Lecture # 15, Nov. 6

0.1. First-order Proofs and Theorems

A Hilbert-style proof from Γ (Γ-proof) is *exactly as defined in the case of Boolean Logic*. Namely:

It is a finite sequence of *wff*

$$A_1, A_2, A_3, \ldots, A_i, \ldots, A_n$$

such that each A_i is ONE of

1. Axiom from Λ_1 OR a member of Γ

   OR

2. Is obtained by *MP* from $X \to Y$ and X that appear to the LEFT of A_i (*A_i is the same string as Y then.*)

However, here "*wff*" is *1st-order*, and Λ_1 is a DIFFERENT set of axioms than the old Λ. Moreover we have ONLY one rule up in front.

As in Boolean definitions, a 1st-order theorem from Γ (Γ-theorem) is *a formula that occurs in a 1st-order Γ-proof*.

As before we write "$\Gamma \vdash A$" to say "A is a Γ-theorem" and write "$\vdash A$" to say "A is an absolute theorem".

Hilbert proofs in 1st-order logic are written vertically as well, with line numbers and annotation.

The metatheorems about proofs and theorems

- proof tail removal,
- proof concatenation,
- a wff is a Γ-theorem iff it occurs at the end of a proof
- hypothesis strengthening,
- hypothesis splitting,
- usability of derived rules,
- usability of previously proved theorems

hold with the same metaproofs *as in the Boolean case*.

We trivially have Post's Theorem (the weak form that we proved for Boolean logic).

0.1.1 Theorem. (Weak Post's Theorem for 1st-order logic)
If $A_1, \ldots, A_n \models_{taut} B$ *then* $A_1, \ldots, A_n \vdash B$

Proof. Exactly the same as in Boolean logic. □

Thus we may use

$$A_1, \ldots, A_n \vdash B$$

as a *DERIVED rule in any 1st-order proof*, *if we know that*

$$A_1, \ldots, A_n \models_{taut} B$$

.

0.2. Deduction Theorem

This Metatheorem of First-Order Logic says:

0.2.1 Metatheorem. *If* $\Gamma, A \vdash B$*, then also* $\Gamma \vdash A \to B$

Proof. Induction on the proof length L we used for $\Gamma, A \vdash B$:

1. $L = 1$ (*Basis*). There is only one formula in the proof: The proof must be

$$B$$

   Only two subcases apply:

   - $B \in \Gamma$. Then $\Gamma \vdash B$. But $B \models_{taut} A \to B$, thus by 0.1.1 also $B \vdash A \to B$. So

$$B, A \to B$$

     is a Γ-proof too. That is, $\Gamma \vdash A \to B$.
   - B *IS* A. So, $A \to B$ is a tautology hence axiom hence $\Gamma \vdash A \to B$.
   - $B \in \Lambda_1$. Then $\Gamma \vdash B$. Conclude as above.

2. *Assume (I.H.) the claim for all proofs of lengths* $L \leq n$.

3. *I.S.*: The proof has length $L = n + 1$:

$$\overbrace{\ldots, B}^{n+1}$$

   If $B \in \Gamma \cup \Lambda_1$ then we are done by the argument in 1.

   Assume instead that it is the result of MP on formulas to the left of B:

$$\underbrace{\overbrace{\underbrace{\ldots, X, \ldots, X \to B}_{\leq n}, \ldots, B}^{n+1}}_{n}$$

   By the I.H. we have

$$\Gamma \vdash A \to X \tag{$*$}$$

and

$$\Gamma \vdash A \to (X \to B) \qquad (**)$$

The following Hilbert proof concludes the case and the entire proof:

1) $A \to X$ $\langle$thm by $(*)\rangle$
2) $A \to (X \to B)$ $\langle$thm by $(**)\rangle$
3) $A \to B$ $\langle 1 + 2 +$ taut. implication$\rangle$

The last line proves the metatheorem. □

Comment. In line 3 above, seeing that

$$A \to X, \qquad A \to (X \to B) \models_{taut} A \to B$$

is trivially verifiable, we used the "RULE"

$$A \to X, A \to (X \to B) \vdash A \to B$$

that we obtain from the above via 0.1.1.

The annotation said "1 + 2 + taut. implication".

It could also have said instead "1 + 2 + Post".

0.3. Generalisation and "weak" Leibniz Rule

We learn here HOW exactly to handle the quantifier $\forall$.

0.3.1. Adding and Removing "$(\forall \mathbf{x})$"

0.3.1 Metatheorem. (Weak Generalisation) *Suppose that for any wff X in Γ X* has no free occurrences *of* $\mathbf{x}$.

Then if we have $\Gamma \vdash A$, we will also have $\Gamma \vdash (\forall \mathbf{x})A$.

Proof. Induction on the length L of the Γ-proof used for A.

1. $L = 1$ (*Basis*). There is only one formula in the proof: The proof must be

$$A$$

Only two subcases apply:

- $A \in \Gamma$. Then A has *no free* $\mathbf{x}$. But $\vdash A \to (\forall \mathbf{x})A$ by axiom 3. Thus, we have a Hilbert proof (written horizontally for speed),

$$\overbrace{A}^{\Gamma-proved}, \overbrace{A \to (\forall \mathbf{x})A}^{axiom}, \quad \overbrace{(\forall \mathbf{x})A}^{\text{MP on the previous two}}$$

- $A \in \Lambda_1$. Then then so is $(\forall \mathbf{x})A \in \Lambda_1$ by partial generalisation.

  Hence $\Gamma \vdash (\forall \mathbf{x})A$ once more. (WHY?)

AHA! So that's what "partial generalisation" does for us!

2. *Assume (I.H.) the claim for all proofs of lengths $L \leq n$.*

3. *I.S.*: The proof has length $L = n + 1$:

$$\overbrace{\ldots, A}^{n+1}$$

If $A \in \Gamma \cup \Lambda_1$ then we are done by the argument in 1.

Assume instead that A is the result of MP on formulas to the left of it:

$$\overbrace{\underbrace{\underbrace{\ldots, X, \ldots, X \to A}_{\leq n}, \ldots}_{n}, A}^{n+1}$$

By the I.H. we have

$$\Gamma \vdash (\forall \mathbf{x})X \tag{$*$}$$

and

$$\Gamma \vdash (\forall \mathbf{x})(X \to A) \tag{$**$}$$

The following Hilbert proof concludes the case and the entire proof:

1)	$(\forall \mathbf{x})X$	$\langle$thm by $(*)\rangle$
2)	$(\forall \mathbf{x})(X \to A)$	$\langle$thm by $(**)\rangle$
3)	$(\forall \mathbf{x})(X \to A) \to (\forall \mathbf{x})X \to (\forall \mathbf{x})A$	$\langle$axiom 4$\rangle$
5)	$(\forall \mathbf{x})X \to (\forall \mathbf{x})A$	$\langle 2 + 3 + \text{MP}\rangle$
6)	$(\forall \mathbf{x})A$	$\langle 1 + 5 + \text{MP}\rangle$

The last line proves the metatheorem. □

0.3.2 Corollary. *If* $\vdash A$*, then* $\vdash (\forall \mathbf{x})A$*.*

Proof. The condition that no X in Γ has free $\mathbf{x}$ is met: Vacuously. Γ is empty! □

0.3.3 Remark.

1. So, the Metatheorem says that if A is a Γ-theorem then so is $(\forall \mathbf{x})A$ as long as the restriction of 0.3.1 is met.

   But then, *since I can invoke THEOREMS* (not only axioms and hypotheses) in a proof, I can insert $(\forall \mathbf{x})A$ anywhere AFTER A in any Γ-proof of A where Γ obeys the restriction.

2. Why "weak"? Because I need to know how the A was obtained before I may use $(\forall \mathbf{x})A$. □

0.3.4 Metatheorem. (Specialisation Rule) $(\forall \mathbf{x})A \vdash A[\mathbf{x} := t]$

Goes without saying that *IF* the expression $A[\mathbf{x} := t]$ is undefined, then we have nothing to prove.

Proof.

$$
\begin{array}{lll}
(1) & (\forall \mathbf{x})A & \langle \text{hyp} \rangle \\
(2) & (\forall \mathbf{x})A \to A[\mathbf{x} := t] & \langle \text{axiom } 2 \rangle \\
(3) & A[\mathbf{x} := t] & \langle 1 + 2 + \text{MP} \rangle
\end{array}
$$
□

0.3.5 Corollary. $(\forall \mathbf{x})A \vdash A$

Proof. This is the special case where t is $\mathbf{x}$. □

Really Important! The metatheorems 0.3.5 and 0.3.1 (or 0.3.2) —*which we nickname "spec" and "gen" respectively*— are tools that make our life easy in Hilbert proofs where handling of ∀ is taking place.

0.3.5 *with no restrictions* allows us to REMOVE a leading "$(\forall \mathbf{x})$".

Doing so *we might uncover Boolean glue* and thus benefit from applications of "Post" (0.1.1).

If we need to re-INSERT $(\forall \mathbf{x})$ before the end of proof, we employ 0.3.1 to do so.

This is a good recipe for success in 1st-order proofs!

0.3.2. Examples

Ping-Pong proofs.

Hilbert proofs are not well-suited to handle equivalences.

However, trivially

$$A \to B, B \to A \models_{taut} A \equiv B$$

and —by 0.1.1—

$$A \to B, B \to A \vdash A \equiv B \tag{1}$$

Thus, *to prove* $\Gamma \vdash A \equiv B$ *in Hilbert style* it suffices —by (1)— to offer TWO Hilbert proofs:

$$\Gamma \vdash A \to B \text{ AND } \Gamma \vdash B \to A$$

This back and forth motivates the nickname “ping-pong” for this proof technique.

0.3.6 Theorem. (Distributivity of $\forall$ over $\wedge$) $\vdash (\forall \mathbf{x})(A \wedge B) \equiv (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$

Proof. By Ping-Pong argument.

We will show TWO things:

1. $\vdash (\forall \mathbf{x})(A \wedge B) \rightarrow (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$

   and

2. $\vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \rightarrow (\forall \mathbf{x})(A \wedge B)$

($\rightarrow$) ("1." above)

By DThm, it suffices to prove $(\forall \mathbf{x})(A \wedge B) \vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$.

(1)	$(\forall \mathbf{x})(A \wedge B)$	$\langle$hyp$\rangle$
(2)	$A \wedge B$	$\langle$1 + spec (0.3.5)$\rangle$
(3)	A	$\langle$2 + Post$\rangle$
(4)	B	$\langle$2 + Post$\rangle$
(5)	$(\forall \mathbf{x})A$	$\langle$3 + gen; OK: hyp contains no free $\mathbf{x}\rangle$
(6)	$(\forall \mathbf{x})B$	$\langle$4 + gen; OK: hyp contains no free $\mathbf{x}\rangle$
(7)	$(\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$	$\langle$(5,6) + Post$\rangle$

NOTE. We *ABSOLUTELY MUST* acknowledge for each application of "gen" that *the restriction is met.*

Lecture #16, Nov. 11

($\leftarrow$) ("2." above)

By DThm, it suffices to prove $(\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \vdash (\forall \mathbf{x})(A \wedge B)$.

(1)	$(\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{x})A$	$\langle 1 +$ Post$\rangle$
(3)	$(\forall \mathbf{x})B$	$\langle 1 +$ Post$\rangle$
(4)	A	$\langle 2 +$ spec$\rangle$
(5)	B	$\langle 3 +$ spec$\rangle$
(6)	$A \wedge B$	$\langle (4,5) +$ Post$\rangle$
(7)	$(\forall \mathbf{x})(A \wedge B)$	$\langle 6 +$ gen; OK: hyp has no free $\mathbf{x}\rangle$

□

Easy and Natural! Right?

0.3.7 Theorem. $\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \equiv (\forall \mathbf{y})(\forall \mathbf{x})A$

Proof. By Ping-Pong. $\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \overset{\rightarrow}{\leftarrow} (\forall \mathbf{y})(\forall \mathbf{x})A$.

($\rightarrow$) direction.

By DThm it suffices to prove $(\forall \mathbf{x})(\forall \mathbf{y})A \vdash (\forall \mathbf{y})(\forall \mathbf{x})A$

(1)	$(\forall \mathbf{x})(\forall \mathbf{y})A$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{y})A$	$\langle$1 + spec$\rangle$
(3)	A	$\langle$2 + spec$\rangle$
(4)	$(\forall \mathbf{x})A$	$\langle$3 + gen; OK hyp has no free $\mathbf{x}\rangle$
(5)	$(\forall \mathbf{y})(\forall \mathbf{x})A$	$\langle$4 + gen; OK hyp has no free $\mathbf{y}\rangle$

($\leftarrow$)
Exercise! Justify that you can write the above proof backwards! $\square$

0.3.8 Metatheorem. (Monotonicity of $\forall$) *If* $\Gamma \vdash A \to B$*, then* $\Gamma \vdash (\forall \mathbf{x})A \to (\forall \mathbf{x})B$*, as long as no wff in* Γ *has a free* $\mathbf{x}$*.*

Proof.

(1)	$A \to B$	$\langle$invoking a Γ-thm$\rangle$
(2)	$(\forall \mathbf{x})(A \to B)$	$\langle 1$ + gen; OK no free $\mathbf{x}$ in $\Gamma\rangle$
(3)	$(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\forall \mathbf{x})B$	$\langle$axiom 4$\rangle$
(4)	$(\forall \mathbf{x})A \to (\forall \mathbf{x})B$	$\langle (2,3) + MP \rangle$

$\square$

0.3.9 Corollary. *If* $\vdash A \to B$*, then* $\vdash (\forall \mathbf{x})A \to (\forall \mathbf{x})B$*.*

Proof. Case of $\Gamma = \emptyset$. The restriction is vacuously satisfied. $\square$

0.3.10 Corollary. *If* $\Gamma \vdash A \equiv B$, *then also* $\Gamma \vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{x})B$, as long as Γ *does not contain wff with* $\mathbf{x}$ *free.*

Proof.

(1)	$A \equiv B$	$\langle \Gamma$-theorem$\rangle$
(2)	$A \to B$	$\langle 1$ + Post$\rangle$
(3)	$B \to A$	$\langle 1$ + Post$\rangle$
(4)	$(\forall \mathbf{x})A \to (\forall \mathbf{x})B$	$\langle 2 + \forall$-mon (0.3.8)$\rangle$
(5)	$(\forall \mathbf{x})B \to (\forall \mathbf{x})A$	$\langle 3 + \forall$-mon (0.3.8)$\rangle$
(6)	$(\forall \mathbf{x})A \equiv (\forall \mathbf{x})B$	$\langle (4,5)$ + Post$\rangle$

□

0.3.11 Corollary. *If* $\vdash A \equiv B$, *then also* $\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{x})B$.

Proof. Take $\Gamma = \emptyset$. □