbf{x})(\phi(\mathbf{x},\mathbf{y}) \equiv \neg\phi(\mathbf{x},\mathbf{x})) \qquad (R)$$

By the Metatheorem "Proof by Contradiction" I can show

$$(\exists \mathbf{y})(\forall \mathbf{x})(\phi(\mathbf{x},\mathbf{y}) \equiv \neg\phi(\mathbf{x},\mathbf{x})) \vdash \bot$$

instead. Here it is

(1)	$(\exists \mathbf{y})(\forall \mathbf{x})(\phi(\mathbf{x},\mathbf{y}) \equiv \neg\phi(\mathbf{x},\mathbf{x}))$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{x})(\phi(\mathbf{x},\mathbf{z}) \equiv \neg\phi(\mathbf{x},\mathbf{x}))$	$\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\rangle$
(3)	$\phi(\mathbf{z},\mathbf{z}) \equiv \neg\phi(\mathbf{z},\mathbf{z})$	$\langle$(2) + spec$\rangle$
(4)	$\bot$	$\langle$(3) + Post$\rangle$

If we let the atomic formula $\phi(\mathbf{x},\mathbf{y})$ be Set Theory's "$\mathbf{x} \in \mathbf{y}$" then (R) that we just proved (in fact *for ANY* binaru predicate ϕ not just $\in$) morphs into

$$\vdash \neg(\exists \mathbf{y})(\forall \mathbf{x})(\mathbf{x} \in \mathbf{y} \equiv \mathbf{x} \notin \mathbf{x}) \qquad (R')$$

In plain English (R') says that *there is NO set* $\mathbf{y}$ *that contains ALL x satisfying $x \notin x$*.

This theorem was proved without using even a single axiom of set theory, indeed not even using "$\{\ldots\}$-notation" for sets, or any other symbols from set theory.

After all we proved (R') generally and abstractly in the form (R) and that expression and its proof has NO SYMBOLS from set theory!

In short, Russell's Paradox can be expressed AND demonstrated in PURE LOGIC.

It is remarkable that Pure Logic can tell us that NOT ALL COLLECTIONS are SETS, a fact that escaped Cantor. □

Semantics of First-Order Languages —Simplified

Lecture #19 Nov. 20, 2020

0.1. Interpretations

An *interpretation* of a wff —and of *THE ENTIRE language*, that is, the set of *ALL* Terms and wff— is INHERITED from an **interpretation of all symbols of the Alphabet**.

This tool —*the Interpretation*— Translates *each wff* to some formula of a familiar branch of mathematics that *we choose*, and thus questions such as is the "translated formula true?" can in principle be dealt with (see 0.1.2 below for details).

An interpretation is totally up to us, just as states were in Boolean logic.

The process is only slightly more complex.

Here we need to interpret not only wff but also terms as well.

The latter requires that we *choose a NONEMPTY set of objects to begin with*. We call this set the *Domain* of our Interpretation and *generically* call it "D" but in specific cases it could be $D = \mathbb{N}$ or $D = \mathbb{R}$ (the *reals*) or even something "small" like $D = \{0, 5\}$.

An *Interpretation* of a 1st-order language consists of a *PAIR* of two things:

The aforementioned domain D and a translation *mapping* M —the latter translates the abstract symbols of the Alphabet of logic to concrete mathematical symbols.

► This translation of the ALPHABET INDUCES a translation for each term and wff of the language; thus of ALL THE LANGUAGE. ◄

We write the interpretation "package" as $\mathfrak{D} = (D, M)$ displaying the two ingredients D and M in round brackets.

The unusual calligraphy here is German capital letter calligraphy that is usual in the literature *to name an interpretation package*. The letter for the name chosen is usually the same as that of the Domain.

Let me repeat that both D and M are our choice.

0.1.1 Definition. (Translating the Alphabet $\mathcal{V}_1$)

An *Interpretation* $\mathfrak{D} = (D, M)$ gives concrete *counterparts* (translations) to ALL elements of the *Alphabet* as follows:

In the listed cases below we may use notation $M(X)$ to indicate the concrete translation (mapping) of an abstract *linguistic object* X.

We also may use $X^{\mathfrak{D}}$ as an alternative notation for $M(X)$.

The literature favours $X^{\mathfrak{D}}$ and so will we.

(1) For each *FREE* variable (of a wff) $\mathbf{x}$, $\mathbf{x}^{\mathfrak{D}}$ —that is, the translation $M(\mathbf{x})$— is some *chosen* (BY US!) *FIXED* member of D.

BOUND variables are NOT translated! They stay AS IS.

(2) For each Boolean variable $\mathbf{p}$, $\mathbf{p}^{\mathfrak{D}}$ is a member of $\{\mathbf{t}, \mathbf{f}\}$.

(3) $\top^{\mathfrak{D}} = \mathbf{t}$ and $\bot^{\mathfrak{D}} = \mathbf{f}$.

This is just we did —via states— in the Boolean case. As was the case there, *I will remind the reader once again* that we choose the value $\mathbf{p}^{\mathfrak{D}}$ anyway we please, but for $\top$ and $\bot$ we follow the fixed (Boolean) rule.

(4) For any (object) *constant* of the alphabet, say, c, we choose a *FIXED* $c^{\mathfrak{D}}$, as we wish, in D.

(5) For every *function* symbol f of the alphabet, the translation $f^{\mathfrak{D}}$ is a mathematical function of the metatheory ("real" or "concrete" MATH) of the same arity as f.

$f^{\mathfrak{D}}$ —*which WE choose!*— takes inputs from D and gives outputs in D.

(6) For every predicate ϕ of the alphabet OTHER THAN "=", our CHOSEN translation $\phi^{\mathfrak{D}}$ is a mathematical RELATION of the metatheory with the same arity as ϕ. It takes its inputs from D while its outputs are one or the other of the truth values **t** or **f**.

► NOTE THAT ALL the Boolean glue as well as the equality symbol translate exactly as THEMSELVES: "=" for "equals", $\vee$ for "OR", etc.

Finally, brackets translate as the SAME TYPE of bracket (left or right). □

We have all we need to translate wff, terms and thus the entire Language:

0.1.2 Definition. (The Translation of wff)
Consider a wff A in a[†] first-order language.

Suppose we have chosen an interpretation $\mathfrak{D} = (D, M)$ of the alphabet.

The interpretation or translation of A via $\mathfrak{D}$ *a mathematical ("concrete") formula of the metatheory* or a concrete object of the metatheory that we will denote by

$$A^{\mathfrak{D}}$$

It is constructed as follows one symbol at a time, **scanning A from left to right until no symbol is left**:

[†] A, not THE. For every choice of constant, predicate and function symbols we get a different alphabet, as we know, hence a different first-order language. Remember the examples of Set Theory vs. Peano Arithmetic!

(i) *We replace* every occurrence of $\perp, \top$ in A by $\perp^{\mathfrak{D}}, \top^{\mathfrak{D}}$ —that is, by $\mathbf{f}, \mathbf{t}$— respectively.

(ii) *We replace* every occurrence of $\mathbf{p}$ in A by $\mathbf{p}^{\mathfrak{D}}$ —this is an *assigned by US TRUTH VALUE*; we assigned it *when we translated the alphabet*.

(iii) *We replace* each *FREE* occurrence of an object variable $\mathbf{x}$ of A by the value $\mathbf{x}^{\mathfrak{D}}$ from D that we assigned *when we translated the alphabet*.

(iv) *We replace* every occurrence of $(\forall \mathbf{x})$ in A by $(\forall \mathbf{x} \in D)$, which means "for all values of $\mathbf{x}$ in D".

(v) We emphasise again that Boolean connectives (glue) *translate as themselves*, and so do "=" and the brackets "(" and ")".

Theory-specific symbols in A:

(vi) *We replace* every occurrence of a(n object) constant c in A by the specific fixed $c^{\mathfrak{D}}$ from D —which we chose when translating the alphabet.

(vii) *We replace* every occurrence of a function f in A by the specific fixed $f^{\mathfrak{D}}$ —which we chose when translating the alphabet.

(viii) *We replace* every occurrence of a predicate ϕ in A by the specific fixed $\phi^{\mathfrak{D}}$ —which we chose when translating the alphabet. □

0.1.3 Definition. (Partial Translation of a wff) Given a wff A in a first-order language and an interpretation $\mathfrak{D}$ of the alphabet.

Sometimes *we do NOT wish to translate a FREE variable* $\mathbf{x}$ of A. Then the result of the translation that *leaves* $\mathbf{x}$ *as is* is denoted by $A^{\mathfrak{D}}_{\mathbf{x}}$.

Similarly, if we choose *NOT* to translate *ANY* of

$$\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}, \ldots$$

that (may) occur FREE in A, then we show the result of such "*partial*" translation as

$$A^{\mathfrak{D}}_{\mathbf{x_1},\ldots,\mathbf{x_n}}$$

Thus $A^{\mathfrak{D}}$ has no free variables, but $A^{\mathfrak{D}}_{\mathbf{x}}$ will have $\mathbf{x}$ free IF $\mathbf{x}$ actually DID occur free in A —the notation guarantees that if $\mathbf{x}$ so occurred, then we left it alone.

□

0.1.4 Remark. What is the use of the concept and notation "$A_{\mathbf{x}}^{\mathfrak{D}}$"?

Well, note that when we translate $(\forall \mathbf{x})A$ *FROM LEFT TO RIGHT*, we get "$(\forall \mathbf{x} \in D)$" followed by the translation of A.

However, ANY $\mathbf{x}$ that occur free IN A *BELONG* to $(\forall \mathbf{x})$ in the wff $(\forall \mathbf{x})A$ thus are *NOT FREE* in the latter and hence are NOT translated!

Therefore, "$(\forall \mathbf{x} \in D)$" concatenated with "$A_{\mathbf{x}}^{\mathfrak{D}}$" is what we get: "$(\forall \mathbf{x} \in D)A_{\mathbf{x}}^{\mathfrak{D}}$". □

0.1.5 Example. Consider the AF $\phi(x, x)$, ϕ is a binary predicate.

Here are some possible interpretations:

(a) $D = \mathbb{N}$, $\phi^{\mathfrak{D}} = <$.

Here "$<$" is the "less than" relation on natural numbers. So $\big(\phi(x, x)\big)^{\mathfrak{D}}$, which is the same as $\phi^{\mathfrak{D}}(x^{\mathfrak{D}}, x^{\mathfrak{D}})$ —in familiar notation is the formula over $\mathbb{N}$:

$$x^{\mathfrak{D}} < x^{\mathfrak{D}}$$

More specifically, if we took $x^{\mathfrak{D}} = 42$, then $\big(\phi(x, x)\big)^{\mathfrak{D}}$ is specifically "$42 < 42$".

Incidentally, $\big(\phi(x, x)\big)^{\mathfrak{D}}$ is false <u>for ANY choice of</u> $x^{\mathfrak{D}}$.

We will write $\big(\phi(x, x)\big)^{\mathfrak{D}} = \mathbf{f}$ to denote the above sentence symbolically.

I would have <u>preferred</u> to write something like "$V\Big(\big(\phi(x, x)\big)^{\mathfrak{D}}\Big) = \mathbf{t}$ —"V" for value— but it is *so much <u>easier</u> to agree that writing the above I mean the same thing!* :)

For the sake of practice, here are two *partial* *interpretations.*

In the first we exempt the variables y, z. In the second we exempt x:

(i) $\Big(\phi(x,x)\Big)^{\mathfrak{D}}_{y,z}$ is $x^{\mathfrak{D}} < x^{\mathfrak{D}}$. WHY?

(ii) $\Big(\phi(x,x)\Big)^{\mathfrak{D}}_{x}$ is $x < x$.

(b) $D = \mathbb{N}$, $\phi^{\mathfrak{D}} = \leq$ (the "less than or equal" relation on $\mathbb{N}$). So, $\big(\phi(x,x)\big)^{\mathfrak{D}}$ is the concrete $x^{\mathfrak{D}} \leq x^{\mathfrak{D}}$ on $\mathbb{N}$.

Clearly, independently of the choice of $x^{\mathfrak{D}}$, we have

$$\big(\phi(x,x)\big)^{\mathfrak{D}} = \mathbf{t}$$

□

0.1.6 Example. Consider next the wff

$$f(x) = f(y) \to x = y \qquad (1)$$

where f is a unary function.

Here are some interpretetions:

1. $D = \mathbb{N}$ and $f^{\mathfrak{D}}$ is chosen to satisfy $f^{\mathfrak{D}}(x) = x + 1$, for all values of x in D.

   Thus $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}$ this formula over $\mathbb{N}$:

$$x^{\mathfrak{D}} + 1 = y^{\mathfrak{D}} + 1 \to x^{\mathfrak{D}} = y^{\mathfrak{D}}$$

   Note that *every* choice of $x^{\mathfrak{D}}$ and $y^{\mathfrak{D}}$ makes the above true.

2. $D = \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

   Take $f^{\mathfrak{D}}(x) = x^2$, for all x in $\mathbb{Z}$.
   Then, $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}$ is, more concretely, the following formula over $\mathbb{Z}$:

$$(x^{\mathfrak{D}})^2 = (y^{\mathfrak{D}})^2 \to x^{\mathfrak{D}} = y^{\mathfrak{D}}$$

   The above is true for some choices of $x^{\mathfrak{D}}$ and $y^{\mathfrak{D}}$ but not for others:

   E.g., it is false if we took $x^{\mathfrak{D}} = -2$ and $y^{\mathfrak{D}} = 2$.

   Finally here are two *partial interpretations* of (1) at the beginning of this example:

   (i) $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}_{x}$ is $x^2 = (y^{\mathfrak{D}})^2 \to x = y^{\mathfrak{D}}$.

   (ii) $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}_{x,y}$ is $x^2 = y^2 \to x = y$. $\square$