1. While the following theorem —nicknamed "One-point rule"— will not play a big role in our lectures, still, on one hand it gives us a flavour of how we *use the axioms of equality* (Axioms 5 and 6) and on the other hand every mathematician uses it without even thinking about it, in the form, for example,

$$A(3) \text{ is the same as } (\exists x)(x = 3 \wedge A(x))$$

0.0.1 Theorem. (One point rule —$\forall$ version) *On the condition that* $\mathbf{x}$ *does not occur in* t,[†] *we have* $\vdash (\forall \mathbf{x})(\mathbf{x} = t \to A) \equiv A[\mathbf{x} := t]$.

Proof. By Ping-Pong.

($\to$) Note that since $\mathbf{x}$ does not occur in t, we have

$$(\mathbf{x} = t \to A)[\mathbf{x} := t] \quad \text{means the same thing as} \quad t = t \to A[\mathbf{x} := t]$$

Thus,

(1)	$(\forall \mathbf{x})(\mathbf{x} = t \to A) \to t = t \to A[\mathbf{x} := t]$	$\langle$**Ax2**$\rangle$
(2)	$(\forall \mathbf{x})(\mathbf{x} = \mathbf{x})$	$\langle$**Ax5** —partial gen. of $\mathbf{x} = \mathbf{x}\rangle$
(3)	$t = t$	$\langle(2)$ + spec$\rangle$
(4)	$(\forall \mathbf{x})(\mathbf{x} = t \to A) \to A[\mathbf{x} := t]$	$\langle(1, 3)$ + Post$\rangle$

($\leftarrow$) Recall the **General form of Axiom 6**: $\boxed{s = t \to (A[\mathbf{x} := s] \equiv A[\mathbf{x} := t])}$

(1)	$\mathbf{x} = t \to (A \equiv A[\mathbf{x} := t])$	$\langle$**Ax6**$\rangle$
(2)	$A[\mathbf{x} := t] \to \mathbf{x} = t \to A$	$\langle(1)$ + Post$\rangle$
(3)	$(\forall \mathbf{x})A[\mathbf{x} := t] \to (\forall \mathbf{x})(\mathbf{x} = t \to A)$	$\langle(2)$ + $\forall$-MON —(2) is an absolute theorem$\rangle$
(4)	$A[\mathbf{x} := t] \to (\forall \mathbf{x})(\mathbf{x} = t \to A)$	$\langle(3)$ + **Ax3** + Post$\rangle$

I have done the "Post" in (4) before (previous class). Note that Ax3 is applicable since $\mathbf{x}$ is not free in $A[\mathbf{x} := t]$

[†]We can also say "does not occur free in t", but that is an overkill: A term t has NO bound variables.

2. 0.0.2 Corollary. (One point rule —∃ version) *On the condition that* $\mathbf{x}$ *does not occur in* t*, we have* $\vdash (\exists\mathbf{x})(\mathbf{x} = t \wedge A) \equiv A[\mathbf{x} := t]$.

Proof. Exercise! (*Hint.* Use the ∀ version and an Equational proof to prove the ∃ version (use the "Def of E" Theorem).) □

0.0.3 Theorem. (Bound variable renaming $(\forall)$) *IF* $\mathbf{z}$ *is fresh for* A *—that is, does not occur* as either free or bound *in* A*— then* $\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$.

"Everyday mathematician's" notation is $\vdash (\forall \mathbf{x})A(\mathbf{x}) \equiv (\forall \mathbf{z})A(\mathbf{z})$.
But NOT our notation!

Proof. Ping-Pong.

$(\rightarrow)$

(1)	$(\forall \mathbf{x})A \rightarrow A[\mathbf{x} := \mathbf{z}]$	$\langle$**Ax2** —fresh $\mathbf{z}$; *no capture*: no "$(\forall \mathbf{z})(\ldots, \mathbf{x}, \ldots)$" in $A\rangle$
(2)	$(\forall \mathbf{z})(\forall \mathbf{x})A \rightarrow (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$	$\langle$(1) + $\forall$-mon$\rangle$
(3)	$(\forall \mathbf{x})A \rightarrow (\forall \mathbf{z})(\forall \mathbf{x})A$	$\langle$**Ax3**$\rangle$
(4)	$(\forall \mathbf{x})A \rightarrow (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$	$\langle$(2, 3) + Post$\rangle$

$(\leftarrow)$ Let us first settle a useful "lemma" for the proof below:

0.0.4 Lemma. *Under the assumptions about* $\mathbf{z}$*, we have that* $A[\mathbf{x} := \mathbf{z}][\mathbf{z} := \mathbf{x}]$ *is just the original* A*.*

Proof. Now, $\mathbf{z}$ is *neither*

- *Bound* in A. That is, there is NO "$(\forall \mathbf{z})(\ldots)$" in A. So the substitution $A[\mathbf{x} := \mathbf{z}]$ *GOES THROUGH, AND* "flags" (and replaces) all FREE $\mathbf{x}$ in A as $\mathbf{z}$.

  nor is

- *Free* in A. So NO FREE $\mathbf{z}$ *pre-existed* in A before doing $A[\mathbf{x} := \mathbf{z}]$. That is, ALL FREE $\mathbf{z}$ in $A[\mathbf{x} := \mathbf{z}]$ are EXACTLY the $\mathbf{x}$ that became $\mathbf{z}$. *These* $\mathbf{z}$ *are PLACEHOLDERS for THE ORIGINAL FREE* $\mathbf{x}$ *in* A.

  BUT then! Doing now $[\mathbf{z} := \mathbf{x}]$ changes ALL $\mathbf{z}$ in $A[\mathbf{x} := \mathbf{z}]$ back to $\mathbf{x}$.

  We are back to the original A!

$\square$

(1)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to A[\mathbf{x} := \mathbf{z}][\mathbf{z} := \mathbf{x}]$	$\langle$**Ax2** —$A[\mathbf{x} := \mathbf{z}][\mathbf{z} := \mathbf{x}]$ defined by lemma$\rangle$
(2)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to A$	$\langle$same as (1) —see lemma$\rangle$
(3)	$(\forall \mathbf{x})(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to (\forall \mathbf{x})A$	$\langle$abs. thm (2) + $\forall$ MON$\rangle$
(4)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to (\forall \mathbf{x})(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$	$\langle$Ax3; no free $\mathbf{x}$ in lhs$\rangle$
(5)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to (\forall \mathbf{x})A$	$\langle$(3, 4) + Post$\rangle$ $\square$

Lecture #18, Nov. 18

0.1. Adding and Removing the Quantifier "$(\exists x)$"

First, introducing (adding) $\exists$ is easy via the following tools:

0.1.1 Theorem. (Dual of Ax2) $\vdash A[\mathbf{x} := t] \to (\exists \mathbf{x})A$

Proof.

$$
\begin{aligned}
& A[\mathbf{x} := t] \to (\exists \mathbf{x})A \\
\Leftrightarrow\ & \langle \text{WL + "Def of E" (this is an abs. thm); "Denom:" } A[\mathbf{x} := t] \to \mathbf{p} \rangle \\
& A[\mathbf{x} := t] \to \neg(\forall \mathbf{x})\neg A \\
\Leftrightarrow\ & \langle \text{tautology} \rangle \\
& (\forall \mathbf{x})\neg A \to \neg A[\mathbf{x} := t] \qquad \text{Bingo!}
\end{aligned}
$$

□

0.1.2 Corollary. (The Dual of Specialisation) $A[\mathbf{x} := t] \vdash (\exists \mathbf{x})A$

Proof. 0.1.1 and MP. □

0.1.3 Corollary. $A \vdash (\exists \mathbf{x})A$

Proof. 0.1.2, taking $\mathbf{x}$ as t. □

Either corollaries above we call "*Dual Spec*" in annotating proofs.

But how can I remove a leading (the entire formula) $\exists$?

We need two preliminary results to answer this.

0.1.4 Metatheorem. ($\forall$ Introduction) *If $\mathbf{x}$ does not occur free in Γ nor in A, then $\Gamma \vdash A \to B$ iff $\Gamma \vdash A \to (\forall \mathbf{x})B$.*

Proof. of the "iff".

($\to$) direction.

Assumption gives $\Gamma \vdash (\forall \mathbf{x})(A \to B)$ by valid generalisation.

But we have

$$
\begin{aligned}
& (\forall \mathbf{x})(A \to B) \\
\Leftrightarrow\ & \langle \text{thm from NOTES/Class} \rangle \\
& A \to (\forall \mathbf{x})B
\end{aligned}
$$

So the bottom formula is a Γ-theorem.

($\leftarrow$) direction.

This time we know the bottom of the above short Equational proof is a Γ-theorem.

Then so is the top. But from the latter I get $\Gamma \vdash A \to B$ by spec. $\Box$

0.1.5 Corollary. ($\exists$ Introduction) *IF $\mathbf{x}$ does not occur free in Γ nor in B, then $\Gamma \vdash A \to B$ iff $\Gamma \vdash (\exists \mathbf{x})A \to B$.*

Note how we shifted the condition for $\mathbf{x}$ from A to B.

Proof. of the "*iff*". Well,

$$\Gamma \vdash A \to B \stackrel{\text{Post}}{\textbf{iff}} \Gamma \vdash \neg B \to \neg A \stackrel{0.1.4}{\textbf{iff}} \Gamma \vdash \neg B \to (\forall \mathbf{x})\neg A \stackrel{\text{Post}}{\textbf{iff}} \Gamma \vdash \neg(\forall \mathbf{x})\neg A \to B$$

□