Lecture #5. Sept. 23

Here is the Basic Truth Table again:

x	y	$F_\neg(x)$	$F_\vee(x,y)$	$F_\wedge(x,y)$	$F_\rightarrow(x,y)$	$F_\equiv(x,y)$
f	**f**	**t**	**f**	**f**	**t**	**t**
f	**t**	**t**	**t**	**f**	**t**	**f**
t	**f**	**f**	**t**	**f**	**f**	**f**
t	**t**	**f**	**t**	**t**	**t**	**t**

I repeated above the table that we saw on Friday (Sept. 18) to discuss a bit $F_\rightarrow(x,y)$.

The most "sane" entry in this column is arguably, the one for input $(\mathbf{t},\mathbf{f})$.

Since this function is describing the truth-value of *implications*, and the x input is the *hypothesis* and the y input is the *conclusion*,

Then having $F_\rightarrow(\mathbf{t},\mathbf{f}) = \mathbf{f}$ can be interpreted as saying that *the implication cannot be true if we started with a true hypothesis and ended up with a false conclusion.*

We can easily agree with this!

…Since our *intuition* accepts that *"→" preserves truth from left to right.*

BUT STILL: There are unsettling issues about the *truth table* for "$\rightarrow$", this table known as the *semantics* of *Classical* or *Material Implication*.

For example:

- "*If it is cloudy today, then my name is George*"

  My name *IS* "George" so the above *is true* if it is NOT cloudy today (by row **two** in the table).

  But the hypothesis is *irrelevant* to the conclusion, which does *not FOLLOW*, intuitively speaking, *from the hypothesis*.

- "*If it is cloudy today, then my name is George*"

  My name *IS* "George". Suppose also that it IS cloudy today!

  Then the above *is true* (by row **four** in the table).

  The same comment as above, about the *irrelevance between hypothesis and conclusion* applies!

What do we do?

Answer. Nothing. *This semantics* ("this", not "these"; singular) is what 99% of the literature uses. *If it helps, think of* **f** *as 0 and* **t** *as 1.*

Now if you think of "$\rightarrow$" as "$\le$" (we will have a strong reason to do so; wait for when we introduce our Axioms) then the column under $F_{\rightarrow}$ is all right!

It confirms that $F_{\rightarrow}(x, y)$ *has the semantics of* "$x \le y$", *which* incidentally is totally consistent with the semantics of $\equiv$ as "$x = y$":

That is, the view that "$\equiv$" says "$\rightarrow$ and $\leftarrow$", coincides with the view in the table that it says "$=$" —in other words, "$\le$ and $\ge$".

So far, *states* give meaning (values) *to atomic formulas only*. Let us *extend* this meaning-giving to *any wff*.

0.0.1 Definition. (The value of a wff in some state, v) We *extend any* state v to be meaningful *not only with atomic arguments* but also with any wff arguments.

We will call such an *extension of v* by the same letter, but will "cap" it with a "hat", $\overline{v}$, since it is a different function!

What *IS* an "extension" of v?

It is a function $\overline{v}$ that *on the arguments that v is defined* so is $\overline{v}$ and gives the same output!

But $\overline{v}$ is defined *on more inputs: On all wff found in WFF.*

The definition of $\overline{v}$ is inductive:

The first three lines below simply say that $\overline{v}$ agrees with v on the inputs that the latter is defined on.

The remaining lines trace along **the inductive definition of wff,** and give the value of a wff **using the values —via "recursive calls"— of its UNIQUE i.p.**

You see the **significance** of the uniqueness of i.p.!!!

$$
\begin{aligned}
\overline{v}(\mathbf{p}) &= v(\mathbf{p}) \\
\overline{v}(\top) &= \mathbf{t} \\
\overline{v}(\bot) &= \mathbf{f} \\
\overline{v}\Big((\neg A)\Big) &= F_{\neg}\Big(\overline{v}(A)\Big) \\
\overline{v}\Big((A \wedge B)\Big) &= F_{\wedge}\Big(\overline{v}(A), \overline{v}(B)\Big) \\
\overline{v}\Big((A \vee B)\Big) &= F_{\vee}\Big(\overline{v}(A), \overline{v}(B)\Big) \\
\overline{v}\Big((A \to B)\Big) &= F_{\to}\Big(\overline{v}(A), \overline{v}(B)\Big) \\
\overline{v}\Big((A \equiv B)\Big) &= F_{\equiv}\Big(\overline{v}(A), \overline{v}(B)\Big)
\end{aligned}
$$

□

Truth tables are more convenient to understand, AND misunderstand!

For example the 6-th equality in the previous definition can also be depicted as:

A	B	$A \vee B$
f	**f**	**f**
f	**t**	**t**
t	**f**	**t**
t	**t**	**t**

At a glance the table says that to compute the value of $A \vee B$ you just utilise the values of the i.p. A and B as indicated.

The misunderstanding you MUST avoid is this: The two left columns are NOT values you assign to A and B.

You can assign values ONLY to ATOMIC formulas!

What these two columns DO say is that *the formulas A and B have each two possible values*.

That is 4 **pairs of values**, as displayed!

We say a variable **p** *occurs in a formula meaning the obvious: It is,* as a string, *a substring —a part— of the formula*.

0.0.2 Theorem. *Given a formula A. Suppose that two states, v and s agree on* **all** *the variables of A. Then $\overline{v}(A) = \overline{s}(A)$.*

Proof. We do induction of the formula A:

1. Case where A is atomic. Well if it is $\top$ or $\bot$ then $\overline{v}(A) = \overline{s}(A)$ is true. If A is $\mathbf{p}$, then $\overline{v}(A) = \overline{s}(A)$ by hypothesis of the theorem.

2. Case where A is $(\neg B)$. The value of A —whether under v or under s— is *determined* by a recursive call to $\overline{v}(B)$ and $\overline{s}(B)$. Seeing that all the variables of B are in A, the I.H. yields $\overline{v}(B) = \overline{s}(B)$ and hence $\overline{v}(A) = \overline{s}(A)$.

3. Case where A is $(B \circ C)$. The value of A —whether under v or under s— is *determined* by recursive calls to $\overline{v}(B)$ and $\overline{v}(C)$ on one hand and $\overline{s}(B)$ and $\overline{s}(C)$ on the other.

   Seeing that all the variables of B and C are in A, the I.H. yields

   $$\overline{v}(B) = \overline{s}(B) \text{ and } \overline{v}(C) = \overline{s}(C) \qquad (*)$$

   Hence no matter which one of the $\wedge, \vee, \rightarrow, \equiv$ the symbol $\circ$ stands for, it operating on $\overline{v}(B)$ and $\overline{v}(C)$ or on $\overline{s}(B)$ and $\overline{s}(C)$ will yield the same result by $(*)$.

   That is, $\overline{v}(A) = \overline{s}(A)$. □

0.0.3 Remark. (Finite "appropriate" States) A state v is *by definition an infinite table*.

By the above theorem, the value of any wff A in a state v is determined only by the values of v ON THE VARIABLES OF A, since *any other state that agrees with v on said variables gives the same answer.*

Thus, going forward we will be utilising *finite appropriate states* to compute the truth values of any wff. □

0.0.4 Definition. (Tautologies and other things. . .)

1. A *Tautology* is a formula A which is true in *all* states. That is, for *all* v, we have $\overline{v}(A) = \mathbf{t}$.

   We write "$\models_{taut} A$" for "*A is a tautology*".

2. A *contradiction* is a formula A such that, for *all* v, we have $\overline{v}(\neg A) = \mathbf{t}$.

   Clearly, for *all* v, we have $\overline{v}(A) = \mathbf{f}$.

3. A is *satisfiable* iff for *some* v, we have $\overline{v}(A) = \mathbf{t}$.

   We say that v satisfies A.

Boolean logic for the user helps to discover tautologies. □

We saw that WFF denotes the set of all (well-formed) formulas.

Capital Greek letters that are different from any Latin capital letter are used to denote arbitrary sets of formulas. Such letters are $\Gamma, \Delta, \Phi, \Psi, \Omega, \Pi, \Sigma$. As always, in the rare circumstance you run out of such letters you may use primes and/or (natural number) subscripts.

0.0.5 Definition. (Tautological implication —the binary $\models_{taut}$)

1. Let Γ be a set of wff. *We say that v satisfies Γ iff v satisfies every formula in Γ.*

2. We say that Γ *tautologically implies A* —and we write this as $\Gamma \models_{taut} A$— iff *every state v that satisfies Γ also satisfies A.*

   The configuration

$$\Gamma \models_{\text{taut}} A \tag{1}$$

   is called a tautological implication claim.

   We call Γ *the set of hypotheses* or *premises* of the tautological implication, while A is the *conclusion*. □

IMPORTANT! The task to verify (1) entails work on our part **ONLY** if we found a v that satisfies Γ.

If there is NO such v then the claim (1) is valid! YOU *cannot* contradict its validity for you will *need* a v that *satisfies* Γ but NOT A.

0.0.6 Example.

(1) If $\models_{\text{taut}} A$, then for any Σ, we have $\Sigma \models_{\text{taut}} A$.

The converse is not valid:

(2) We have $\mathbf{p} \models_{\text{taut}} \mathbf{p} \vee \mathbf{q}$. Indeed, for any v such that $v(\mathbf{p}) = \mathbf{t}$ we compute $\overline{v}(\mathbf{p} \vee \mathbf{q}) = \mathbf{t}$ from the truth table for $\vee$.

Yet, $\mathbf{p} \vee \mathbf{q}$ is NOT a tautology. Just take $v(\mathbf{p}) = v(\mathbf{q}) = \mathbf{f}$

Note also the obvious: $A \models_{taut} A \vee B$, for any wff A and B. Again use the truth table of p.5. □

In view of 0.0.2 we can check all of *satisfiability*, *tautology* status, and *tautological implication* with *finite* Γ *using a finite truth table*.

Examples.

Example 1. $\bot \models_{taut} A$.

Because no v satisfies the lhs of "$\models_{taut}$" so according to Definition I rest my case.

Example 2. Let us build a truth table for $A \to B \vee A$ and see what we get.

I wrote sloppily, according to our priorities agreement.

<u>I mean $(A \to (B \vee A))$</u>.

We align our part-work under the glue since it is the glue that causes the output.

Here $\to$ is the last (applied) glue. *Under it we write the final results for this formula*.

Since A and B *are not necessarily atomic*, the values under A and B in the table below are *possible* values *NOT assigned values*! *So* $(A \to (B \vee A))$ *is a*

A	B	A	$\to$	B	$\vee$	A
f	**f**		**t**		**f**	
f	**t**		**t**		**t**	
t	**f**		**t**		**t**	
t	**t**		**t**		**t**	

tautology.

Example 3. Here is another tautology. I will verify this by a shortcut method, WITHOUT building a truth table.

I will show

$$\models_{taut} ((A \to B) \to A) \to A \tag{1}$$

I will do so by arguing that *it is IMPOSSIBLE TO MAKE (1) FALSE*.

- *If (1) is false* then A *is false* and $(A \to B) \to A$ *is true*.
- Given the two blue statements above, it *must* be that $A \to B$ *is false*. *IMPOSSIBLE, since A is false!*

Lecture #6. Sept. 25.

0.0.7 Definition. (Substitution in Formulas)

The *META*notation

$$A[\mathbf{p} := B] \tag{1}$$

where A and B are formulas and $\mathbf{p}$ is any variable means

- **As an Action**: "*Find and replace by B ALL occurrences of $\mathbf{p}$ in A*".
- **As a Result**: *The STRING resulting from the action* described in the previous bullet.

□

1. In the *META*theory of Logic where we use the exprssion "$[\mathbf{p} := B]$" we Agree to Give it The Highest priority: Thus, $A \wedge B[\mathbf{q} := C]$ means $A \wedge \Big(B[\mathbf{q} := C]\Big)$ and $\neg A[\mathbf{p} := B]$ means $\neg\Big(A[\mathbf{p} := B]\Big)$
2. Clearly if $\mathbf{p}$ does NOT occur in A, then the "action" found nothing to replace, so the resulting string —according to (1)— in this case is just A; NO CHANGE.

We observe the following, according to the inductive definition of formulas.

With reference to (1) of the previous page, say

1. A is atomic. In particular, *using "=" for equality of strings*,
   - A is $\mathbf{p}$. Then $A[\mathbf{p} := B] = B$
   - A is $\mathbf{q}$ —where by $\mathbf{q}$ we denote a *variable other than the one* $\mathbf{p}$ *stands for*. Then $A[\mathbf{p} := B] = A$ —no change.
   - A is $\bot$ or $\top$. Then $A[\mathbf{p} := B] = A$ —no change.
2. A is $(\neg C)$. *Then all occurrences of* $\mathbf{p}$ *are in* C. *All Action happens with* C.

   Thus $A[\mathbf{p} := B]$ is effected by doing first $S = C[\mathbf{p} := B]$.

   Above I named the result S for convenience.

   Now $A[\mathbf{p} := B]$ is $(\neg S)$.
3. A is $(C \circ D)$. *Then all occurrences of* $\mathbf{p}$ *are in* C *or* D.

   All Action happens with C *and also* D.

   Thus $A[\mathbf{p} := B]$ *is effected by doing*

   (a) $S = C[\mathbf{p} := B]$

   (b) $T = D[\mathbf{p} := B]$

   Where I named the two above results S *and* T *for convenience.*

   (c) To conclude, *use concatenation —in the order indicated below—* to obtain the *string*

$$(S \circ T)$$

0.0.8 Proposition. *For every wff A and wff B and any variable $\mathbf{p}$, $A[\mathbf{p} := B]$ is* a wff.*

Proof. Induction on A using the observations 1.–3. of the previous page.

Cases for A:

- A is Atomic. *So we are under Case 1 of the previous page.*
  Regardless of subcase (we get as the result of substitution) A or B. This result is a wff.

- Case where A is $(\neg C)$. The I.H. on i.p. applies to C, so $S = C[\mathbf{p} := B]$ *is a formula —where we used a new name S for convenience.*

  But $A[\mathbf{p} := B]$ is $(\neg S)$. Done.

- Case where A is $(C \circ D)$. The I.H. on i.p. applies to C and D, so $S = C[\mathbf{p} := B]$ *and* $T = D[\mathbf{p} := B]$ *are formulas* —using the notation of the previous page.

  But $A[\mathbf{p} := B]$ is $(S \circ T)$. Done. □

*We are purposely sloppy with jargon here —like everybody else in the literature: "IS" means "results into".

We are poised to begin describing the proof system of Boolean logic.

To this end we will need the notation that is called *formula schemata* or *formula schemas* (if you consider "schema" an English word).

0.0.9 Definition. (Schema, Schemata) *Add to the alphabet $\mathcal{V}$* the following symbols:

1. "[", "]", and ":="

2. All NAMES of formulas: $A, B, C, \ldots$, *with or without primes and/or subscripts*.

3. All *metasymbols* for variables: $\mathbf{p}, \mathbf{q}, \mathbf{r}$, *with or without primes and/or subscripts*.

Then a *formula schema is a STRING over the augmented alphabet, which becomes a wff whenever all metasymbols of types 2 and 3 above, which occur in the string, are replaced by wff and actual variables respectively*.

A formula that we obtain by the process described in the paragraph above is called an **Instance of the Schema**. □

Three examples of schemata.

(1) A: This Schema stands for a wff! So trivially, *if I plug into A an actual wff, I get that wff as an* ***instance!***

(2) $(A \equiv B)$: Well, whatever formulas I substitute into A and B (metavariables) I get a wff by the inductive definition of wff.

(3) $A[\mathbf{p} := B]$: We know that if I substitute A and B by formulas and $\mathbf{p}$ by a Boolean variable I get a wff (0.0.8).

Next stop is Proofs!

In *proofs* we use *Axioms and Rules* (of *Inference*).

It is the habit in the literature to write Rules as *fractions*:

$$\frac{P_1, P_2, \ldots, P_n}{Q} \qquad (R)$$

where *all* of $P_1, \ldots, P_n, Q$ are schemata.

An *Instance of the Rule* is a *common instance* of all $P_1, \ldots, P_n, Q$, that is, a metavariable A is *replaced by the same wff throughout*, and a metavariable **p** is *also replaced by the same Boolean variable throughout*.

We call *the schema* (if one, or *schemata* if many) on the numerator the *premise(s)* but also *hypothes(is/es)*.

The single schema in the denominator we call the *conclusion* (also "*result*").

I note that the fraction (R) above, *the RULE, is meant as an input / output device*.

► For every instance of (R)

$$\text{all the } P_i \text{ and the } Q \text{ become } \underline{\text{wff}}\ P'_1, \ldots, P'_n, Q'$$

We say

the Rule, with input $P'_1, \ldots, P'_n$ yields output (result, conclusion) Q'.

We also say that Q' is the *result* of the *application* of (R) to $P'_1, \ldots, P'_n$.

0.0.10 Definition. (Rules of Inference of OUR version of Boolean Logic) There are just two:

Rule1

$$\frac{A \equiv B}{C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]} \qquad (Leibniz)$$

There are NO restrictions in the use of "Leibniz".

In particular, it is NOT required that $\mathbf{p}$ *actually occurs in* C*.*

If it does not, then the denominator is $C \equiv C$*.*

Rule2

$$\frac{A, A \equiv B}{B} \qquad (Eqn)$$

□

