Contents

Chapter 1

The Beginning

1.1 Russell's Paradox

...or, when things (in MATH) go "sideways" ...

1.1.1 Example. (Briefly about set notation) We represent sets either by explicit *listing*,

- $\{0\}$
- $\{\$, \#, 3, 42\}$
- $\{0, 1, 2, 3, 4, \ldots\}$

or by some "*defining property*": *The set of all x** *that make P(x) true*, in symbols

$$S = \{x : P(x)\} \tag{1}$$

*Strictly speaking you *DON'T* collect the various shapes and colours of the letter x. There is only ONE x. The expression "set of all x such that $P(x)$ is true" is sloppy for "the set of all VALUES of x such that $P(x)$ is true".

As we know from discrete MATHs, (1) says the *same thing* as the statement

$$x \in S \equiv P(x) \tag{2}$$

read "for any value of x, $x \in S$ is equivalent to $P(x)$"

Why so? Because $P(x)$ is an *entrance condition*! A value of x is included in the set S IF and *ONLY* IF (iff) said value *passes the test* $P(x)$.

Wait! Shouldn't I have written (2) as

$$x \in S \equiv P(x) \text{ is true} \tag{2'}$$

Nope. When mathematicians state $P(x)$ for some unspecified *fixed* x they mean "$P(x)$ is true" for that value.

Cantor believed (as did the philosopher Frege) that, for any property $P(x)$, (1) defines a set.

Which is neither here nor there because they never said what a set is. They allowed ANY collection of (mathematical) objects to be set!

Russell begged to differ, so he said: "Oh, yeah? How about"

$$R = \{x : x \notin x\}$$

where the property "$P(x)$" here is "$x \notin x$"

Now, by (2) we have

$$x \in R \equiv x \notin x$$

If R *IS* a set, then we can plug it in the set variable x above to obtain

$$R \in R \equiv R \notin R$$

How do we avoid this *contradiction*?

By *admitting* that R is *NOT* a set so we do not allow the substitution! □

1.2 Enters Logic!

So Cantor was sloppy about what a set is and **how sets get formed**.

Formal —*meaning SYNTACTICALLY PERFORMED; Based on FORM*— logic was invented by Russell and Whitehead, and **Hilbert** to *salvage* Mathematics from "antinomies" and "paradoxes", both words derived from Greek, and both meaning **contradictions**.

How does formal logic *salvage* Mathematics?

By helping you *stay on track* in your argumentation.

You cannot pull facts and fake facts off the air, but your facts *MUST* be *axioms* or *PREVIOUSLY proved* theorems, and the *rules* of logic that you use *MUST* NOT *DEPEND* on an *Interpretation* (your's, Cantor's or mine).

The rule is the rule: How helpful would an excuse like "but Officer, I may have not stopped at the stop sign but it is the middle of nowhere and nobody is here except you and me".

Burned! You just interpreted the rule, didn't you!

Connection of Formal Logic with Programming

(1) In programming we use *syntactic rules* to write a *program* in order to solve some problem computationally.

(2) In logic you use the syntactic rules to write a *proof* that establishes a theorem.

Kinds of logic reasoning that we will thoroughly examine and use in this course.

1. Equational logic —also known as *calculational logic*.

   Introduced by [DS90] and simplified by [GS94] and later by [Tou08] to make it accessible to undergraduates. Software Engineers like it.

2. Hilbert-style logic. This is the logic which most people use to write their mathematical arguments in publications, lectures, etc.

Logic is meant to certify mathematical truths syntactically.

Logic is normally learnt by

- *A LOT* of practice.

- By presenting and teaching it *gradually*, namely

  1. **First, learning the *Propositional Logic* (also known as *Boolean Logic*).**

     Here one learns how logical truths *combine* using *connectives* familiar from programming like OR, AND, and NOT.

     Boolean logic is *not expressive enough* to formulate statements about mathematical objects. Naturally, if you cannot ask it —a question about such objects— then you cannot answer it either.

2. **Next, learning *Predicate Logic* (also known as *First-Order Logic*).**

   This is the full logic of the mathematician and computer scientist as it lets you formulate and explore statements that involve *mathematical objects* like numbers, strings and trees, and many others.

The following is a fundamental *BELIEF* of the great David Hilbert, which he formulated in the early 30s:

"We should be able to *solve* the Decision Problem of *Mathematical Theories* by **mechanical means**".

It triggered a lot of research in the 30s and also led to the birth of "computability", a branch of logic that studies "mechanical processes" and their properties.

Decision Problem of Logic (*Entscheidungsproblem* of Hilbert's): It asks: **Is this formula a theorem of logic**?

Here we are ahead of ourselves: What is a "**formula**"? What is a "**theorem**"?

I will tell you soon!

But in short and superficially,

- A "formula" is a syntactically well-formed *STATEMENT*.
- A "theorem" is a **statement** of which *I can certify its truth syntactically*.

BTW:

☡ Enclosing text between ☡ symbols means that **this** text is important; *pay attention!* ☡

while

☡☡ means that **this** is *rather esoteric and not pressing to learn; it can be skipped.* ☡☡

1. **Boolean Logic:** Its Decision Problem, because of *Post's theorem* that we will learn in this course, *does* have an *algorithmic —or "mechanical"— solution*, for example, via *truth tables*.

   There is a catch: The solution is in general useless because the algorithm takes tons of time to give an answer.

   I am saying that at the present state of our knowledge about algorithms, the truth table method is unpractically slow. To get an answer from a $n \times n$ table it takes 2^n steps.

2. **Predicate Logic:** Things get desperate here: We have a totally negative answer to the Decision Problem. *There is NO algorithm* at all that will solve it! This result is due to Church ([Chu36])

So it *makes sense* to *find ways to certify truth, which rely on human ingenuity and sound methodology* rather than on some machine and a computer program, in short *we must learn to work syntactically BOTH* in

- Boolean logic where the decision problem *currently* has an *unfeasible* algorithm that solves it,

  and

- Predicate logic where the decision problem has *provably no algorithm waiting to be discovered* —ever.

This we will learn in this course: How to certify truth by syntactic means, through practice and sound methodology.

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1.3 A look back at strings

1.3.1 Definition. (Strings; also called Expressions)

1. What is a *string* over some *alphabet* of symbols?

   It is an *ordered* finite sequence of symbols from the alphabet — with no gaps between symbols.

   1.3.2 Example. If the alphabet is $\{a, b\}$ then here are a few strings:

   (a) a

   (b) $aaabb$

   (c) $bbaaa$

   (d) $bbbbbbb$

   □

   What do we mean by "ordered"? We mean that *order matters*! For example, $aaabb$ and $bbaaa$ are different strings. We indicate this by writing $aaabb \neq bbaaa$.

Two strings are equal iff [†] **they have the same length n and at *each* position —from 1 to n— both strings have the same symbol. So, $aba = aba$, but $aa \neq a$ and $aba \neq baa$.**

[†] If and only if.

1.3.1 A Bad Alphabet

Consider the alphabet $B = \{a, aa\}$.

This is bad. WHY?

Because if we write the string **aaa** over this alphabet we do not know what we mean by just **looking** at the string!

Do we mean 3 a like

$$a \quad a \quad a$$

Or do we mean

$$a \quad aa$$

Or perhaps

$$aa \quad a$$

We say that alphabet B leads to *ambiguity*.

Since we use NO separators —like a space or a comma— between symbols in denoting strings we MUST ALWAYS choose alphabets with single-symbol items.

2. *Names* of strings: A, A'', A_5, B, C, S, T.

   What for? CONVENIENCE AND EASE OF EXPRESSION.

   Thus $A = bba$ gives the string bba the name A.

   Names vs *IS*: Practicing mathematicians and computer scientists take a sloppy attitude towards using the verb "IS".

   When they say "let A be a string" they mean "let A name a string".

   Same as in "let x be a rational number". Well x is not a number at all! It is a letter! We mean "let x **STAND** for, or **NAME**, a rational number"

3. *Operations* on strings: *Concatenation.* From strings aab and baa, concatenation in the order given yields the string $aabbaa$.

If A is a string (meaning names a string) and B is another, then their concatenation AB is not a concatenation of the names but *is a concatenation of the contents*. If $A = aaaa$ and $B = 101$ then $AB = aaaa101$.

Incidentally,

$$BA = 101aaaa \neq aaaa101 = AB$$

Thus *in* general *concatenation is* not commutative *as we say*.

Why "in general"?

Well, if $X = aa$ and $Y = a$ then $XY = aaa = YX$.

Special cases where concatenation commutes exist!

4. *Associativity* of concatenation.

   It is expressed as $(AB)C = A(BC)$ where bracketing here denotes invisible METAsymbols (*they are NOT part of any* string!) that simply INDICATE the order in which we GROUP, from left to right.

   At the left of the "=" we first concatenate A and B and then glue C at the *right* end.

   $$A \quad B \quad C$$

   if $A = 1, B = 2, C = 3$ then $A(BC) = 123$ NOT $1(23)$

   To the right of "=" we first glue B and C and then glue A to the *left* of the result.

   In either case we did not change the relative positions of A, B and C.

   The property is self-evident.

   I can now skip brackets and write $ABCD$ and you know what I mean!

5. *Empty* string. A string with *no symbols*, hence with *length* 0. Denoted by λ.

$$\emptyset \cap A = \emptyset, \lambda Q = Q$$

How is λ different than $\emptyset$ the empty *set*?

Well one is of *string type* and the other is of *set type*. So? The former is an ORDERED empty set, the latter is an UNORDERED empty set that moreover is *oblivious to repetitions*.

I mean, $aaa \neq a$ but $\{a, a, a\} = \{a\}$.

6. Clearly, for any string A we have $A\lambda = \lambda A = A$ as concatenation of λ adds nothing to either end.

7. *Substrings*. A string A is a *substring* of B iff A appears *as is* as a *part* of B.

   So if $A = aa$ and $B = aba$ then A is NOT a substring of B.

   Its members both appear in B (the two a) but are *not* together as they are in A. *A does not appear "as is"*.

   Can we get rid of all this bla-bla with a proper definition? Sure:

   1.3.3 Definition. A is a substring of B iff for some strings (named) U and V we have $B = UAV$. □

We also say A is *part* of B.

8. *Prefix* and *suffix*. A is a *prefix* of B if for some string V, $B = AV$.

   So A is part of B up in front!

   A is a *suffix* of B if for some string U, $B = UA$. □

Example: λ is a prefix and a suffix, indeed a part, of *any* string B. Here are the "proofs" of the two cases I enumerated:

- $B = \lambda B$
- $B = B\lambda$

WHAT ABOUT THE THIRD CASE?

Split a sting A you got any way you please: Say, $A = UV$.

But you have also $A = U\lambda V$.

1.4 The Formulas or well-formed-formulas (wff)

The Syntax of logic. Boolean Logic at first!

Boolean logic is the "Algebra of statements". We start with *atomic* statements and build complex statements using "glue" as I call the Boolean *connectives* $\neg, \wedge, \vee, \rightarrow, \equiv$.

Atomic statements have NO glue!

Examples of statements that Boolean logic can express:

p, $(\neg p)$ and also $((p \vee q) \wedge r)$. And more!

Can I *see inside* atomic statements like p to see what they *mean*?

NO!! We cannot! But we can assign *arbitrarily* "true" or "false" *values* to atomic statements and then proceed to see how these *truth values* propagate when I *apply glue*.

That is all Boolean logic can do.

And this ends up being useful! Read on!

1.4.1 Definition. (Alphabet of Boolean Symbols)

A1. *Names* for *variables*, which we call "propositional" or "Boolean" variables.

These are p, q, r, with or without primes or subscripts (indices) (e.g., $p, q, r, p', q_{13}, r'''_{51}$ are all names for Boolean variables).

A2. Two *symbols* denote the Boolean *constants*, $\top$ and $\bot$. We pronounce them "top" and "bot" respectively.

What are $\top$ and $\bot$ good for? We will soon see!

A3. (Round) brackets, i.e., "(" and ")" (employed without the quotes, of course).

A4. Boolean "*connectives*" that I will usually call "*glue*".

We use glue to put a formula together much like we do so when we build model cars or airplanes or houses.

The symbols for Boolean connectives are

$$\neg \ \wedge \ \vee \ \rightarrow \equiv \tag{1}$$

and are read from left to right as "negation, conjunction, disjunction, implication, equivalence". □

We stick to the above symbols for glue (no pun!) in this course! *Just as in programming*.

You *cannot use* any symbols you **please** or **like**.

SPEAKING BY ANALOGY, You use *THE* symbols of the programming language as *GIVEN*.

If not, your program does NOT work and your GRADE bottoms!

Same holds for logic!

1.4.2 Definition. (Formula Construction (process)) A *formula construction* (in the text called "*formula calculation*") is any *finite (ordered) sequence of strings over the alphabet*[†] *of Boolean logic* $\mathcal{V}$ that obeys the following three specifications:

C1. At any step we may write precisely one symbol from categories **A1.** or **A2.** above (1.4.1).

C2. At any step we may write precisely *one* **string** *of the form* $(\neg A)$, *as long as* we have written the string (*named*) A already at a previous step.

So, "$(\neg A)$" is a string that has "$(\neg$" (no quotes) as a prefix, then it has a part we named A, and then it has ")" (no quotes) as a suffix.

I must stress that the letter A names the string that we write down. Just as in a *program*: When you issue the command "**print** X" you mean to print what the X contains as value —what it names. You do *not* mean to print the **letter** "X"!

C3. At any step we may write precisely *one* **of the strings** $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \equiv B)$, *as long as* we have already written *each* of the strings A and B earlier.

We do not care which we wrote first, A or B.

□

[†] "Over the Alphabet": Using exclusively symbols from the Alphabet $\mathcal{V}$ that we adopted.

1.4.3 Definition. (Boolean formulas (wff)) Any string A over the alphabet $\mathcal{V}$ (**A1.–A4.**) is called *a Boolean formula* or *propositional formula* —in short **wff**— *iff* A is a string that appears in some formula construction. □

1.4.4 Example. First off, the above says more than it pretends to:

For example, it says that *every* string that appears in a formula construction is a wff. **The definition also says**,

"*do you want to know if A is a wff? **Just make sure you can build a formula construction** where A appears.*"

We normally write formula constructions **vertically**. Below I use numbering and annotation (in "$\langle \ldots \rangle$" brackets) to explain each step.

•

(1)	$\bot$	$\langle$C1$\rangle$	"C" for Construction
(2)	p	$\langle$C1$\rangle$	
(3)	$(\neg\bot)$	$\langle$(1) + (C2)$\rangle$	
(4)	$\bot$	$\langle$C1$\rangle$	
(5)	$\top$	$\langle$C1$\rangle$	

Note that we *can* have *redundancy* and *repetitions*.

Ostensibly the only nontrivial info in the above is that $(\neg\bot)$ is a formula. But it also establishes that $\bot$ and $\top$ and p are formulas.

•

$$\begin{array}{lll}
(1) & \bot & \langle \text{C1} \rangle \\
(2) & p & \langle \text{C1} \rangle \\
(3) & (\neg\top) & \langle \text{oops!} \rangle \\
(4) & \bot & \langle \text{C1} \rangle \\
(5) & \top & \langle \text{C1} \rangle
\end{array}$$

Why the "oops"? The above is wrong at step (3). I have not written $\top$ in the construction *before* I attempted to use it!

□

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Chapter 2

Properties of the wff

Here we **speak** about wff —and discover useful properties— before we get to our main task, eventually, of USING wff *in proofs*.

2.1 Boolean Wff

Let us repeat

2.1.1 Definition. (Boolean formulas or wff) A *string* (or *expression*) A over the alphabet of Boolean symbols $\mathcal{V}$ is called a *Boolean formula* or a *Boolean well-formed formula* (in short *wff*) ***iff*** prim-recopit occurs in *some* formula construction.

The set of all wff we denote by the all-capitals **WFF**.

The wff that are either propositional variables $p, q, p'', r_{123}, \ldots$ or $\perp$ or $\top$, in short, *glue-less* , we call *Atomic* wff. □

Notation. We often want to say things such as "…bla-bla … *all variables p* …".

► Well this is not exactly right! **There is only ONE variable p!**

We get around this difficulty by having *informal names* (in the *metatheory* as we say) for Boolean variables: $\mathbf{p}, \mathbf{q}, \mathbf{r}'$, etc.

Any such bold face informal variable can stand for *any* actual variable of our alphabet $\mathcal{V}$ whatsoever.

So "all variables $\mathbf{p}$" means "any of the actual variables $p, q, r_{1110001}, \ldots$ that $\mathbf{p}$ may stand for" while "all p" is *meaningless*!

We can give a definition of formulas that is *independent* from formula constructions: OK, the above Definition 1.4.3 says that A is a wff iff it appears in a construction as

1. Atomic: $\bot, \top, \mathbf{p}$
2. A *negation* $(\neg B)$, where B $\overbrace{\text{appeared earlier in the construction}}$
3. An expression $(B \wedge C)$ or $(B \vee C)$ or $(B \to C)$ or $(B \equiv C)$, where B *and* C $\overbrace{\text{appeared earlier in the construction}}$ and

BUT we can say "B Appeared EARLIER" differently:

"B is a wff"

So,

we have

2.1.2 Definition. (The Inductive Definition of wff) An expression A over $\mathcal{V}$ is a wff just in case A is:

(1) Atomic $(\mathbf{p}, \bot, \top)$

or one of

(2) $(\neg B)$, $(B \wedge C)$, $(B \vee C)$, $(B \to C)$, $(B \equiv C)$, *where B and C are wff*. □

2.1.3 Remark. The formulas $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \equiv B)$ are read in English, from left to right, "not A", "A and B", "A or B", "if A then B" (but also " A implies B"), "A is equivalent to B".

$$\begin{array}{c} \text{last} \\ \downarrow \\ ((A \rightarrow B) \ \vee \ C) \end{array}$$

The wff in the remark have the same names as their "last glue", namely, *negation, conjunction, disjunction, implication* and *equivalence.*

Pause. Why did I say "LAST" glue?

□

2.1.4 Example. Using 1.4.3 let us verify that $((p \vee q) \vee r)$ is a wff.

Well, here is a formula construction written with annotations:

(1)	p	$\langle$atomic$\rangle$
(2)	q	$\langle$atomic$\rangle$
(3)	r	$\langle$atomic$\rangle$
(4)	$(p \vee q)$	$\langle 1 + 2 + \vee$-glue$\rangle$
(5)	$((p \vee q) \vee r)$	$\langle 4 + 3 + \vee$-glue$\rangle$

Do we have to write down *all* the atomic wff at the very *beginning*? *Not really*, but it is important to write them BEFORE they are *used* in the construction!

So, this works too:

(1)	p	$\langle$atomic$\rangle$
(2)	q	$\langle$atomic$\rangle$
(3)	$(p \vee q)$	$\langle 1 + 2 + \vee$-glue$\rangle$
(4)	r	$\langle$atomic$\rangle$
(5)	$((p \vee q) \vee r)$	$\langle 4 + 3 + \vee$-glue$\rangle$

□

Intuitively, immediate predecessors of a wff are the formulas on which we applied the last glue.

2.1.5 Definition. (Immediate predecessors (i.p.)) No atomic formula has immediate predecessors.

Any of the following wff $(A \wedge B), (A \vee B), (A \rightarrow B), (A \equiv B)$ has as i.p. A *and* B.

A is an i.p. of $(\neg A)$. □

2.1.6 Example.

- The i.p. of $((p \vee q) \vee r)$ are $(p \vee q)$ and r
- The i.p. of $(p \vee q)$ are p and q
- The only i.p. of $(\neg \top)$ is $\top$

□

2.1.7 Remark. (Priorities of glue (connectives)) The *priorities* of glue, from left to right in (1) below, go from *strongest* to *weakest*.

$$\neg, \wedge, \vee, \rightarrow, \equiv \tag{1}$$

□

Why do we care? What does "priority" do?

Well, *suppose* we do not want to always write wff down with all the brackets that Definitions 1.4.3 and 2.1.2 require.

Why wouldn't we? For better readability!

Thus we **agree** to judiciously omit brackets in a manner that *we can reinsert them correctly* if we are required to!

That is, we ***agree*** *on how to write formulas* ***sloppily*** *and get away with it!*

Is there any other way to agree *on priorities*?

Yes, BUT: As it is with **any agreement between any two parties**, there can be ONLY ONE agreement.

Remember. We are learning a "programming" language!!

So please *do* follow (1) above and the clarifications that follow below. Anything else will be wrong.

The "algorithm" is that whenever two pieces of glue compete for a variable as in, *for example*,

$$\ldots \vee p \wedge \ldots$$

then the *stronger glue wins* (higher priority). In this case it is $\wedge$ that wins and "gets" the p.

This means brackets were intended —and hence are reinserted— this way:

$$\ldots \vee (p \wedge \ldots$$

What if we have the situation

$$\ldots \vee p \vee \ldots \tag{2}$$

i.e., same glue left and right of p?

We have the agreement that all glue is right-associative, that is, in a chain like (2) *the glue on the right wins*! We insert brackets this way:

$$\ldots \vee (p \vee \ldots$$

In particular

$$\neg\neg\neg p$$

means

$$\Big(\neg\big(\neg(\neg p)\big)\Big)$$

$$p \to q \to r \to \bot$$

means

$$(p \to (q \to (r \to \bot)))$$

In $((p \to q) \to r)$ cannot remove the brackets; all are needed.

2.1.8 Definition. (Complexity of a wff) The *complexity* of a wff is *the number of occurrences* of connectives (glue) in it. Counting occurrences means that *multiplicity matters* and counts! □

2.1.9 Example. Clearly we can compute complexity correctly whether we wrote a formula with all its brackets or not.

For example, the complexity of $p \to \perp \to r$ is 2 whether we wrote it with no brackets or wrote it as Definitions 1.4.3 and 2.1.2 want: $(p \to (\perp \to r))$.

Directly from the definition above, every atomic formula has complexity zero. □

All the theorems (and their corollaries) in this section are **ABOUT** formulas of Boolean logic, and their *FORM*.

They are not *theorems OF* Boolean logic. This concept we have not defined yet!!

Theorems that are ABOUT logic we call METAtheorems.

2.1.10 Metatheorem. *Every formula A has equal numbers of left and right brackets.*

Proof. Induction on the complexity, let's call it n, of A.

1. *Basis.* $n = 0$. Then A has no glue, so it is atomic. But an atomic formula has no left or right brackets!

   Since 0=0 we are good!

2. *Induction Hypothesis*, in short "I.H." Fix an n and assume the statement for all A of complexity $\leq n$.

3. *Induction Step*, in short "I.S.", is for any A of complexity $n + 1$. As $n + 1 > 0$, A is NOT atomic THEREFORE it has one of *TWO* forms:

(a) A is $(\neg B)$ —where B is a wff.

By I.H. —applicable since A has complexity $n+1$ hence the complexity of B is $\leq n$— B has equal number of left and right brackets. Forming A we added one left and one right. So, total left=total right for A.

(b) A is $(B \circ C)$, where we wrote "$\circ$" as a *metasymbol* that stands for any *binary glue* among $\wedge, \vee, \rightarrow, \equiv$.

By I.H.

$$Blefts = k, Brights = k, Clefts = k', Crights = k'$$

So, after gluing,

$$BandClefts = k + k', BandCrights = k + k'$$

Overall (after adding external brackets for A),

we have $k + k' + 1$ lefts and $k + k' + 1$ rights. **Bingo**!

□

IMPORTANT! You will note that the induction for the formula A above essentially went like this:

- Prove the property for the atomic formulas $\mathbf{p}, \bot, \top$

  Then we assumed the I.H. that all the i.p. of A have the property.

  and *we proved* (I.S.)

- If A is $(\neg B)$, then A has the property *since the i.p. B does* (**WHY B does?**).

- If A is $(B \circ C)$, then A has the property *since the two i.p. B and C do*.

The technique above is called **Induction on** (the *shape of*) **formulas** and **does not need the concept of complexity.**

This is how we will do it *in our inductions going forward*.

2.1.11 Corollary. *Every* nonempty proper prefix *of a wff A has an excess of left (compared to right) brackets.*

Proof. I will do induction on formulas A.

- *Basis.* A is atomic. Then we are done since A has NO nonempty proper prefix!

  People also say "**then there is nothing to prove**" or "**the statement is vacuously satisfied**".

  What just happened here?! Well, I am claiming "the statement is true" and suppose that you are claiming "**the statement is false**".

  It is for you to give me a *counterexample* to what I said in order to show that you are right: Namely,

  > **You must produce a *nonempty proper prefix* of A that fails the property.**

  BUT there is no way! There is NO *nonempty proper prefix* of A!

  So I win!

- *Assume* the I.H. that *all the i.p. of A have the property.*

- For the I.S. we examine *ALL possible forms* of nonempty proper prefixes. These are:

  1. Case where A is $(\neg B)$. A nonempty proper prefix of A has one of the four **forms** below:

     (a) (Then clearly we have an excess of "(" The I.H. was NOT needed.

     (b) $(\neg$ Then clearly we have an excess of "(" The I.H. again was NOT needed.

     (c) $(\neg D$, where D is an nonempty proper prefix of B. D already has an excess of "(" by the I.H. that applies since B is an i.p. of A.
     So, adding to them the leading red "(" does no harm!

     (d) $(\neg B$ Now (2.1.10) B has equal number of lefts and rights. The leading (red)"(" contributes an excess. The I.H. again was NOT needed.

2. A is $(B \circ C)$. A nonempty proper prefix of A has one of the six **forms** below:

   (a) (Then clearly we have an excess of "(" The I.H. was NOT needed.

   (b) $(B'$, where B' is an nonempty proper prefix of B. B' already has an excess of "(" by the I.H. that applies since B is an i.p. of A. So, adding to them the leading "(" does no harm!

   (c) $(B$ B has balanced bracket numbers by 2.1.10, thus the leading "(" creates a majority of "(".

   (d) $(B\circ$ As $\circ$ adds no brackets we are done by the previous case.

   (e) $(B \circ C'$ Here B is a formula so it contributes 0 excess. C' is *a nonempty proper prefix of C* and the I.H. applies to the latter as it is an i.p. of A.

   So C' has an excess of "(" and the leading "(" of A helps too.

   (f) $(B \circ C$ Neither B nor C contribute an excess of "(" as both are formulas. The leading red "(" breaks the balance in favour of "(". □

This is easy:

2.1.12 Theorem. *Every formula A begins with an atomic wff, or with a "(".*

Proof. By 2.1.2, A is one of

- Atomic $\mathbf{p}, \bot, \top$
- $(\neg B)$
- $(B \circ C)$ where $\circ \in \{\wedge, \vee, \rightarrow, \equiv\}$

So, in the first case A begins with an atomic wff, and in the other two begins with an "(".

No Induction was used or needed! □

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2.1.13 Theorem. (Unique Readability) *The i.p. of any formula A are unique.*

So we can "deconstruct" or "parse" a formula in a unique way: It is **exactly one of** *atomic*, a *negation*, a *disjunction*, a *conjunction*, an *implication*, an *equivalence*.

Proof.

- Clearly *no atomic formula* can be read also *as one of*

  a *negation*, a *disjunction*, a *conjunction*, an *implication*, an *equivalence*

  since the atomic *contains no glue*, but all the others do.

- Can we read a formula A as two *distinct* negations? That is, *using here "=" as equality of strings, can we have*

  $$A = (\neg B) = (\neg C)?$$

  No, since $(\neg B) = (\neg C)$ implies that after we match the first two symbols (left to right) then we will continue matching all symbols —by position— until we match all of B with C and finally match the rightmost ")".

- Can we read a formula A as a *negation* **and** as a *disjunction*, or a *conjunction*, or an *implication*, or an *equivalence*? That is, can I have

  $$A = (\neg B) = (C \circ D)?$$

  No, since if we have $(\neg B) = (C \circ D)$, then from left to right the first position is OK (match) but the 2nd is NOT: *C cannot begin* *with "¬"* (see 2.1.12).

- Can we read a formula A as a $(B \circ C)$ and also *differently* as a $(D \diamond Q)$, *where $\diamond$ stands for any binary glue (including "$\circ$")?*

  Let's **assume** *that* we can *and get a* **contradiction**.

  Well, note first that if $(B \circ C) = (D \diamond Q)$ then if we have $B = D$ then this forces $\circ = \diamond$ and hence also that $C) = Q)$ which trivially (remove the ending ")") leads to $C = Q$.

  BUT this is not the case that we are looking at.

  So, assume that $B \neq D$. There are **two cases**.

Case 1. B is shorter than D, so is a nonempty proper prefix of D. Then, by 2.1.11, B has an excess of left brackets. But being a wff it also has balanced numbers of left/right brackets. **Contradiction**!

Case 2. D is shorter than B so is a nonempty proper prefix of B. Then, by 2.1.11, D has an excess of left brackets. But being a wff it also has balanced numbers of left/right brackets. **Contradiction**!

□

Why do we care about unique readability?

Well, there is an old programming language called "PL/1" (from "Programming Language 1").

The language defines "*statement*" to be *any instruction*.

It has two kinds of **if-statements**, namely

- **IF** Con **THEN** St

  and

- **IF** Con **THEN** St_1 **ELSE** St_2

  where "*Con*" stands for any *condition* and "*St*", "St_1" and "St_2" can be any *statements*.

So what does the following syntactically correct instruction do?

► Why is it "syntactically correct"?

$$\textbf{IF}\ \text{Con}_1\ \textbf{THEN}\ \textbf{IF}\ \text{Con}_2\ \textbf{THEN}\ \text{St}_1\ \textbf{ELSE}\ \text{St}_2 \tag{1}$$

We *DON'T KNOW!* The above is *ambiguous*! (No *unique* way to go backwards to figure out what it says)

So who cares? Not knowing which syntax is intended we *do not know* what action is intended!

For example, say Con_1 evaluates as *false*.

Now, *one* meaning of (1) —i.e., ONE WAY to choose I.P.— is

$$\textbf{IF } \mathrm{Con}_1 \textbf{ THEN } \left\{\textbf{IF } \mathrm{Con}_2 \textbf{ THEN } \mathrm{St}_1 \textbf{ ELSE } \mathrm{St}_2\right\} \tag{2}$$

which does *nothing* (*skips all*) and control goes to the next statement, whatever that is.

Another meaning of (1) —ANOTHER WAY TO CHOOSE I.P.— is

$$\textbf{IF } \mathrm{Con}_1 \textbf{ THEN } \left\{\textbf{IF } \mathrm{Con}_2 \textbf{ THEN } \mathrm{St}_1\right\} \textbf{ ELSE } \mathrm{St}_2 \tag{3}$$

which —still under the assumption that Con_1 is *false*— causes the execution of St_2.

POSTSCRIPT The PL/1 language ***was NOT redefined/corrected to remove the ambiguity***! Rather, the **Compiler** was programmed to "believe" that (2)) above was meant (*that is, the keyword* "ELSE matches the closest IF")

2.2 Boolean Semantics

*Boolean Logic is about the **behaviour of glue**. That is, we use Boolean logic to find out how glue* influences *the truth-value of a formula, assuming values are arbitrarily assigned to the atomic formulas.*

What values do we have in mind?

The so-called truth-values, *true* and *false*.

These values are OUTSIDE Boolean Logic.

Did *you* see them in the alphabet $\mathcal{V}$? **Nor did I**!!

They are in the metatheory of Boolean Logic, that is, in the domain where we are speaking about the logic, rather than using the logic.

2.2.1 Definition. A *state* v (or s) is a *function that assigns the value* $\mathbf{f}$ (*false*) or $\mathbf{t}$ (*true*) to *every* Boolean variable, while the *constants* $\bot$ and $\top$, *necessarily, always* get the values $\mathbf{f}$ and $\mathbf{t}$ respectively.

None of these symbols —$v, s, \mathbf{t}, \mathbf{f}$— are in the Boolean logic alphabet $\mathcal{V}$. They are *all metasymbols* in the *meta*theory.

The $\mathbf{f}$ and $\mathbf{t}$ we call *truth values*.

On paper or on the chalk board one usually *underlines* rather than *bolds* —as bolding is cumbersome— so one denotes $\mathbf{f}$ as $\underline{\mathrm{f}}$ and $\mathbf{t}$ as $\underline{\mathrm{t}}$

respectively.

The fact that v gives (assigns) the value $\mathbf{f}$ to the variable q'' is denoted by $v(q'') = \mathbf{f}$. $\square$

Therefore a state v is (think of MATH 1019/1028 here!) an *infinite* input/output *table* like the one below

input	output
$\bot$	$\mathbf{f}$
$\top$	$\mathbf{t}$
p	$\mathbf{t}$
q	$\mathbf{f}$
$\vdots$	$\vdots$

where *no two rows can have the same input but different outputs*.

Again in the jargon of MATH1019/1028 the table is what we call a *function*! This observation justifies the notation

$$\overset{\text{function}}{\overset{\downarrow}{v}} \left(\underset{\text{input}}{\underset{\uparrow}{q''}} \right) = \overset{\text{output}}{\overset{\downarrow}{\mathbf{f}}}$$

in the last sentence of Definition 2.2.1.

► Why an *infinite* table?

Because our *Boolean logic language* has *infinitely many variables* and a state, by definition, assigns a value to *each of them*.

Why are $\mathbf{f}, \mathbf{t}$ outside logic? Aren't they **symbols**?

- Yes, but not ALL symbols belong to our Boolean logic!!
- Compare: "3" and "5" are **informal** symbols standing for the concepts "three objects" (or "3rd position") and "5 objects "(or "fifth position"). Equally well the earlier used (by ancient Greeks) γ, ε and III, V (by Romans) meant the same thing as three objects (or 3rd position) and 5 objects (or fifth position) (respectively).
- Formally, in number theory, "3" is denoted by "$SSS0$" and "5" is denoted by "$SSSSS0$".
- Same here: $\perp, \top$ are our *formal* "false", "true", while $\mathbf{f}, \mathbf{t}$ are our informal ones. PL/1 uses $0B$ and $1B$ respectively ("B" stands for "bit") while C uses 0 and any *nonzero* number respectively.

2.2.2 Definition. (Truth tables) In the *metatheory* of Boolean logic there are five *operations* we are *interested* in that can be applied on the members of the set of *truth values* $\{\mathbf{t}, \mathbf{f}\}$.

Each operation takes its input(s) from the above set, and its outputs are also in this set.

We have one operation for each connective (glue) and in order to keep track of which is formal *and which is* not *we use the generic letter F (for "function") subscripted by the name of the corresponding glue*.

These functions of the metatheory are called Boolean functions and are the following.

$$F_{\neg}(x), F_{\vee}(x, y), F_{\wedge}(x, y), F_{\rightarrow}(x, y), F_{\equiv}(x, y)$$

So, "$\vee$" does *NOT* operate on *inputs* $\mathbf{f}, \mathbf{t}$. $F_{\vee}$ does; in the metatheory!

What "$\vee$" does operate on? What does it glue? *FORMULAS!*

The behaviour of these functions —input/output behaviour, that is— is fully described by the following table that goes by the nickname "*truth table*".

x	y	$F_{\neg}(x)$	$F_{\vee}(x,y)$	$F_{\wedge}(x,y)$	$F_{\rightarrow}(x,y)$	$F_{\equiv}(x,y)$
f	**f**	**t**	**f**	**f**	**t**	**t**
f	**t**	**t**	**t**	**f**	**t**	**f**
t	**f**	**f**	**t**	**f**	**f**	**f**
t	**t**	**f**	**t**	**t**	**t**	**t**

□

Chapter 3

What makes our Logic "Classical"

3.1 States and Truth tables

Refer to the truth table on p.58 and let us discuss the column of $F_{\rightarrow}(x, y)$.

The most "straightforward" entry in this column is arguably, the one for input $(\mathbf{t}, \mathbf{f})$.

This function is describing the truth-value of *implications*, and the x input is the *hypothesis* while the y input is the *conclusion*.

Thus having $F_{\rightarrow}(\mathbf{t}, \mathbf{f}) = \mathbf{f}$ can be **interpreted** as saying that *the implication cannot be RIGHT, i.e.,* $\mathbf{t}$*, IF we start with a* ***true*** hypothesis *and end up with a* ***false*** conclusion.

We can easily agree with the statement in red above since our intuition *accepts that "$\rightarrow$" preserves truth from left to right.*

The *same principle supports* the behaviour of $F_{\rightarrow}$ in the other three rows.

For example you would be wrong to tell me: "Hey, $F_{\rightarrow}(\mathbf{f}, \mathbf{t})$ is not right". I will respond: "Oh yea? Show me that it does *not preserve* truth from hypothesis to conclusion! What truth?!"

So far, *states* give meaning (values) *to atomic formulas only*. Let us *extend* this meaning-giving to *any wff*.

3.1.1 Definition. (The value of a wff in some state, v) We *extend any* state v to be meaningful *not only with atomic arguments* but also with any wff arguments.

We will call such an *extension of* v by the same letter, but will "cap" it with a "hat", $\overline{v}$, since it is a different function!

What *IS* an "extension" of v?

It is a function $\overline{v}$ that *on the arguments that v is defined* so is $\overline{v}$ and gives the same output!

But $\overline{v}$ is defined *on more inputs: On ALL wff found in WFF.*

The definition of $\overline{v}$ is inductive:

The first three lines below simply say that $\overline{v}$ agrees with v on the inputs that the latter is defined on.

The remaining lines trace along **the inductive definition of wff,** and give the value of a wff **using the values —via "recursive calls"— of its UNIQUE i.p.**

You see now the **significance** of the uniqueness of i.p.!!!

$$
\begin{aligned}
\overline{v}(\mathbf{p}) &= v(\mathbf{p}) \\
\overline{v}(\top) &= \mathbf{t} \\
\overline{v}(\bot) &= \mathbf{f} \\
\overline{v}\Big((\neg A)\Big) &= F_{\neg}\Big(\overline{v}(A)\Big) \\
\overline{v}\Big((A \wedge B)\Big) &= F_{\wedge}\Big(\overline{v}(A), \overline{v}(B)\Big) \\
\overline{v}\Big((A \vee B)\Big) &= F_{\vee}\Big(\overline{v}(A), \overline{v}(B)\Big) \\
\overline{v}\Big((A \to B)\Big) &= F_{\to}\Big(\overline{v}(A), \overline{v}(B)\Big) \\
\overline{v}\Big((A \equiv B)\Big) &= F_{\equiv}\Big(\overline{v}(A), \overline{v}(B)\Big)
\end{aligned}
\qquad \square
$$

Truth tables are more convenient to <u>understand</u>, AND <u>misunderstand</u>!

For example the 6-th equality in the previous definition can also be depicted as:

A	B	$A \vee B$
f	**f**	**f**
f	**t**	**t**
t	**f**	**t**
t	**t**	**t**

Says

$$\overline{v}((A \vee B)) = F_{\vee}(\overline{v}(A), \overline{v}(B))$$

At a glance the table says that to compute the value of $A \vee B$ you just utilise the values of the i.p. A and B as indicated.

The <u>misunderstanding</u> you MUST avoid is this: The two left columns are NOT values you <u>assign</u> to A and B.

You can <u>assign</u> values <u>ONLY</u> to <u>ATOMIC</u> formulas!

What these two columns DO say is that *the formulas A and B have <u>each</u> two <u>possible</u> values*.

<u>That is 4 **pairs of values**, as displayed!</u>

3.2 Finite States

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We say a variable **p** *occurs in a formula meaning the obvious: It is,* as a string, *a substring —a part— of the formula*.

3.2.1 Theorem. *Given a formula A. Suppose that two states, v and s agree on* **all** *the variables of A. Then $\overline{v}(A) = \overline{s}(A)$.*

Proof. We do induction on the formula A:

1. Case where A is atomic. Well if it is $\top$ or $\bot$ then $\overline{v}(A) = \overline{s}(A)$ is true. If A is $\mathbf{p}$, then

$$\overline{v}(A) = v(A) \stackrel{Hypothesis}{=} s(A) = \overline{s}(A)$$

I.H.: Claim is true for all i.p. of A.

2. Case where A is $(\neg B)$. The value of A —whether under v or under s— is *determined* by a recursive call to $\overline{v}(B)$ and $\overline{s}(B)$. Seeing that all the variables of B are in A, the I.H. yields $\overline{v}(B) = \overline{s}(B)$ and hence $\overline{v}(A) = \overline{s}(A)$.

3. Case where A is $(B \circ C)$. The value of A —whether under v or under s— is *determined* by recursive calls to $\overline{v}(B)$ and $\overline{v}(C)$ on one

hand and $\overline{s}(B)$ and $\overline{s}(C)$ on the other.

Seeing that all the variables of B and C are in A, the I.H. yields

$$\overline{v}(B) = \overline{s}(B) \text{ and } \overline{v}(C) = \overline{s}(C) \qquad (*)$$

Hence no matter which one of the $\wedge, \vee, \rightarrow, \equiv$ the symbol $\circ$ stands for, it operating on $\overline{v}(B)$ and $\overline{v}(C)$ or on $\overline{s}(B)$ and $\overline{s}(C)$ will yield the same result by $(*)$.

That is, $\overline{v}(A) = \overline{s}(A)$. □

$F(p, q, r), G(r, r', p)$keep only p, q, r, r' in the state

3.2.2 Remark. (Finite "appropriate" States) A state v is *by definition an infinite table*.

By the above theorem, the value of any wff A in a state v *is determined only by the values of v* ON THE VARIABLES OF A, since *any other state that agrees with v on said variables gives the same answer.*

Thus, going forward we will be utilising *finite appropriate states* to compute the truth values of any wff.

That is, we discard from the infinite state all the rows that contain variables not occurring in the formulas of interest. □

3.3 Tautologies and Tautological Implication

3.3.1 Definition. (Tautologies and other things. . .)

1. A *Tautology* is a formula A which is true in *all* states. That is, for *all* v, we have $\overline{v}(A) = \mathbf{t}$.

   We write "$\models_{taut} A$" for "*A is a tautology*".

2. A *contradiction* is a formula A such that, for *all* v, we have $\overline{v}(A) = \mathbf{f}$.

   Clearly, for *all* v, we have $\overline{v}(\neg A) = \mathbf{t}$.

3. A is *satisfiable* iff for *some* v, we have $\overline{v}(A) = \mathbf{t}$.

   We say that v satisfies A.

► *Boolean logic for the user helps to discover tautologies.*

□

We saw that WFF denotes the set of all (well-formed) formulas.

Capital Greek letters that are different from any Latin capital letter are used to denote arbitrary sets of formulas. Such letters are $\Gamma, \Delta, \Phi, \Psi, \Omega, \Pi, \Sigma$. As always, in the rare circumstance you run out of such letters you may use primes and/or (natural number) subscripts.

3.3.2 Definition. (Tautological implication: binary $\models_{taut}$)

1. Let Γ be a set of wff. *We say that v satisfies Γ iff v satisfies every formula in Γ.*

2. We say that Γ *tautologically implies A* —and we write this as $\Gamma \models_{taut} A$— iff *every state v that satisfies Γ also satisfies A.*

   The configuration

   $$\Gamma \models_{\text{taut}} A \tag{1}$$

   is called a tautological implication claim.

   We call Γ *the set of hypotheses* or *premises* of the tautological implication, while A is the *conclusion*. □

IMPORTANT! The task to verify (1) needs work on our part **ONLY** with v that satisfy Γ.

If there is NO such v then the claim (1) is VACUOUSLY valid! YOU *cannot* contradict its validity for you **will *need* a v that *satisfies*** Γ **but NOT A.**

You have NO COUNTEREXAMPLE.

3.3.3 Example.

(1) If $\models_{\text{taut}} A$, then for any Σ, we have $\Sigma \models_{\text{taut}} A$.

The converse is not valid:

(2) We have $\mathbf{p} \models_{\text{taut}} \mathbf{p} \vee \mathbf{q}$. Indeed, for any v such that $v(\mathbf{p}) = \mathbf{t}$ we compute $\overline{v}(\mathbf{p} \vee \mathbf{q}) = \mathbf{t}$ from the truth table for $\vee$.

Yet, $\mathbf{p} \vee \mathbf{q}$ is NOT a tautology. Just take $v(\mathbf{p}) = v(\mathbf{q}) = \mathbf{f}$

Note also the obvious: $A \models_{taut} A \vee B$, for any wff A and B. Again use the truth table of p.63. □

In view of 3.2.1 we can check all of *satisfiability*, *tautology* status, and *tautological implication* with *finite* Γ *using a finite truth table*.

Examples.

Example 1. $\bot \models_{taut} A$.

Because no v satisfies the lhs of "$\models_{taut}$" so according to Definition, I rest my case.

Example 2. Let us build a truth table for $A \to B \vee A$ and see what we get.

I wrote sloppily, according to our priorities agreement.

I mean $(A \to (B \vee A))$.

We align our part-work under the glue since it is the glue that causes the output.

Here $\to$ is the last (applied) glue. *Under it we write the final results for this formula*.

Since A and B *are not necessarily atomic*, the values under A and B in the table below are *possible* values *NOT assigned values*! *So $(A \to (B \vee A))$ is a tautology.*

A	B	A	$\rightarrow$	B	$\vee$	A
f	**f**		**t**		**f**	
f	**t**		**t**		**t**	
t	**f**		**t**		**t**	
t	**t**		**t**		**t**	

Example 3. Here is another tautology. I will verify this by a shortcut method, WITHOUT building a truth table.

I will show

$$\models_{taut} ((A \rightarrow B) \rightarrow A) \rightarrow A \tag{1}$$

I will do so by arguing that *it is IMPOSSIBLE TO MAKE (1) FALSE*.

- *If (1) is false* then A *is false* and $(A \rightarrow B) \rightarrow A$ *is true*.
- Given the two blue statements above, it *must* be that $A \rightarrow B$ *is false*. *IMPOSSIBLE, since A is false!*

Chapter 4

Substitution and Schemata

4.0.1 Definition. (Substitution in Formulas)

The *META*notation

$$A[\mathbf{p} := B] \tag{1}$$

where A and B are formulas and $\mathbf{p}$ is any variable means

- **As an Action**: "*Find and replace by B ALL occurrences of* $\mathbf{p}$ *in A*".

- **As a Result**: *The STRING resulting from the action* described in the previous bullet. □

1. In the *META*theory of Logic where we use the exprssion "[$\mathbf{p} := B$]" we Agree to Give it The Highest priority: Thus, $A \wedge B[\mathbf{q} := C]$ means $A \wedge \Big(B[\mathbf{q} := C]\Big)$ and $\neg A[\mathbf{p} := B]$ means $\neg\Big(A[\mathbf{p} := B]\Big)$

2. Clearly if $\mathbf{p}$ does NOT occur in A, then the "action" found nothing to replace, so the resulting string —according to (1)— in this case is just A; NO CHANGE.

We observe the following, according to the inductive definition of formulas.

With reference to (1) of page 73, we prove that the result of (1) is a wff.

1. A is atomic. In particular, *using "=" for equality of strings*,

   - A is $\mathbf{p}$. Then $A[\mathbf{p} := B] = B$

   - A is $\mathbf{q}$ —where by $\mathbf{q}$ we denote a *variable other than the one* $\mathbf{p}$ *stands for*. Then $A[\mathbf{p} := B] = A$ —*no change*.

   - A is $\perp$ or $\top$. Then $A[\mathbf{p} := B] = A$ —*no change*. *But A, B are wff, thus so is $A[\mathbf{p} := B]$ for A atomic.*

   Now take an I.H. on the i.p. of A and argue two cases:

2. A is $(\neg C)$. *Then all occurrences of* $\mathbf{p}$ *are in* C. *All Action happens with* C.

   Thus $A[\mathbf{p} := B]$ is effected by doing first $S = C[\mathbf{p} := B]$.

   Above I named the result S for convenience. This is a wff by I.H.

Now $A[\mathbf{p} := B]$ is $(\neg S)$. A wff.

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3. $\underline{A \text{ is } (C \circ D)}$. *Then all occurrences of* $\mathbf{p}$ *are in* C *or* D.

   All Substitution Action happens with C *and also* D.

   Thus $A[\mathbf{p} := B]$ *is effected by doing*

   (a) $S = C[\mathbf{p} := B]$

   (b) $T = D[\mathbf{p} := B]$

      Where I named the two above results S *and* T *for convenience.*

      S and T are wff by the I.H.

   (c) To conclude, *use concatenation —in the order indicated below—* to obtain the *wff*

$$(S \circ T)$$

4.0.2 Proposition. *For every wff A and wff B and any variable $\mathbf{p}$, $A[\mathbf{p} := B]$ is* a wff.*

Proof. See the preceding argument. □

We are poised to begin describing the proof system of Boolean logic.

To this end we will need the notation that is called *formula schemata* or *formula schemas* (if you consider "schema" an English word —but it is not!).

$$(A \vee (B \rightarrow \mathbf{p}))$$

$$A[\mathbf{p} := B]$$

4.0.3 Definition. (Schema, Schemata) *Add to the alphabet $\mathcal{V}$* the following symbols:

1. "[", "]", and ":="

2. All NAMES of formulas: $A, B, C, \ldots$, *with or without primes and/or subscripts*.

3. All *metasymbols* for variables: $\mathbf{p}, \mathbf{q}, \mathbf{r}$, *with or without primes and/or subscripts*.

Then a *formula schema is a STRING over the augmented alphabet, which becomes a wff whenever all metasymbols of types 2 and 3 above, which occur in the string, are replaced by wff and actual variables (like*

*We are purposely sloppy with jargon here —like everybody else in the literature: "IS" means "results into".

non bold p, q, r'', q_{13}''') *respectively, and all actions indicated by* $[\mathbf{p} := B]$ *are performed.*

A formula that we obtain by the process described in the paragraph above *is called an* **Instance of the Schema**. □

Three examples of schemata.

(1) A: This Schema stands (is a placeholder) for a wff! So trivially, *if I plug into A an actual wff, I get that wff as an* ***instance!***

(2) $(A \equiv B)$: Well, whatever formulas I substitute into A and B (metavariables) I get a wff by the inductive definition of wff.

(3) $A[\mathbf{p} := B]$: We know that if I substitute A and B by formulas and $\mathbf{p}$ by a Boolean variable I get a wff (4.0.2).

Next stop is Proofs!

In *proofs* we use *Axioms and Rules* (of *Inference*).

It is the habit in the literature to write Rules as *fractions*:

$$\frac{P_1, P_2, \ldots, P_n}{Q} \tag{R}$$

where *all* of $P_1, \ldots, P_n, Q$ are schemata.

Example,

$$\frac{\mathbf{p}}{(\mathbf{p} \rightarrow q)}$$

I note that the fraction (R) above, *the RULE, is meant as an input / output device*.

► An *Instance of the Rule* is a *common instance* of all $P_1, \ldots, P_n, Q$, that is, every wff-metavariable A and variable-metavariable $\mathbf{p}$ are *replaced by the same wff and actual variable throughout* respectively.

Jargon. We call *the schema* (if one, or *schemata* if many) on the numerator the *premise(s)* but also *hypothes(is/es)*.

Jargon. The single schema in the denominator we call the *conclusion* (also "*result*" or "output").

▶ **More Jargon**. For every instance of (R)

all the P_i and the Q become wff $P'_1, \ldots, P'_n, Q'$

We say

the Rule, with input $P'_1, \ldots, P'_n$ yields output (result, conclusion) Q'.

▶ We also say that Q' is the *result* of the *application* of (R) to $P'_1, \ldots, P'_n$.

4.1 Rules and Axioms of Boolean Logic

4.1.1 Definition. (Rules of Inference of Boolean Logic) There are just two:

Rule1

$$\frac{A \equiv B}{C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]} \qquad (Leibniz)$$

There are NO restrictions in the use of "Leibniz".

In particular,

(a) *it is NOT required that* **p** *actually occurs in* C.

If it does not, *then the denominator is* $C \equiv C$.

(b) *The single hypothesis can be ANY equivalence.*

Rule2 "Equanimity" Rule.

$$\frac{A, A \equiv B}{B} \qquad (Eqn)$$

There are NO restrictions in the use of "Equanimity" other than

"A" must be the left part of the equivalence on the numerator.

Does it matter "left" or "right"? FOR NOW YES!, as we have NO basis to decide otherwise and will NOT be caught "importing" so-called "knowledge" (from other courses) whose validity we did NOT prove in our Logic; YET!!!

□

4.1.2 Definition. (Axioms of Boolean Logic) In the following, (1)–(11), A, B, C *name* or *stand for* arbitrary wff.

Properties of $\equiv$

Associativity of $\equiv$ $\quad ((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C))$ (1)

Commutativity of $\equiv$ $\quad (A \equiv B) \equiv (B \equiv A)$ (2)

Properties of $\bot, \top$

$\top$ and $\bot$ $\quad \top \equiv \bot \equiv \bot$ (3)

Properties of $\neg$

Introduction of $\neg$ $\quad \neg A \equiv A \equiv \bot$ (4)

Properties of $\vee$

Associativity of $\vee$ $\quad (A \vee B) \vee C \equiv A \vee (B \vee C)$ (5)

Commutativity of $\vee$ $\quad A \vee B \equiv B \vee A$ (6)

Idempotency of $\vee$ $\quad A \vee A \equiv A$ (7)

Distributivity of $\vee$ over $\equiv$ $\quad A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C \quad (8)$

"Excluded Middle" $\quad A \vee \neg A \quad (9)$

Properties of $\wedge$

"Golden Rule" $\quad A \wedge B \equiv A \equiv B \equiv A \vee B \quad (10)$

Properties of $\rightarrow$

Implication $\quad A \rightarrow B \equiv A \vee B \equiv B \quad (11)$

All of the above (1)–(11) except (3) are schemata for axioms. We call them *Axiom Schemata*, while (3) is an *Axiom*. Each axiom schema above defines *infinitely many axioms* that are its *Instances*.

So our axioms are (3) and all the instances of the Axiom Schemata (1), (2), (4)–(11).

We reserve the Greek letter Λ for the set of all Axioms of Boolean Logic. □

4.1.3 Definition. (Proofs) Let Γ (could use Σ, Θ, Ψ etc., instead of Γ —there is nothing special about the letter Γ!) be some set of wff.

A *proof* from Γ is any *finite ordered* sequence of formulas that satisfy the following two *specifications*:

At every step of the *Construction (that we call "Proof")* we *may write*

Proof 1. *Any ONE formula* from Λ or Γ.

Proof 2. *Any wff A which is the RESULT of an Application* of the rule *Leib* or rule *Eqn* to wff(s) that appeared in *THIS* proof *before* A.

A proof from Γ is also called "Γ-proof". □

4.1.4 Remark. (1) So, *a proof is a totally syntactic construct*, *totally devoid of semantic concepts*.

(2) Γ is a *convenient* set of "*additional hypotheses*".

Syntactically the elements of Γ "behave" like the Axioms from Λ —as it is clear from 4.1.3, item 1— but *semantically they are NOT the same*:

While every member of Λ is a *tautology* by choice,

this *need NOT be the case for the members of* Γ.

(3) Since *every proof* (from some Γ) has *finite length*,

only a finite part of Γ and Λ can ever appear in some proof.

□

4.1.5 Definition. (Theorems) Any wff A that appears in a Γ-proof is called a Γ-theorem.

We also say, "A is a theorem *from* Γ".

In symbols, the sentence "A is a Γ-theorem", is denoted by "$\Gamma \vdash A$".

If $\Gamma = \emptyset$ then we write $\vdash A$.

That is, Λ never appears to the left of the turnstile "$\vdash$".

We call an A such that $\vdash A$ an absolute or logical theorem. □

4.1.6 Remark. That A is a Γ-theorem is certified by a Γ-proof like this

$$B_1, \ldots, B_n, A, C_1, \ldots, C_m \tag{1}$$

the sequence (1) obeying the *specifications* of 4.1.3.

Clearly, the sequence (2) below also satisfies the specifications, since each specification for a B_i or A that utilises *rules* refers to formulas *to the left only*.

Thus the sequence (2) is also a Γ-proof of A!

$$B_1, \ldots, B_n, A \tag{2}$$

The bottom line of this story is expressed as either

1. *If you are proving a theorem A*, *just stop as soon as you wrote it down with justification* in a proof!

   OR

2. *A Γ-theorem is a wff that appears at the end of some proof.* □

Concatenating two Γ*-proofs*

$$A_1, \ldots, A_n$$

and

$$B_1, B_2, \ldots, B_r$$

results in a Γ*-proof.*

Indeed, checking

$$B_1, B_2, \ldots, B_r, A_1, \ldots, A_n$$

from left to right we give EXACTLY the same reasons that we gave for writing the formulas down in each standalone proof.

The reader did not miss to note the similarity between a Γ*-proof and a formula construction.*

Let us develop an *Inductive definition* for the concept “theorem” just as we did before for the concept “wff”.

So we learnt that a Γ-theorem, *let's call it A*, satisfies

1. *A is member of Λ or Γ*

2. *A* appears *in a Γ-proof as the result of an application of Eqn to wff to its left in the proof.*

3. *A* appears *in a Γ-proof as the result of an application of Leib to wff to its left in the proof.*

Let us rephrase the *blue* "appears" above, *remembering that a Γ-theorem IS a formula that appears in a Γ-proof*.

1. *A is member of Λ or Γ*

2. *A is the result of an application of Eqn to two Γ-theorems.*

3. *A is the result of an application of Leib to one Γ-theorem.*

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4.1.7 Exercise. How do we do this?

By providing a Γ-proof where our target theorem appears, OR by using the Inductive Definition of the previous page.

(1) $A, B, C \vdash A$, for any wff A
(2) More generally, if $A \in \Sigma$, then $\Sigma \vdash A$
(3) $\vdash B$, for all $B \in \Lambda$

$\square$

4.1.8 Remark. (Hilbert-style Proofs)

A Γ-proof is also called a "*Hilbert-style proof*" —in honour of the great mathematician *David Hilbert*, who was *the first big supporter* of the idea *to use SYNTACTIC (FORMAL) logic as a* TOOL *in order to do CORRECT mathematics*.

We arrange Hilbert proofs *vertically*, one formula per line, *numbered by its position number*, adding "*annotation*" *to the right of the formula we just wrote*, articulating briefly *HOW exactly* we followed the spec of Definition 4.1.3.

Practical Note. Forget numbering or annotation, or that *each line contains ONE wff ONLY* and the result is *a very bad grade*! :) □

4.1.9 Example. (Some very simple Hilbert Proofs)

(a) We verify that "$A, A \equiv B \vdash B$" (goes without saying, *for all wff A and B*).

Well, just write a proof of B with "Γ" being $\{A, A \equiv B\}$.

BTW, we indicate a finite "Γ" like $\{A, A \equiv B\}$ without the braces "{ }" when writing it to the left of "$\vdash$".

(1)	A	$\langle$hypothesis$\rangle$
(2)	$A \equiv B$	$\langle$hypothesis$\rangle$
(3)	B	$\langle (1) + (2) + (Eqn) \rangle$

Incidentally, *members of Γ are annotated as "hypotheses" and going forward we just write "*hyp*".*

Members of Λ we annotate as "Axioms".

Since A and B are arbitrary *undisclosed* wff, *the expression* $A, A \equiv B \vdash B$ *is a Theorem Schema* (*a theorem, no matter what formulas we plug into A and B*).

(b) Next verify the Theorem Schema

$$A \equiv B \vdash C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$$

Here you go:

$$\begin{array}{lll} (1) & A \equiv B & \langle \text{hyp} \rangle \\ (2) & C[\mathbf{p} := A] \equiv C[\mathbf{p} := B] & \langle (1) + \text{Leib} \rangle \end{array}$$

C can be any wff (and p any actual Boolean variable) so from ONE hypothesis for fixed A and B we can derive an infinite number of theorems of the "shape" $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$.

(c) Something more substantial. *Our First Derived Rule!*

We establish the following *Theorem Schema* that we will refer to as *Transitivity* of $\equiv$—or simply "*Trans*". **How**? *We write a Hilbert proof!*

$$A \equiv B, B \equiv C \vdash A \equiv C \qquad (Trans)$$

(1)	$A \equiv B$	$\langle$hyp$\rangle$
(2)	$B \equiv C$	$\langle$hyp$\rangle$
(3)	$(A \equiv B) \equiv (A \equiv C)$	$\langle$(2) + $(Leib)$, Denom. "$A \equiv \mathbf{p}$" where $\mathbf{p}$ is "*fresh*"$\rangle$
(4)	$A \equiv C$	$\langle$(1) + (3) + $(Eqn)\rangle$

Why must $\mathbf{p}$ *be fresh?*

Say A *is* $\mathbf{p} \wedge \mathbf{q}$.

Then, feeding B *to* $\mathbf{p}$, "$A \equiv \mathbf{p}$" *becomes* $B \wedge \mathbf{q} \equiv B$, *which is NOT the SAME STRING AS* $A \equiv B$.

$$\overbrace{B \wedge q}^{this\ is\ NOT\ A} \equiv B$$

(d) *And a Tricky One!* Verify that "$A \equiv A$" is an absolute theorem for all A. That is,

$$\vdash A \equiv A$$

No "HYP" in the proof below!!

(1) $A \vee A \equiv A$ ⟨axiom⟩

(2) $A \equiv A$ ⟨(1) + (*Leib*): $A[\mathbf{p} := A \vee A] \equiv A[\mathbf{p} := A]$ where $\mathbf{p}$ is "fresh"⟩

A shorter way to say this might be to invoke the ☡-remark after Leibniz in 4.1.1. I can prove

$$A \equiv B \vdash C \equiv C$$

if C has no free $\mathbf{p}$.

Well, if I take, *say* "$A \equiv B$" to be $\overbrace{\top}^{A} \equiv \overbrace{(\bot \equiv \bot)}^{B}$ then I get $\vdash C \equiv C$. □

4.1.10 Metatheorem. (Hypothesis Strengthening) If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$, then also $\Delta \vdash A$.

Proof. *A Γ-proof for A is also a Δ-proof*, since every time we say about a formula B in the proof "legitimate since $B \in \Gamma$" we can say instead "legitimate since $B \in \Delta$". □

4.1.11 Metatheorem. (Transitivity of $\vdash$) Assume $\Gamma \vdash B_1$, $\Gamma \vdash B_2, \ldots, \Gamma \vdash B_n$. Let also $B_1, \ldots, B_n \vdash A$. Then we have $\Gamma \vdash A$.

Proof.

We have Γ-proofs

$$\boxed{\ldots, B_1} \tag{1}$$

$$\boxed{\ldots, B_2} \tag{2}$$

$$\vdots$$

$$\boxed{\ldots, B_n} \tag{n}$$

We also have a $\{B_1, \ldots, B_n\}$-proof

$$\boxed{\ldots, B_i, \ldots, A} \tag{n+1}$$

Concatenate all proofs (1)–(n) (in any order) and to the right of the result glue the proof $(n+1)$.

We have the following proof:

$$\begin{array}{c} \text{DON'T say "hyp"} \\ \downarrow \end{array}$$

$$\boxed{\ldots, B_1}, \boxed{\ldots, B_2}, \ldots, \boxed{\ldots, B_n}, \boxed{\ldots, B_i, \ldots, A}$$

$$\begin{array}{c} \uparrow \\ \text{SAY: obtained earlier; \underline{see box } \underline{i}} \end{array}$$

□

> So if we view $B_1, \ldots, B_n \vdash A$ as a (derived or "macro" rule) then this "rule" is applicable!
>
> If the B_i are Γ-theorems and $B_1, \ldots, B_n \vdash A$, then we can apply the latter as a "rule" to obtain the Γ-theorem A.

4.1.12 Corollary. *If* $\Gamma \vdash A$ *and also* $\Gamma \cup \{A\} \vdash B$*, then* $\Gamma \vdash B$*.*

In words, the conclusion says that A drops out as a hypothesis and we get $\Gamma \vdash B$.

That is, a THEOREM A can be invoked just like an axiom OR a hyp in a proof!

Proof. We have *two* proofs:

$$\boxed{\overbrace{\ldots A}^{\text{from } \Gamma}}$$

and

$$\boxed{\overbrace{\ldots A \ldots B}^{\text{from } \Gamma \cup \{A\}}}$$

When the second box is *standalone*, the justification for A is "hyp".

Now concatenate the two proofs above in the order

$$\boxed{\overbrace{\ldots A}^{\text{from } \Gamma}}\boxed{\overbrace{\ldots A \ldots B}^{\text{from } \Gamma \cup \{A\}}}$$

Now change all the justifications for the red A in the right box from "hyp" to the same exact reason you gave to the A in box one —OR, as in the proof of 4.1.11 say about the red A: "obtained earlier in box 1".

Thus, the status of A as "hyp" is removed and B is proved from Γ alone. □

4.1.13 Corollary. If $\Gamma \cup \{A\} \vdash B$ and $\vdash A$, then $\Gamma \vdash B$.

Proof. By hyp strengthening, I have $\Gamma \vdash A$. Now apply the previous corollary. □

4.1.14 Theorem. $A \equiv B \vdash B \equiv A$

Proof.

$$
\begin{array}{lll}
(1) & A \equiv B & \langle \text{hyp} \rangle \\
(2) & (A \equiv B) \equiv (B \equiv A) & \langle \text{axiom} \rangle \\
(3) & B \equiv A & \langle (1,2) + \text{Eqn} \rangle
\end{array}
$$

4.1.15 Theorem. $\vdash (A \equiv (B \equiv C)) \equiv ((A \equiv B) \equiv C)$

NOTE. This is the mirror image of Axiom (1).

Proof.

$$
\begin{array}{ll}
(1) & ((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C)) \ \langle \text{axiom} \rangle \\
(2) & (A \equiv (B \equiv C)) \equiv ((A \equiv B) \equiv C) \ \langle (1)\text{+4.1.14} \rangle
\end{array}
$$
□

4.1.16 Remark. Thus, *in a chain of <u>two</u> "≡" we can shift brackets from left to right (axiom) but also right to left (above theorem).*

So it does not matter how brackets are inserted in such chain.

An induction proof on <u>chain length</u> (see course URL, bullet #4 under Notes:

http://www.cs.yorku.ca/~gt/courses/MATH1090F22/1090.html *) extends this remark to any chain of "≡", of any length.* □

4.1.17 Theorem. (The other (Eqn)) $B, A \equiv B \vdash A$

Proof.

$$
\begin{array}{lll}
(1) & B & \langle \text{hyp} \rangle \\
(2) & A \equiv B & \langle \text{hyp} \rangle \\
(3) & B \equiv A & \langle (2) + 4.1.14 \rangle \\
(4) & A & \langle (1, 3) + (Eqn) \rangle
\end{array}
$$

□

4.1.18 Corollary. $\vdash \top$

Proof.

$$
\begin{array}{lll}
(1) & \top \equiv \bot \equiv \bot & \langle \text{axiom} \rangle \\
(2) & \bot \equiv \bot & \langle \text{theorem} \rangle \\
(3) & \top & \langle (1, 2) + (Eqn) \rangle
\end{array}
$$

□

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4.1.19 Theorem. $\vdash A \equiv A \equiv B \equiv B$

(1) $(A \equiv B \equiv B) \equiv A$ ⟨axiom; brackets as I please!⟩
(2) $A \equiv (A \equiv B \equiv B)$ ⟨(1) + 4.1.14⟩ □

4.1.20 Corollary. $\vdash \bot \equiv \bot \equiv B \equiv B$ *and* $\vdash A \equiv A \equiv \bot \equiv \bot$

NOTE *absence of brackets in theorem AND corollary!*

4.1.21 Corollary. (Redundant $\top$ Theorem)
$\vdash \top \equiv A \equiv A$ *and* $\vdash A \equiv A \equiv \top$.

Proof.

(1)	$\top \equiv \bot \equiv \bot$	$\langle$axiom$\rangle$
(2)	$\bot \equiv \bot \equiv A \equiv A$	$\langle$absolute theorem 4.1.20$\rangle$
(3)	$\top \equiv A \equiv A$	$\langle (Trans) + (1, 2) \rangle$

4.1.22 Metatheorem. (Redundant $\top$ METAtheorem) *For any Γ and A, we have $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$.*

Proof. Say $\Gamma \vdash A$.

Thus

	Γ	
	$\vdots$	
(1)	A	$\langle \Gamma$-theorem$\rangle$
(2)	$A \equiv A \equiv \top$	$\langle$Red. $\top$ theorem; 4.1.21$\rangle$
(3)	$A \equiv \top$	$\langle$(1, 2) +Eqn$\rangle$

The other direction is similar. □

4.2 Equational Proofs

Example from high school trigonometry.

Prove that $1 + (\tan x)^2 = (\sec x)^2$ given the identities

$$\tan x = \frac{\sin x}{\cos x} \qquad (i)$$

$$\sec x = \frac{1}{\cos x} \qquad (ii)$$

$$(\sin x)^2 + (\cos x)^2 = 1 \text{ (Pythagoras' Theorem)} \qquad (iii)$$

Equational proof with annotation

$$
\begin{aligned}
& 1+(\tan x)^2 \\
= & \langle \text{by } (i) \rangle \\
& 1+(\sin x/\cos x)^2 \\
= & \langle \text{arithmetic} \rangle \\
& \frac{(\sin x)^2+(\cos x)^2}{(\cos x)^2} \\
= & \langle \text{by } (iii) \rangle \\
& \frac{1}{(\cos x)^2} \\
= & \langle \text{by } (ii) \rangle \\
& (\sec x)^2
\end{aligned}
\qquad (E)
$$

An equational proof looks like:

$$\overbrace{A_1 \equiv A_2}^{\text{reason}}, \overbrace{A_2 \equiv A_3}^{\text{reason}}, \overbrace{A_3 \equiv A_4}^{\text{reason}} \ldots, \overbrace{A_n \equiv A_{n+1}}^{\text{reason}} \tag{1}$$

4.2.1 Metatheorem. (Important!)

$$A_1 \equiv A_2, A_2 \equiv A_3, \ldots, A_{n-1} \equiv A_n, A_n \equiv A_{n+1} \vdash A_1 \equiv A_{n+1} \tag{2}$$

Proof. *By repeated application of (derived) rule (Trans).*

For example to show the "special case"

$$A \equiv B, B \equiv C, C \equiv D, D \equiv E \vdash A \equiv E \tag{3}$$

the proof is

(1)	$A \equiv B$	$\langle$hyp$\rangle$
(2)	$B \equiv C$	$\langle$hyp$\rangle$
(3)	$C \equiv D$	$\langle$hyp$\rangle$
(4)	$D \equiv E$	$\langle$hyp$\rangle$
(5)	$A \equiv C$	$\langle 1 + 2 +$ Trans$\rangle$
(6)	$A \equiv D$	$\langle 3 + 5 +$ Trans$\rangle$
(7)	$A \equiv E$	$\langle 4 + 6 +$ Trans$\rangle$

For the "general case (2)" do induction on n with Basis at $n = 1$ (see text; better still do it without looking!) □

All Equational Proofs are based on Metatheorem 4.2.1:

4.2.2 Corollary. *In an Equational proof (from* Γ*) like the one in (1) of p.112 we have* $\Gamma \vdash A_1 \equiv A_{n+1}$*.*

Proof. So we have n Γ-proofs, for $i = 1, \ldots, n$,

$$\boxed{\ldots A_i \equiv A_{i+1}}$$

Concatenate them all to get ONE Γ-proof

$$\boxed{\ldots A_1 \equiv A_2} \ldots \boxed{\ldots A_i \equiv A_{i+1}} \ldots \boxed{\ldots A_n \equiv A_{n+1}}$$

By the DERIVED RULE 4.2.1 the following is a Γ-proof of $A_1 \equiv A_{n+1}$

$$\boxed{\ldots A_1 \equiv A_2} \ldots \boxed{\ldots A_i \equiv A_{i+1}} \ldots \boxed{\ldots A_n \equiv A_{n+1}} \quad A_1 \equiv A_{n+1}$$

□

4.2.3 Corollary. *In an Equational proof (from Γ) like the one in (1) of p.112 we have $\Gamma \vdash A_1$ iff $\Gamma \vdash A_{n+1}$.*

Proof. *From the above Corollary we have*

$$\Gamma \vdash A_1 \equiv A_{n+1} \qquad (\dagger)$$

Now split the "iff" in two directions:

- IF: So we have
$$\Gamma \vdash A_{n+1}$$
This plus (†) plus Eqn yield $\Gamma \vdash A_1$.
- ONLY IF: So we have
$$\Gamma \vdash A_1$$
This plus (†) plus Eqn yield $\Gamma \vdash A_{n+1}$.

□

Equational Proof Layout

Successive equivalences like "$A_i \equiv A_{i+1}$ and $A_{i+1} \equiv A_{i+2}$" we write vertically, without repeating the shared formula A_{i+1}.

WITH annotation in $\langle \ldots \rangle$ brackets

$$
\begin{aligned}
& A_1 \\
\equiv\ & \langle \text{annotation} \rangle \\
& A_2 \\
\equiv\ & \langle \text{annotation} \rangle \\
& \vdots \\
& A_{n-1} \\
\equiv\ & \langle \text{annotation} \rangle \\
& A_n \\
\equiv\ & \langle \text{annotation} \rangle \\
& A_{n+1}
\end{aligned} \tag{ii}
$$

EXCEPT FOR ONE THING!

(ii) is just ONE FORMULA, namely

$$A_1 \equiv A_2 \equiv \ldots \equiv A_n \equiv A_{n+1}$$

where I can put brackets anywhere I please.

which is NOT the same as (1) of p.112.

For example, "$\top \equiv \bot \equiv \bot$" *is NOT the same as* "$\top \equiv \bot$ *AND* $\bot \equiv \bot$"

The former (blue) is true but the latter (red) is false.

What do we do?

We introduce a metasymbol for an equivalence that acts ONLY on TWO formulas!

AND

Cannot be chained to form ONE formula.

The symbol is "$\Leftrightarrow$" and thus

"$A \Leftrightarrow B \Leftrightarrow C$" MEANS "$A \equiv B$ AND $B \equiv C$".

We say that "$\Leftrightarrow$" is CONJUNCTIONAL while "$\equiv$" is associative.

So the final layout is:

$$
\begin{aligned}
& A_1 \\
\Leftrightarrow\ & \langle \text{annotation} \rangle \\
& A_2 \\
\Leftrightarrow\ & \langle \text{annotation} \rangle \\
& \vdots \\
& A_{n-1} \\
\Leftrightarrow\ & \langle \text{annotation} \rangle \\
& A_n \\
\Leftrightarrow\ & \langle \text{annotation} \rangle \\
& A_{n+1}
\end{aligned}
$$

Examples.

4.2.4 Theorem. $\vdash \neg(A \equiv B) \equiv \neg A \equiv B$

Proof. (Equational)

$$\begin{aligned}
& \neg(A \equiv B) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \equiv B \equiv \bot \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } B \equiv \bot \equiv \bot \equiv B\text{; Denom: } A \equiv \mathbf{p}\text{; } \mathbf{p} \text{ fresh} \rangle \\
& A \equiv \bot \equiv B \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } A \equiv \bot \equiv \neg A\text{; Denom: } \mathbf{q} \equiv B\text{; } \mathbf{q} \text{ fresh} \rangle \\
& \neg A \equiv B
\end{aligned}$$

□

Why do I need Leib above? Why not <u>just</u> use the Axiom?

4.2.5 Corollary. $\vdash \neg(A \equiv B) \equiv A \equiv \neg B$

Proof. (Equational)

$$\begin{aligned}
& \neg(A \equiv B) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \equiv B \equiv \bot \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } B \equiv \bot \equiv \neg B\text{; Denom: } A \equiv \mathbf{p}\text{; } \mathbf{p} \text{ fresh} \rangle \\
& A \equiv \neg B
\end{aligned}$$

□

4.2.6 Theorem. (Double Negation) $\vdash \neg\neg A \equiv A$

Proof. (Equational)

$$
\begin{aligned}
& \neg\neg A \\
\Leftrightarrow\ & \langle \text{axiom “}\neg X \equiv X \equiv \bot\text{”} \rangle \\
& \neg A \equiv \bot \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } \neg A \equiv A \equiv \bot;\ \text{Denom: } \mathbf{p} \equiv \bot\ \rangle \\
& A \equiv \bot \equiv \bot \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } \top \equiv \bot \equiv \bot;\ \text{Denom: } A \equiv \mathbf{q};\ \mathbf{q} \text{ fresh} \rangle \\
& A \equiv \top \\
\Leftrightarrow\ & \langle \text{red. } \top \text{ thm.} \rangle \\
& A
\end{aligned}
$$

□

4.2.7 Theorem. $\vdash \top \equiv \neg\bot$

Proof. (Equational)

$$
\begin{array}{ll}
 & \top \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & \bot \equiv \bot \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & \neg\bot
\end{array}
$$

□

4.2.8 Theorem. $\vdash \bot \equiv \neg\top$

Proof. (Equational)

$$
\begin{array}{ll}
 & \neg\top \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & \top \equiv \bot \\
\Leftrightarrow & \langle \text{red. } \top \rangle \\
 & \bot
\end{array}
$$

□

Practical Advise. In Equational Proofs move from the most complex side towards the least complex one.

4.2.9 Theorem. $\vdash A \vee \top$

Proof.

$$
\begin{aligned}
& A \vee \top \\
\Leftrightarrow & \langle (\textit{Leib}) + \text{axiom: } \top \equiv \bot \equiv \bot;\ \text{“Denom:” } A \vee \mathbf{p};\ \textbf{Mind the brackets!} \rangle \\
& A \vee (\bot \equiv \bot) \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
& A \vee \bot \equiv A \vee \bot \quad \text{Bingo!}
\end{aligned}
$$

□

Recall about $\equiv$ that, by axiom (1) and a theorem we proved in the NOTES posted in `http://www.cs.yorku.ca/~gt/courses/MATH1090F22/1090.html` (4th bullet), we have that

in a chain of any number of $\equiv$ we may omit brackets.

The same holds for a chain of $\vee$ (and $\wedge$) using *the same kind of proof, in the same source mentioned above.*:

That is,

we do not need to show bracketing in a chain of $\vee$.

How about moving formulas around in such a chain? (Permuting them).

It is OK! I prove this for $\vee$-chains HERE. The proof is identical for $\equiv$-chains and $\wedge$-chains (EXERCISE!!)

Prove first this theorem:

$$\vdash B \vee C \vee D \equiv D \vee C \vee B$$

Indeed here is a proof:

$$
\begin{aligned}
& B \vee C \vee D \\
\Leftrightarrow\ & \langle \vee \text{ commutes axiom} \rangle \\
& D \vee B \vee C \qquad (*) \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \vee \text{ commutes axiom. “Denom:” } D \vee \mathbf{p} \rangle \\
& D \vee C \vee B
\end{aligned}
$$

More generally we CAN DO an arbitrary swap (not only the END-FORMULAS), that is, we have the theorem

$$\vdash A \vee B \vee C \vee D \vee E \equiv A \vee D \vee C \vee B \vee E$$

Follows by an application of the previous special case:

$$\begin{aligned} & A \vee \overbrace{B \vee C \vee D} \vee E \\ \Leftrightarrow\ & \langle (Leib) + \text{special case. "Denom:"}\ A \vee \mathbf{p} \vee E \rangle \\ & A \vee \underbrace{D \vee C \vee B} \vee E \end{aligned}$$

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4.2.10 Theorem. $\vdash A \vee \bot \equiv A$

Proofs. (Equational)

This time we work with the entire formula, not just one of the sides of "$\equiv$".

How do we know? We don't! It is just a matter of practice.

$$
\begin{array}{ll}
 & A \vee \bot \equiv A \\
\Leftrightarrow & \langle (Leib) + \text{axiom } A \equiv A \vee A; \text{ "Denom:" } A \vee \bot \equiv \mathbf{p} \rangle \\
 & A \vee \bot \equiv A \vee A \\
\Leftrightarrow & \langle \text{axiom } \vee \text{ over } \equiv \rangle \\
 & A \vee (\bot \equiv A) \\
\Leftrightarrow & \langle (Leib) + \text{axiom: } \bot \equiv A \equiv \neg A; \text{ "Denom:" } A \vee \mathbf{p} \rangle \\
 & A \vee \neg A \qquad \textit{Bingo}!
\end{array}
$$
$\square$

Comment on "same mouth" p in above proof.

4.2.11 Theorem. $\vdash A \to B \equiv \neg A \vee B$

Proof.

$$
\begin{array}{ll}
 & A \to B \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & A \vee B \equiv B \qquad \text{HERE} \\
\Leftrightarrow & \langle (Leib) + 4.2.10;\ \text{“Denom:”}\ \ A \vee B \equiv \mathbf{p} \rangle \\
 & A \vee B \equiv \bot \vee B \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & (A \equiv \bot) \vee B \\
\Leftrightarrow & \langle (Leib) + \text{axiom};\ \text{“Denom:”}\ \ \mathbf{p} \vee B \rangle \\
 & \neg A \vee B
\end{array}
$$

$\square$

4.2.12 Corollary. $\vdash \neg A \vee B \equiv A \vee B \equiv B$

Proof. Start the above proof from “HERE”. $\square$

4.2.13 Theorem. (de Morgan 1)
$\vdash A \wedge B \equiv \neg(\neg A \vee \neg B)$

Proof.

Long but obvious. Start with the most complex side!

$\neg(\neg A \vee \neg B)$
$\Leftrightarrow$ ⟨axiom⟩
$\neg A \vee \neg B \equiv \bot$
$\Leftrightarrow$ ⟨(*Leib*) + 4.2.12; "Denom:" $\mathbf{p} \equiv \bot$⟩
$A \vee \neg B \equiv \neg B \equiv \bot$
$\Leftrightarrow$ ⟨(*Leib*) + axiom; "Denom:" $A \vee \neg B \equiv \mathbf{p}$ —<u>order</u> does not matter!⟩
$A \vee \neg B \equiv B$
$\Leftrightarrow$ ⟨(*Leib*) + 4.2.12; "Denom:" $\mathbf{p} \equiv B$⟩
$A \vee B \equiv A \equiv B$
$\Leftrightarrow$ ⟨GR axiom —<u>order</u> does not matter⟩
$A \wedge B$ □

4.2.14 Corollary. (de Morgan 2) $\vdash A \vee B \equiv \neg(\neg A \wedge \neg B)$

Proof. See Text. <u>Better still</u>, EXERCISE!

MORE About "$\wedge$"

4.2.15 Theorem. $\vdash A \wedge A \equiv A$

Proof.

$$
\begin{aligned}
& A \wedge A \equiv A \\
\Leftrightarrow\ & \langle \text{GR axiom —}\underline{\text{order}}\text{ does not matter} \rangle \\
& A \vee A \equiv A \qquad \textit{Bingo!}
\end{aligned}
$$

□

4.2.16 Theorem. $\vdash A \wedge \top \equiv A$

Proof.

$$
\begin{aligned}
& A \wedge \top \equiv A \\
\Leftrightarrow\ & \langle \text{GR axiom} \rangle \\
& A \vee \top \equiv \top \\
\Leftrightarrow\ & \langle \text{Red. } \top \text{ Thm.} \rangle \\
& A \vee \top \qquad \textit{Bingo!}
\end{aligned}
$$

□

4.2.17 Theorem. $\vdash A \wedge \bot \equiv \bot$

Proof.

$$
\begin{aligned}
& A \wedge \bot \equiv \bot \\
\Leftrightarrow\ & \langle \text{GR axiom} \rangle \\
& A \vee \bot \equiv A \quad \textit{Bingo!}
\end{aligned}
$$

□

READ this theorem and its proof!

4.2.18 Theorem. (Distributive Laws)

(i) $$\vdash A \lor B \land C \equiv (A \lor B) \land (A \lor C)$$

and

(ii) $$\vdash A \land (B \lor C) \equiv A \land B \lor A \land C$$

The above are written in least parenthesised notation!

Proof.

We just prove (i).

$$\begin{aligned}
& (A \lor B) \land (A \lor C) \\
\Leftrightarrow\ & \langle \text{GR} \rangle \\
& A \lor B \lor A \lor C \equiv A \lor B \equiv A \lor C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{scramble an } \lor\text{-chain; “Denom:” } \mathbf{p} \equiv A \lor B \equiv A \lor C \rangle \\
& A \lor A \lor B \lor C \equiv A \lor B \equiv A \lor C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom; “Denom:” } \mathbf{p} \lor B \lor C \equiv A \lor B \equiv A \lor C \rangle \\
& A \lor B \lor C \equiv A \lor B \equiv A \lor C
\end{aligned}$$

HERE WE STOP, and try to reach this result from the other side: $A \lor B \land C$.

$$
\begin{aligned}
& A \vee B \wedge C \\
\Leftrightarrow\ & \langle (Leib) + \text{GR; “Denom:” } A \vee \mathbf{p}\text{; mind brackets!} \rangle \\
& A \vee (B \vee C \equiv B \equiv C) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \vee B \vee C \equiv A \vee (B \equiv C) \\
\Leftrightarrow\ & \langle (Leib) + \text{axiom; “Denom:” } A \vee B \vee C \equiv \mathbf{p} \rangle \\
& A \vee B \vee C \equiv A \vee B \equiv A \vee C
\end{aligned}
$$

□

4.2.19 Theorem. ("Proof by cases")
$\vdash A \vee B \to C \equiv (A \to C) \wedge (B \to C)$

Proof.

$$
\begin{aligned}
& A \vee B \to C \\
\Leftrightarrow\ & \langle 4.2.11 \rangle \\
& \neg(A \vee B) \vee C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + 4.2.14;\ \text{“Denom:”}\ \neg\mathbf{p} \vee C \rangle \\
& \neg\neg(\neg A \wedge \neg B) \vee C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{double neg.};\ \text{“Denom:”}\ \mathbf{p} \vee C \rangle \\
& (\neg A \wedge \neg B) \vee C \\
\Leftrightarrow\ & \langle 4.2.18 \rangle \\
& (\neg A \vee C) \wedge (\neg B \vee C) \\
\Leftrightarrow\ & \langle \text{obvious } (\mathit{Leib}), \text{ twice} + 4.2.11 \rangle \\
& (A \to C) \wedge (B \to C)
\end{aligned}
$$

□

Until now we only proved absolute theorems Equationally.

How about theorems with HYPOTHESES?

To do so we use the Redundant $\top$ METAtheorem:

$\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$

The Technique is demonstrated via Examples!

4.2.20 Example. (1) $A, B \vdash A \wedge B$

(2) $A \vee A \vdash A$

(3) $A \vdash A \vee B$

(4) $A \wedge B \vdash A$

For (1):

$$
\begin{aligned}
& A \wedge B \\
\Leftrightarrow\ & \langle (Leib) + \text{hyp } B + \text{Red. } \top \text{ META; “Denom:” } A \wedge \mathbf{p} \rangle \\
& A \wedge \top \\
\Leftrightarrow\ & \langle 4.2.16 \rangle \\
& A \qquad\qquad \text{Bingo!}
\end{aligned}
$$

NOTES:

▶ $A, B \vdash B$. Hence $A, B \vdash \boxed{B \equiv \top}$

For (2):

$$
\begin{aligned}
& A \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \vee A \quad \text{Bingo!}
\end{aligned}
$$

For (3):

$$
\begin{aligned}
& A \vee B \\
\Leftrightarrow\ & \langle (Leib) + \text{Hyp } A + \text{Red-}\top\text{-META; “Denom:” } \mathbf{p} \vee B \rangle \\
& \top \vee B \qquad \langle \text{Bingo!} \rangle
\end{aligned}
$$

(4) is a bit trickier:

$$
\begin{aligned}
& A \\
\Leftrightarrow\ & \langle 4.2.16 \rangle \\
& A \wedge \top \\
\Leftrightarrow\ & \langle (Leib) + \text{Hyp } A \wedge B + \text{Red-}\top\text{-META; "Denom:" } A \wedge \mathbf{p} \rangle \\
& A \wedge A \wedge B \\
\Leftrightarrow\ & \langle (Leib) + 4.2.15\text{; "Denom:" } \mathbf{p} \wedge B \rangle \\
& A \wedge B \qquad \text{Bingo!}
\end{aligned}
$$

□

4.2.21 Metatheorem. (Hypothesis splitting/merging)
For any wff A, B, C and hypotheses Γ, we have $\Gamma \cup \{A, B\} \vdash C$ iff $\Gamma \cup \{A \wedge B\} \vdash C$.

Proof. (Hilbert-style)
(I) *ASSUME* $\Gamma \cup \{A, B\} \vdash C$ and *PROVE* $\Gamma \cup \{A \wedge B\} \vdash C$.

So, armed with Γ and $A \wedge B$ as *hypotheses* I have to prove C.

(1)	$A \wedge B$	$\langle$hyp$\rangle$
(2)	A	$\langle$(1) + $A \wedge B \vdash A$ rule $\rangle$
(3)	B	$\langle$(1) + $A \wedge B \vdash B$ rule $\rangle$
(4)	C	$\langle$using HYP Γ + (2) and (3) $\rangle$

(II) *ASSUME* $\Gamma \cup \{A \wedge B\} \vdash C$ and *PROVE* $\Gamma \cup \{A, B\} \vdash C$.

Exercise, or see Text. □

4.2.22 Theorem. (Modus Ponens) $A, A \to B \vdash B$

Proof.

$$
\begin{aligned}
& A \to B \\
\Leftrightarrow\ & \langle \neg\vee\text{-theorem} \rangle \\
& \neg A \vee B \\
\Leftrightarrow\ & \langle (Leib) + \text{hyp } A + \text{Red-}\top\text{-META; ``Denom:'' } \neg\mathbf{p} \vee B \rangle \\
& \neg \top \vee B \\
\Leftrightarrow\ & \langle (Leib) + \text{theorem from class; ``Denom:'' } \mathbf{p} \vee B \rangle \\
& \bot \vee B \\
\Leftrightarrow\ & \langle \text{thm from class} \rangle \\
& B
\end{aligned}
$$

□

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4.2.23 Theorem. (Cut Rule) $A \vee B, \neg A \vee C \vdash B \vee C$

Proof. We start with an AUXILIARY theorem —a Lemma— which makes the most complex hypothesis $\neg A \vee C$ usable (an EQUIVALENCE).

$$\begin{aligned}
& \neg A \vee C \\
\Leftrightarrow\ & \langle \text{how to lose a NOT} \rangle \\
& A \vee C \equiv C
\end{aligned}$$

Since $\neg A \vee C$ is a HYP hence also a THEOREM, the same is true for $A \vee C \equiv C$ from the Equational proof above.

$$\begin{aligned}
& B \vee C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{Lemma; "Denom:" } B \vee \mathbf{p} \rangle \\
& B \vee (A \vee C) \\
\Leftrightarrow\ & \langle \text{shifting brackets to our advantage AND swapping wff} \rangle \\
& (A \vee B) \vee C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{HYP } A \vee B + \text{Red-}\top\text{-Meta; "Denom:" } \mathbf{p} \vee C \rangle \\
& \top \vee C \qquad \text{Bingo!}
\end{aligned}$$

□

SPECIAL CASES of CUT:

4.2.24 Corollary. $A \vee B, \neg A \vee B \vdash B$

Proof. From 4.2.23 we get $A \vee B, \neg A \vee B \vdash B \vee B$.

We have also learnt the rule $B \vee B \vdash B$.

Apply this rule to the proof above that ends with "$B \vee B$" to get B. □

4.2.25 Corollary. $A \vee B, \neg A \vdash B$

Proof. Apply the rule $\neg A \vdash \neg A \vee B$.

We now can use the above Corollary! □

4.2.26 Corollary. $A, \neg A \vdash \bot$

Proof. Hilbert-style.

(1)	A	$\langle$hyp$\rangle$
(2)	$\neg A$	$\langle$hyp$\rangle$
(3)	$A \vee \bot$	$\langle 1 + \text{rule } X \vdash X \vee Y \rangle$
(4)	$\neg A \vee \bot$	$\langle 2 + \text{rule } X \vdash X \vee Y \rangle$
(5)	$\bot$	$\langle 3 + 4 + \text{rule } 4.2.24 \rangle$

$\square$

Skip this proof, but memorise the result!

4.2.27 Corollary. (Transitivity of $\rightarrow$) $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

Proof. (Hilbert style)

(1)	$A \rightarrow B$	$\langle$hyp$\rangle$
(2)	$B \rightarrow C$	$\langle$hyp$\rangle$
(3)	$A \rightarrow B \equiv \neg A \vee B$	$\langle \neg\vee$ thm$\rangle$
(4)	$B \rightarrow C \equiv \neg B \vee C$	$\langle \neg\vee$ thm$\rangle$
(5)	$\neg A \vee B$	$\langle(1, 3) + (Eqn)\rangle$
(6)	$\neg B \vee C$	$\langle(2, 4) + (Eqn)\rangle$
(7)	$\neg A \vee C$	$\langle(5, 6) +$ CUT$\rangle$

□

The last line is provably equivalent to $A \rightarrow C$ by the $\neg\vee$ theorem.

Chapter 5

A Weak Post's Theorem and the Deduction Theorem Retold

This note is about the *Soundness* and *Completeness* (the latter is also known as "Post's Theorem") in Boolean logic.

5.1 Soundness of Boolean Logic

Soundness is the Property expressed by the statement of the *metatheory* below —which in English says "**Boolean Logic tells ONLY the truth**":

$$\text{If } \Gamma \vdash A, \text{ then } \Gamma \models_{taut} A \tag{1}$$

5.1.1 Definition. The statement "Boolean logic is Sound" means that Boolean logic satisfies (1). □

To prove soundness is an easy induction on the length of Γ-proofs:

We prove that proofs preserve truth.

5.1.2 Lemma. Eqn *and* **Leib** *preserve truth, that is,*

$$A, A \equiv B \models_{taut} B \tag{2}$$

and

$$A \equiv B \models_{taut} C[\mathbf{p} := A] \equiv C[\mathbf{p} := B] \tag{3}$$

Proof. (2) is trivial.

We prove (3) here:

So, let a state s make $A \equiv B$ true ($\mathbf{t}$).

Thus,

$$\overline{s}(A) = \overline{s}(B)$$

We will show that

$$C[\mathbf{p} := A] \equiv C[\mathbf{p} := B] \text{ is } \mathbf{t} \text{ in state } s \tag{4}$$

If $\mathbf{p}$ is not in C then (4) is $C \equiv C$, a tautology, so is true under s in particular.

Let then the distinct $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{r}', \mathbf{r}'', \ldots$ all occur in C.

Now *in the lhs of (4)* $\mathbf{p}$ gets the value $\overline{s}(A)$, while $\mathbf{q}, \mathbf{r}, \mathbf{r}', \mathbf{r}'', \ldots$ get their values DIRECTLY from s.

Similarly, *in the RHS of (4)* $\mathbf{p}$ gets the value $\overline{s}(B)$, while $\mathbf{q}, \mathbf{r}, \mathbf{r}', \mathbf{r}'', \ldots$ STILL get their values DIRECTLY from s.

► But $\overline{s}(A) = \overline{s}(B)$.

So both the lhs and rhs variables of (4) end up with the same truth value assignments after the indicated substitutions, and from s.

Then the computed values of lhs and rhs in (4) are the same.

In short, the equivalence is true. □

We can now prove:

5.1.3 Metatheorem. *Boolean logic is Sound, that is, (1) on p.143 holds.*

Proof. By induction on the length of proof, n, needed to obtain $\Gamma \vdash A$ we prove

$$\Gamma \models_{taut} A \qquad (\dagger)$$

So pick a state s that satisfies Γ. $(\ddagger)$

1. *Basis*. $n = 1$. Then we have just A in the proof.

   If $A \in \Lambda$, then it is a tautology, so in particular *is true under* s. We have $(\dagger)$.

   If $A \in \Gamma$, then s satisfies A by $(\ddagger)$. Again we have $(\dagger)$.

   I.H. Assume claim for all proofs of length $\leq n$.

   I.S. Prove the theorem in the case $\Gamma \vdash A$ needed a proof of length $n + 1$:

$$\underbrace{\overbrace{\ldots}^{length\ =n}, A}_{length\ =n+1}$$

   Now if A is in $\Lambda \cup \Gamma$ we are back to the Basis. Done.

If not

- Case where A is the result of *EQN* on X and $X \equiv Y$ that are in the "...-area".

  By the I.H. s satisfies X and $X \equiv Y$ hence, by the Lemma, satisfies A.

- Case where A is the result of *LEIB* on $X \equiv Y$ that is in the "...-area".

  By the I.H. s satisfies $X \equiv Y$ hence, by the Lemma, satisfies A. □

5.1.4 Corollary. *If $\vdash A$ then $\models_{taut} A$. A is a tautology.*

Proof. $\Gamma = \emptyset$ here. By the above, $\emptyset \models_{taut} A$.
BUT, $\emptyset \models_{taut} A$ says EXACTLY $\models_{taut} A$ (EXERCISE). □

5.1.5 Example. Soundness allows us to disprove formulas: To show they are NOT theorems.

- The statement "$\vdash \mathbf{p}$" is false. If this were true, then $\mathbf{p}$ would be a tautology!
- The statement "$\vdash \perp$" is false! Because $\perp$ is not a tautology!
- The statement "$p \vdash p \wedge q$" is false. Because if it were true I'd have to have $p \models_{taut} p \wedge q$.

  Not so: Take a state s such that $s(p) = \mathbf{t}$ and $s(q) = \mathbf{f}$. □

5.2 Completeness of Boolean logic ("Post's Theorem")

We prove here

(1) A weak form of Post's theorem: If Γ is finite and $\Gamma \models_{\text{taut}} A$, then $\Gamma \vdash A$

and derive as a corollary the *Deduction Theorem*:

(2) If $\Gamma, A \vdash B$, then $\Gamma \vdash A \rightarrow B$.

We will employ ONE TOOL from class below:

5.2.1 Theorem. $\neg C \vee E, \neg D \vee E \vdash \neg(C \vee D) \vee E.$

Proof. Translating via the "¬∨-theorem", the above says

$$C \to E, D \to E \vdash (C \vee D) \to E \tag{†}$$

Here is an Equational proof of (†):

$$
\begin{aligned}
& (C \vee D) \to E \\
\Longleftrightarrow & \langle 4.2.19 \rangle \\
& (C \to E) \wedge (D \to E) \\
\Longleftrightarrow & \langle \text{Leib + Red. } \top \text{ META + hyp } C \to E\text{; Denom:}\mathbf{p} \wedge (D \to E) \rangle \\
& \top \wedge (D \to E) \\
\Longleftrightarrow & \langle \text{thm} \rangle \\
& D \to E \qquad \text{bingo!}
\end{aligned}
$$

□

5.2.2 Main Lemma. *Suppose that A contains none of the symbols $\top, \bot, \to, \wedge, \equiv$. If $\models_{taut} A$, then $\vdash A$.*

Proof. *The proof is long but easy!*

Under the assumption, A is *an $\vee$-chain*, that is, it has the form

$$A_1 \vee A_2 \vee A_3 \vee \ldots \vee A_i \vee \ldots \vee A_n \tag{1}$$

where none of the A_i has the form $B \vee C$.

In (1) we assume without loss of generality that $n > 1$, due to the axiom $X \vee X \equiv X$ —that is, *in the contrary case* we can use $A \vee A$ instead, which is a tautology as well.

Moreover, (1), that is, A, is written in *least parenthesised notation.*

Let us call an A_i *reducible* iff it has the form $\neg(C \vee D)$ or $\neg(\neg C)$.

"Reducible", since A_i is not alone in the $\vee$-chain, will be synonymous to simplifiable without changing the meaning of A_i.

Otherwise A_i is *irreducible*. *Not* simplifiable.

Thus, the only possible irreducible A_i have the form $\mathbf{p}$ or $\neg\mathbf{p}$ (where $\mathbf{p}$ is a variable).

By **definition** *we will say that A is irreducible iff* ***all*** *its A_i are.*

We define the *reducibility degree*, of EACH A_i —in symbols, $rd(A_i)$— to be the total number, counting repetitions of the $\neg$ and $\vee$ connectives in it, **not counting a possible leading** $\neg$.

The reducibility degree of the entire A is the sum of the reducibility degrees of all its A_i.

For example, $rd(p) = 0$, $rd(\neg p) = 0$, $rd(\neg(\neg p \vee q)) = 2$, $rd(\neg(\neg p \vee \neg q)) = 3$, $rd(\neg p \vee q) = 0$.

We say that $\mathbf{p}$ "*occurs positively* in $\ldots \vee \mathbf{p} \vee \ldots$", while it "*occurs negatively* in $\ldots \vee \neg \mathbf{p} \vee \ldots$".

In, for example, $\mathbf{p} \vee \neg \mathbf{p}$ it occurs *both* positively and negatively.

By induction on $rd(A)$ we now prove the main lemma, that $\vdash A$ follows the stated hypothesis that $\models_{taut} A$.

For the *Basis*, let A be an *irreducible* tautology —so, $rd(A) = 0$.

It must be that A is a string of the form

"$\cdots \vee \mathbf{p} \vee \cdots \neg \mathbf{p} \vee \cdots$"

for some $\mathbf{p}$, otherwise,

if no $\mathbf{p}$ appears *both* "positively" and "negatively",

then we can find a truth-assignment that makes A false ($\mathbf{f}$) —a contradiction to its *tautologyhood*.

To see that we can do this, just assign $\mathbf{f}$ to $\mathbf{p}$'s that occur ***positively only***, and $\mathbf{t}$ to those that occur ***negatively only***.

Now

$$
\begin{aligned}
& A \\
\Leftrightarrow & \Big\langle \text{commuting the terms of an } \vee\text{-chain} \Big\rangle \\
& \mathbf{p} \vee \neg \mathbf{p} \vee B \quad (\text{what is “}B\text{”?}) \\
\Leftrightarrow & \Big\langle \text{Leib + axiom + Red. } \top \text{ META; Denom: } \mathbf{r} \vee B\text{; fresh } \mathbf{r} \Big\rangle \\
& \top \vee B \quad \text{bingo!}
\end{aligned}
$$

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Thus $\vdash A$, which settles the *Basis*-case: $rd(A) = 0$.

We now argue the case where $rd(A) = m + 1$, on the I.H. that for any formula Q with $rd(Q) \leq m$, we *do* have that $\models_{taut} Q$ implies $\vdash Q$.

Since we can shufle an $\vee$-chain any way we please, we assume *without restricting generality* that $rd(A_1) > 0$.

We have two cases:

(1) A_1 is the string $\neg\neg C$, **hence** A **has the form** $\neg\neg C \vee D$.

Clearly $\models_{taut} C \vee D$ as well.

Moreover, $rd(C \vee D) < rd(\neg\neg C \vee D)$, hence (by I.H.)

$$\vdash C \vee D \tag{†}$$

But,

$$\begin{aligned} & \neg\neg C \vee D \\ \Leftrightarrow & \Big\langle \text{Leib } + \vdash \neg\neg X \equiv X\text{; Denom: } \mathbf{r} \vee D\text{; fresh } \mathbf{r} \Big\rangle \\ & C \vee D \quad \text{"bingo" by (†) above!} \end{aligned}$$

Hence, $\vdash \neg\neg C \vee D$, that is, $\vdash A$ in this case.

One more case to go:

(2) A_1 is the string $\neg(C \vee D)$, **hence A has the form $\neg(C \vee D) \vee E$.**

$$\text{We want: } \vdash \neg(C \vee D) \vee E \qquad (i)$$

We are given

$$\models_{taut} \neg(C \vee D) \vee E \qquad (i')$$

which says $\models_{taut} A$. We immediately get that

$$\models_{taut} \neg C \vee E \qquad (ii)$$

and

$$\models_{taut} \neg D \vee E \qquad (iii)$$

from truth tables.

Check it!!!

Hint. To show (ii) let v be any state. Consider cases: (1) where $\overline{v}(C) = \mathbf{f}$ and (2) where $\overline{v}(C) = \mathbf{t}$. In the second case use (i') to show $\overline{v}(E) = \mathbf{t}$.

Since the rd of each of (ii) and (iii) is $< rd(A)$, the I.H. yields $\vdash \neg C \vee E$ AND $\vdash \neg D \vee E$.

Apply the TOOL 5.2.1 to the above two theorems to get (i).

We are done, **except for one small detail:**

If we had changed the "original" A into $A \vee A$ (cf. the "without loss of generality" remark just below (1) on p.151), then all we proved above is $\vdash A \vee A$.

The *contraction* rule from Notes, and Text then yield $\vdash A$. □

Do you see now why we wanted $n \geq 2$?

But ALL this only proves "$\models_{taut} A$ implies $\vdash A$"

▶ when A does *not* contain any of $\wedge, \rightarrow, \equiv, \bot, \top$.

WHAT IF IT DOES?

We are now removing the restriction on A regarding its connectives and constants:

5.2.3 Metatheorem. (Post's Theorem) *If* $\models_{taut} A$, *then* $\vdash A$.

Proof. First, we note the following *theorems* stating equivalences, *where* $\mathbf{p}$ *is fresh for* A.

The proof of the last one is in the notes and text but it was too long (but easy) to cover in class.

$$
\begin{aligned}
&\vdash \top \equiv \neg \mathbf{p} \vee \mathbf{p} \\
&\vdash \bot \equiv \neg(\neg \mathbf{p} \vee \mathbf{p}) \\
&\vdash C \to D \equiv \neg C \vee D \\
&\vdash C \wedge D \equiv \neg(\neg C \vee \neg D) \\
&\vdash (C \equiv D) \equiv ((C \to D) \wedge (D \to C))
\end{aligned}
\tag{2}
$$

Using (2) above, we eliminate, in order, all the $\equiv$, then all the $\wedge$, then all the $\to$ and finally all the $\bot$ and all the $\top$.

Let us assume that our process eliminates **one** unwanted symbol at a time.

This leads to *the Equational Proof below*.

Starting from A we will generate a sequence of formulae

$$F_1\,(A), F_2, F_3, \ldots, F_n\,(A')$$

where the last, F_n, contains no $\top, \bot, \wedge, \rightarrow, \equiv$.

I am using here F_1 as an alias for A. We will also give to F_n an alias A'.

$$
\begin{array}{ll}
 & A \\
\Leftrightarrow & \Big\langle \text{Leib from (2)} \Big\rangle \\
 & F_2 \\
\Leftrightarrow & \Big\langle \text{Leib from (2)} \Big\rangle \\
 & F_3 \\
\Leftrightarrow & \Big\langle \text{Leib from (2)} \Big\rangle \\
 & F_4 \\
 & \vdots \\
\Leftrightarrow & \Big\langle \text{Leib from (2)} \Big\rangle \\
 & A'\ (F_n)
\end{array}
$$

$$\text{Thus, } \vdash A' \equiv A \qquad (*)$$

$$\text{By } \textbf{soundness}\text{, we also have } \models_{taut} A' \equiv A \qquad (**)$$

So, say $\models_{taut} A$. By $(**)$ we have $\models_{taut} A'$ as well, and by the Main Lemma 5.2.2 we obtain $\vdash A'$.

So bottom line, A', being a theorem is a "*bingo*!" hence the top line, A, is also a theorem. □

Post's theorem is the "*Completeness Theorem*"† of Boolean Logic.

It shows that the syntactic manipulation apparatus —proofs— *DOES certify* the "whole truth" (tautologyhood) in the Boolean case.

†Which is really a *Meta*theorem, right?

5.2.4 Corollary. *If*

$$\overbrace{A_1, \ldots, A_n}^{finite\ \Gamma} \models_{taut} B$$

then

$$A_1, \ldots, A_n \vdash B$$

Proof. It is an easy semantic exercise to see that the assumption implies

$$\models_{taut} A_1 \to \ldots \to A_n \to B$$

By 5.2.3,

$$\vdash A_1 \to \ldots \to A_n \to B$$

hence (hypothesis strengthening)

$$A_1, A_2 \ldots, A_n \vdash A_1 \to A_2 \to \ldots \to A_n \to B \tag{1}$$

Applying modus ponens n times to (1) we get

$$A_1, \ldots, A_n \vdash B \qquad \square$$

The above corollary is very convenient.

It says that every (correct) schema $A_1, \ldots, A_n \models_{taut} B$ leads to a *derived rule of inference*, $A_1, \ldots, A_n \vdash B$.

In particular, combining with the transitivity of $\vdash$ metatheorem, we get

5.2.5 Corollary. *If* $\Gamma \vdash A_i$, *for* $i = 1, \ldots, n$, *and if* $A_1, \ldots, A_n \models_{taut} B$, *then* $\Gamma \vdash B$.

Thus —*unless otherwise required!*— we can, from now on, *rigorously* mix syntactic with semantic justifications of our proof steps.

For example, we have at once $A \wedge B \vdash A$, because (trivially) $A \wedge B \models_{taut} A$ (compare with our earlier, much longer, proof given in class).

5.3 Deduction Theorem and Proof by Contradiction

5.3.1 Metatheorem. (The Deduction Theorem) *If* $\Gamma, A \vdash B$*, then* $\Gamma \vdash A \to B$*, where "*Γ, A*" means "all the assumptions in* Γ*, plus the assumption* A*"* (in set notation this would be $\Gamma \cup \{A\}$).*

Proof. Let $G_1, \ldots, G_n \subseteq \Gamma$ be a *finite* set of formulae used in a Γ, A-proof of B.

Thus we also have $G_1, \ldots, G_n, A \vdash B$.

By *soundness*, $G_1, \ldots, G_n, A \models_{taut} B$.
But then,

$$\overbrace{G_1, \ldots, G_n}^{finite!} \models_{taut} A \to B$$

By 5.2.4, $G_1, \ldots, G_n \vdash A \to B$ and hence $\Gamma \vdash A \to B$ by hypothesis strengthening. □

*We can also write $\Gamma + A$.

The mathematician, or indeed the mathematics practitioner, uses the Deduction theorem all the time, without stopping to think about it. Metatheorem 5.3.1 above makes an honest person of such a mathematician or practitioner.

The everyday "style" of applying the Metatheorem goes like this:

Say we have all sorts of assumptions and we want, *under these assumptions*, to "prove" that "if A, then B" (verbose form of "$A \to B$").

We start by ***adding*** *A to our assumptions*, often with the words, "*Assume A*". We then proceed and prove *just* B (not $A \to B$), and at that point we rest our case.

Thus, we may view an application of the Deduction theorem as a simplification of the proof-task. It allows us to "split" an implication $A \to B$ that we want to prove, moving its premise to join our other assumptions. We now have to prove a *simpler formula*, B, with the help of *stronger* assumptions (that is, all we knew so far, plus A). That often makes our task so much easier!

An Example. Prove

$$\vdash (A \to B) \to A \vee C \to B \vee C$$

By DThm, suffices to prove

$$A \to B \vdash A \vee C \to B \vee C$$

instead.

Again By DThm, suffices to prove

$$A \to B, A \vee C \vdash B \vee C$$

instead.

Let's do it:

1.	$A \to B$	$\langle$hyp$\rangle$
2.	$A \vee C$	$\langle$hyp$\rangle$
3.	$A \to B \equiv \neg A \vee B$	$\langle\neg\vee$ thm$\rangle$
4.	$\neg A \vee B$	$\langle 1 + 3 + \text{Eqn}\rangle$
5.	$B \vee C$	$\langle 2 + 4 + \text{Cut}\rangle$

□

5.3.2 Definition. A set of formulas Γ is *inconsistent* or *contradictory* iff Γ *proves* *every* A *in* WFF. □

Why "contradictory"? For if Γ proves everything, then it also proves $\mathbf{p} \wedge \neg \mathbf{p}$.

5.3.3 Lemma. Γ *is inconsistent iff* $\Gamma \vdash \bot$

Proof. only if-part. If Γ is as in 5.3.2, then, in particular, it proves $\bot$ since the latter is a well formed formula.

if-part. Say, conversely, that we have

$$\Gamma \vdash \bot \tag{9}$$

Let now A be any formula in WFF whatsoever. We have

$$\bot \models_{taut} A \tag{10}$$

Pause. Do you believe (10)?

By 5.2.5, $\Gamma \vdash A$ follows from (9) and (10). □

5.3.4 Metatheorem. (Proof by contradiction) $\Gamma \vdash A$ *iff* $\Gamma \cup \{\neg A\}$ *is inconsistent.*

Proof. if-part. So let (by 5.3.3)

$$\Gamma, \neg A \vdash \bot$$

Hence

$$\Gamma \vdash \neg A \to \bot \tag{1}$$

by the Deduction theorem. However $\neg A \to \bot \models_{taut} A$, hence, by Corollary 5.2.5 and (1) above, $\Gamma \vdash A$.

only if-part. So let

$$\Gamma \vdash A$$

By hypothesis strengthening,

$$\Gamma, \neg A \vdash A \tag{2}$$

Moreover, trivially,

$$\Gamma, \neg A \vdash \neg A \tag{3}$$

Since $A, \neg A \models_{taut} \perp$, (2) and (3) yield $\Gamma, \neg A \vdash \perp$ via Corollary 5.2.5, and we are done by 5.3.3. □

5.3.4 legitimises the tool of "proof by contradiction" that goes all the way back to the ancient Greek mathematicians: To prove A assume instead the "opposite", $\neg A$. Proceed then to obtain a contradiction. This being accomplished, it is as good as having proved A.

Chapter 6

Resolution

Proof by Resolution is an easy and self-documenting 2-dimensional proof style.

It is essentially a Hilbert style proof that needs no numbering and the *annotation is depicted by drawing certain lines*.

The technique is used in "automatic theorem proving", i.e., special computer systems (programs) that prove theorems.
It is based on the *proof by contradiction metatheorem*:

6.0.1 Metatheorem.

$$\Gamma, \neg A \vdash \bot \tag{1}$$

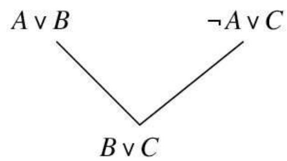
iff

$$\Gamma \vdash A \tag{2}$$

Thus, instead of proving (2) prove (1).

(1) is proved using (almost) exclusively the CUT Rule.

The *self-annotating* diagram below says "apply the CUT rule to premises $A \vee B$ and $\neg A \vee C$ to obtain $B \vee C$".



The technique can be best learnt via examples:

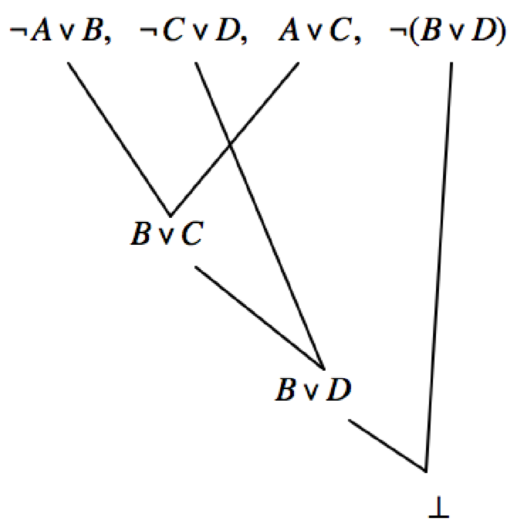
6.0.2 Example. Use Resolution to prove (1) below:

$$A \to B, C \to D \vdash A \vee C \to B \vee D \tag{1}$$

by DThm prove instead:

$$A \to B, C \to D, A \vee C \vdash B \vee D$$

By 6.0.1 prove instead that the "Γ" in the top line below proves $\perp$



□

6.0.3 Example. Next prove

$$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

By the DThm prove instead

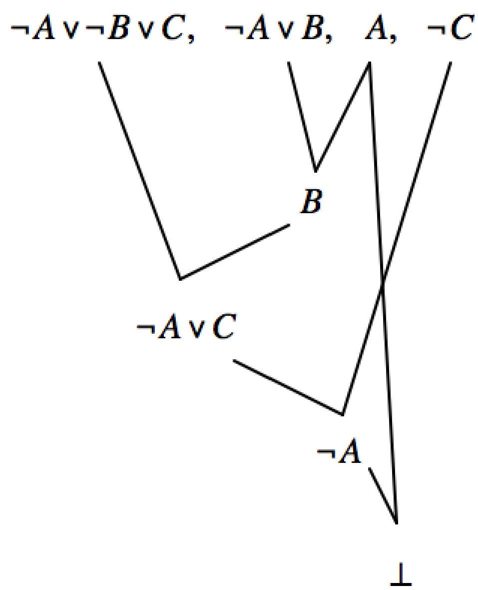
$$A \to (B \to C) \vdash (A \to B) \to (A \to C)$$

Two more applications of the DThm simplify what we will prove into the following:

$$A \to (B \to C), A \to B, A \vdash C$$

By 6.0.1, prove instead that $\Gamma \vdash \perp$ where

$$\Gamma = \{\neg A \vee \neg B \vee C, \neg A \vee B, A, \neg C\}$$



□

6.0.4 Example. Prove

$$\vdash (A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$$

By DThm do instead: $A \wedge \neg B \vdash \neg(A \rightarrow B)$.

By 6.0.1 do instead

$$A \wedge \neg B, A \rightarrow B \vdash \bot$$

or

$$A \wedge \neg B, \neg A \vee B \vdash \bot$$

Use HYP Splitting, so do instead

$$A, \neg B, \neg A \vee B \vdash \bot$$

$$A, \neg B, \neg A \vee B$$

To this end, cut 1st and 3rd to get B.

Cut the latter with $\neg B$ to get $\bot$.

□

6.0.5 Example. *Annotating* *hypothesis splitting* and equivalence graphically: We do not annotate the equivalence or split lines any more than we annotate the CUT lines! □

Chapter 7

Predicate Logic

Extending Boolean Logic

Boolean Logic can deal only with the Boolean glue: properties and behaviour.

Can certify tautologies, but it misses *many other truths* as we will see, *like* $x = x$ *where* x *stands for a mathematical object like a matrix, string, array, number, etc.*

One of the obvious reasons is that Boolean logic cannot even "see" or "speak" about mathematical objects.

If it cannot see or speak about them, then naturally cannot reason about them either!

E.g, we *cannot even state* inside Boolean logic the sentence "every natural number greater than 1 has a prime factor".

Boolean Logic *does not know* what "every" means or what a "number" is, what "natural" means, what is "1", what "greater" means, what "prime" is, or what "factor" is.

In fact it is worse than not "knowing": It cannot even say any one of the concepts listed above.

Its alphabet and language are extremely limited.

We need a richer language!

7.0.1 Example. Look at these two math statements. The first says that *two sets are equal* IF *they have the same elements*. The second says that *any object is equal to itself*.

We read "$(\forall x)$" below as "*for all values of x*", usually said MORE SIMPLY as, "*for all x*".

$$(\forall y)(\forall z)\Big((\forall x)(x \in y \equiv x \in z) \to y = z\Big) \tag{1}$$

and

$$x = x \tag{2}$$

Boolean Logic is a *very high level* (= very non-detailed) abstraction of Mathematics.

Since Boolean Logic cannot see object variables x, y, z, cannot see $\forall$ or $=$, *nor can penetrate inside the so-called "scope" of* $(\forall z)$ —that is, the big brackets above— it myopically *understands* (*perceives*) each of (1) and (2) as atomic statements p and q (not seeing inside the scope it sees NO "glue").

Thus Boolean logic, if forced to opine about the above it will say none of the above is a theorem (by soundness).

Yet, (1) is an axiom of *Set Theory* and (2) is an *axiom in ALL mathematics*.

Says: "Every object is equal to itself." □

Enter First-Order Logic or Predicate Logic.

Predicate logic is *the language AND logic* of mathematics and mathematical sciences.

In it we CAN "speak" (1) and (2) above and *reason about them*.

7.1 The language of First-Order Logic

What symbols are *absolutely necessary* to *include* in the Alphabet, $\mathcal{V}_1$ —the subscript "1" for "1st-order"— of Predicate Logic?

Well, let us enumerate:

7.1.1 Definition. (The 1st-order alphabet; first part)

1. First of all, we are *EXTENDING*, *NOT* discarding, *Boolean Logic*. So we include in $\mathcal{V}_1$ *all of Boolean Logic's symbols* $\mathbf{p}, \bot, \top, (,), \neg, \wedge, \vee, \rightarrow, \equiv$, where $\mathbf{p}$ stands for any of the infinitely many Boolean variables.

2. Then we need *object variables* —that is variables that stand for *mathematical objects*— x, y, z, u, v, w *with or without primes or subscripts*. So, these are infinitely many.

   Metanotation that stands for any of them will be bold face, but using the same letters with or without primes or subscripts: $\mathbf{x}, \mathbf{x}''_{\mathbf{5}}, \mathbf{y}, \mathbf{w}'''_{\mathbf{123}}$, etc.

3. *Equality* between mathematical objects: $=$

4. *New glue*: $\forall$

   We call this glue *universal quantifier*. It is pronounced "for all".

Is that all? No. *But let's motivate with two examples.* □

7.1.2 Example. (Set theory) The language of set theory needs also a binary relation or *predicate* up in front: Denoted by "$\in$". BUT nothing else.

All else is "*manufactured*" in the theory, that is, introduced by definitions.

The manufactured symbols include *constants* like our familiar $\mathbb{N}$ (the *set of natural numbers*, albeit set theorists often prefer the symbol "ω"), our familiar *constant* "$\emptyset$" (the empty set).

Also include *functions* like $\cup, \cap$ and relations or *predicates* like $\subset, \subseteq$.

So set theory needs no constants or functions up in front to start "operating" (proving theorems, that is). □

7.1.3 Example. (Number theory) The language of Number theory —also called Peano arithmetic— needs —in order to get started:

- A *constant*, the *number zero*: 0
- A *predicate* ("less than"): $<$
- A unary *function*: "S". (This, informally/intuitively is the "successor function" which with input x produces output $x + 1$.)
- Two binary *functions*, "$+, \times$" with the obvious meaning.

Oct. 31, 2022

All else is "*manufactured*" in the theory, that is, introduced by definitions.

The manufactured symbols include *constants* like our familiar $1, 2, 1000234000785$.

Also include *functions* like x^y, $\lfloor x/y \rfloor$ and more relations or *predicates* like $\leq$. □

We will do logic for the user, that is, we are aiming to *teach the USE of logic*.

But will do so *without having to do set theory or number theory or any specific mathematical theory* (geometry, algebra, etc.).

So equipped with our observations from the examples above, we note that various theories start up with *DIFFERENT* sets of constants, functions and predicates —according to their specific needs.

So we will complete the Definition 7.1.1 in a way that *APPLIES TO ANY AREA OF MATHEMATICAL APPLICATION*.

7.1.4 Definition. (The 1st-order alphabet; part 2) Our 1st-order alphabet also includes the following symbols

(1) Symbols for zero or more *constants*. *Generically*, we use a, b, c, d with or without primes or subscripts for constants.

(2) Symbols for zero or more *functions*. *Generically* we use f, g, h with or without primes or subscripts for functions.

Each such symbol will have the need for a certain number of arguments, this number called the function's "*arity*" (must be ≥ 1). For example, S has arity 1; it is unary. Each of $+, \times$ have arity two; they are binary.

You see where the word "arity" comes from?

(3) Symbols for zero or more *predicates*, *generically* denoted as ϕ ("$f\bar{e}$", as in "see"), ψ ("$ps\bar{e}$"), with or without primes or subscripts.

Each predicate symbol will have the need for a certain number of arguments called its "*arity*" (must be ≥ 1). For example, $<$ has arity 2. □

The first-order *LANGUAGE* is a set of strings of two types —*terms* and *formulas*— over the *alphabet* 7.1.1 AND 7.1.4.

By now we should feel comfortable with *first-order inductive definitions.*

In fact we gave inductive definitions of *first-order Boolean formulas* and used it quite a bit, but also more recently gave an inductive definition of Boolean *proofs*.

Thus we inductively introduce first-order Terms that denote objects, and first-order formulas, that denote statements, in two separate definitions.

First terms:

7.1.5 Definition. (Terms)

A term is a string over the alphabet $\mathcal{V}_1$ that satisfies one of:

(1) It is just an *object variable* $\mathbf{x}$ (recall that $\mathbf{x}$ is metanotation and stands for *any* object variable).

BTW, we drop the qualifier "object" from "object variable" from now on, but *RETAIN* the qualifier "Boolean" in "Boolean variable".

(2) An *object constant* a (this stands for any constant —generically).

BTW, we ALSO drop the qualifier "object" from "object constant" from now on, but *RETAIN* the qualifier "Boolean" in "Boolean constant".

(3) **General case**. It is a string of the form $ft_1t_2 \ldots t_n$ where the function symbol f has *arity* n.

We will denote arbitrary terms generically by the metasymbols t, s *with or without primes or subscripts.* □

We will often abuse notation and write "$f(t_1, t_2, \ldots, t_n)$" for "$ft_1t_2 \ldots t_n$".

This is one (rare) case where *the human eye prefers extra brackets*! Be sure to note that the comma "," is not in our alphabet!

Examples from number theory.
$x, 0$ are terms. $x + 0$ is a term (abuse of the actual "$+x0$" notation).

$(x + y) \times z$ is a term (abuse of the actual $\times + xyz$).

With the concept of terms out of the way we now define 1st-order formulas:

First the Atomic Case:

7.1.6 Definition. (1st-order Atomic formulas) The following are the *atomic —that is, glue-less— formulas of 1st-order logic*:

(i) Any *Boolean* atomic formula.

(ii) The *expression* (*string*) "$t = s$", for any choice of t and s (probably, the t and s name the same term).

(iii) For any predicate ϕ of arity n, and any n terms $t_1, t_2, \ldots, t_n$, the string "$\phi t_1 t_2 \ldots t_n$".

We denote *the set of all atomic formulas* here defined **AF**. □

In practice, we prefer writing $x < y$ (infix) rather than $< xy$ (prefix)

7.1.7 Remark.

(1) As in the case of "complex" terms $f t_1 t_2 \ldots t_n$, we often abuse notation using "$\phi(t_1, t_2, \ldots, t_n)$" in place of the correctly written "$\phi t_1 t_2 \ldots t_n$".

(2) The symbol "=" is a binary predicate and is always written as it is here (never "ϕ, ψ").

(3) We absolutely NEVER confuse "$=$" with the Boolean "glue" "$\equiv$".

They are more different than apples and oranges! $\square$

7.1.8 Definition. (1st-order formulas) A first-order formula *A* — or **wff** *A*— is one of

We let context fend for us as to *what formulas we have in mind when we say "wff"*.

From here on it is 1st-order ones!

If we want to talk about Boolean wff we *WILL USE* the qualifier "Boolean"!

(1) A *member of 1st-order* **AF** *set*—in particular it could be a *Boolean* atomic wff!

(2) $(\neg B)$ if B is a wff.

(3) $(B \circ C)$ if *B and C* are wff, and $\circ$ is one of $\wedge, \vee, \rightarrow, \equiv$.

(4) $((\forall \mathbf{x})B)$, where B is a wff and $\mathbf{x}$ any variable.

TWO things: (1) we already agreed that "variable" means *object variable* otherwise I'd say "Boolean variable". (2) *Nowhere in the definition is required that* $\mathbf{x}$ *occurs in* B *as a substring*.

We call "$\forall$" the *universal quantifier*.

The configuration $(\forall \mathbf{x})$ is pronounced "for all $\mathbf{x}$" —intuitively meaning "*for all values of* $\mathbf{x}$" rather than "for all variables $x, y'', z'''_{1234009}, \ldots$ that $\mathbf{x}$ may stand for".

We say that the part of A between the two red brackets is the scope of $(\forall \mathbf{x})$.

Thus the $\mathbf{x}$ in $(\forall \mathbf{x})$ and the entire B are in this scope. □

The "in particular" observation in case (1) along with the cases (2) and (3) make it clear that every Boolean wff is also a (1st-order) wff.

Thus first-order logic can "speak" Boolean (but not the other way around, as we made abundantly clear!)

7.1.9 Example. $x = y$, $\bot$ and p are wff. In fact, Atomic.

The last two are also Boolean wff.

$((\forall x)((\forall y)(\neg x = y)))$ is a wff. Note that $\neg$ in $(\neg x = y)$ *applies to* $x = y$ NOT to x!

Glue cannot apply to an object like x. Must apply to a *statement* (a wff)!

$((\forall y)((\neg x = y) \wedge p))$ and $(((\forall y)(\neg x = y)) \wedge p)$ are also formulas.

BTW, in the two last examples: *p is in the scope of $(\forall y)$ in the first*, but not so in the second. □

7.1.10 Definition. (Existential quantifier)
It is convenient —but NOT NECESSARY— to introduce the "*existential quantifier*", $\exists$.

This is only a *metatheoretical* abbreviation symbol that we introduce by this *Definition*, that is, by a "*naming*"

For any wff A, we define $((\exists \mathbf{x})A)$ to be *a short name for*

$$\Big(\neg\Big((\forall \mathbf{x})(\neg A)\Big)\Big) \tag{1}$$

We pronounce $((\exists \mathbf{x})A)$ "for some (value of) $\mathbf{x}$, A holds".

The intuition behind this $((\exists \mathbf{x})A)$ *naming* is captured by the diagram below

$$\Big(\overbrace{\neg}^{\textit{it is not the case that}}\Big(\underbrace{(\forall \mathbf{x})}_{\textit{all values of } x}\overbrace{(\neg A)}^{\textit{make } A \textit{ false}}\Big)\Big)$$

The *scope of* $(\exists \mathbf{x})$ in

$$\Big((\exists \mathbf{x})A\Big) \tag{2}$$

is the area between the two red brackets.

In particular, the leftmost $\mathbf{x}$ in (2) is in the scope. □

Priorities Revisited

We *augment* our priorities *table*, *from highest to lowest*:

$$\overbrace{\forall, \exists, \neg}^{\textit{equal priorities}}, \wedge, \vee, \rightarrow, \equiv$$

Associativities *remain right*! Thus, $\neg(\forall x)\neg A$ is *a short form* of (1) in 7.1.10.

Another example: $(u = v \rightarrow (((\forall x)x = a) \wedge p))$ simplifies into

$$u = v \rightarrow (\forall x)x = a \wedge p$$

More examples:

(2) Instead of $((\forall z)(\neg x = y))$ we write

$$(\forall z)\neg x = y$$

(3) Instead of $((\forall x)((\forall x)x = y))$ we write

$$(\forall x)(\forall x)x = y$$

BOUND vs FREE.

7.1.11 Definition. A variable $\mathbf{x}$ *occurs free* in a wff A iff *it is NOT inside the scope of a* $(\forall \mathbf{x})$ *or* $(\exists \mathbf{x})$ —otherwise it occurs bound.

We say that a bound variable $\mathbf{x}$ in $\boxed{(\forall \mathbf{x})A}$ other than the one in the displayed $(\forall \mathbf{x})$, belongs to the displayed leftmost "$(\forall \mathbf{x})$" iff $\mathbf{x}$ *occurs free in* A —thus it was this leftmost "$(\forall \mathbf{x})$", which we added to the left of A that *did the bounding*!

The terminology "belongs to" is now clear.

We apply this criterion to *subformulas* of A of the form $(\forall \mathbf{x})(\ldots)$ to determine where various bound $\mathbf{x}$ found inside A belong. □

7.1.12 Example. Consider

$$(\forall x)\overbrace{(x = y \to (\forall x)x = z)}^{A})$$

Here the red x in A belongs to the red $\forall x$. The black x belongs to the black $\forall x$. □

Nov. 2, 2022

7.1.13 Remark. We saw that *a Boolean wff, is also a 1st-order wff.* *We view Boolean formulas as abstractions of 1st-order ones*.

How is this Abstraction accomplished?

Well, in any given 1st-order wff we just "hide" all 1st-order features.

That is, view *any wff among the following three* forms as Boolean variables since we are unable (from within Boolean Logic) to *understand the language they are in, and therefore what they say*.

1. $\boxed{t = s}$
2. $\boxed{\phi t_1 t_2 t_3 \ldots t_n}$
3. $\boxed{(\forall x)A}$

Why so? You see, if you "live" inside Boolean logic, you know these configurations are "*statements*" but you *cannot say what they say*:

You do not understand the symbols, and you do not see any glue.

WAIT! I do see "glue" in $\boxed{(\forall x)(A \to B)}$; don't I ???

No, you don't if you are a citizen of Boolean! The "$\to$" is hidden inside the scope of $(\forall x)$.

So an inhabitant of Boolean logic can USE the above "Boolean variables" if and only if they are connected with *VISIBLE* Boolean glue.

Of course, Boolean logic whose job is to certify tautologies —by either truth tables or proofs— has no use for isolated Boolean variables, that is, ones that are not glued to anything!

Examples.

- In Boolean Country you see this "$x = y \to x = y \vee x = z$" as "$\boxed{x = y} \to \boxed{x = y} \vee \boxed{x = z}$" where the first and second box is the same —say variable p— while the last one is different. You recognize a tautology!

- You see this "$x = x$" as "$\boxed{x = x}$". Just a Boolean variable.
Not a tautology.

- The same goes for this "$(\forall x)x = y \to x = y$" which the Boolean citizen views as "$\boxed{(\forall x)x = y} \to \boxed{x = y}$", that is, a Boolean wff $p \to q$. Not a tautology.

> **Process of abstraction**: We only abstract (that is, we see as "Boolean variables") the expressions 1.–3. above in order to turn a 1st-order wff into a Boolean wff.

The three forms above are known in logic as **Prime Formulas**.

More Boolean abstraction examples:

- If A is

  $p \to x = y \vee (\forall x)\phi x \wedge q$ (note that q *is not* in the scope of $(\forall x)$)

  then we abstract as

  $$p \to \boxed{x = y} \vee \boxed{(\forall x)\phi x} \wedge q \tag{1}$$

  so the Boolean citizen sees

  $$p \to p' \vee p'' \wedge q$$

  If we ask "show ALL the prime formulas in A by boxing them" then we —*who understand 1st-order language and we can see inside scopes*— would have also boxed ϕx above. The Boolean citizen cannot see ϕx in the scope of $(\forall x)$ anyway so the boxing done by such a citizen would be exactly as we gave it in (1)

- First box all prime formulas in (2) below.

  $$(\forall x)(x = y \to (\forall z)z = a \vee q) \tag{2}$$

  Here it is.

  $$\boxed{(\forall x)(\boxed{x = y} \to \boxed{(\forall z)\boxed{z = a}} \vee q)}$$

  Now abstract the above as if you were a Boolean citizen:

  $$\boxed{(\forall x)(x = y \to (\forall z)z = a \vee q)}$$

  You see no glue at all because you cannot see inside the scope of the leftmost $(\forall x)$!

  The abstraction is something like

  $$\text{"}p\text{"}$$

- $x = y \to x = y$ abstracts as $\boxed{x = y} \to \boxed{x = y}$. That is, $p \to p$ —*a tautology.*

Why bother with abstractions? Well, the last example is a tautology so a Boolean citizen can prove it.

However $\boxed{x = x}$ and $\boxed{(\forall x)x = y} \to \boxed{x = y}$ are not tautologies and we need predicate logic techniques to settle their theoremhood.

$\square$

We can now define:

7.1.14 Definition. (Tautologies and Tautological Implications) We say that a (1st-order) wff, A, *is a tautology and write* $\models_{taut} A$ iff *its Boolean abstraction is*.

In 1st-order Logic $\Gamma \models_{\mathrm{taut}} A$ is applied to the Boolean abstraction of A and to the abstractions of the wff in Γ.

Goes without saying that ALL the *identical* occurrences of *Prime Formulas* ... in $\Gamma \cup \{A\}$ will stand for the same Boolean variale.

For example, $x = y \models_{taut} x = y \vee z = v$ is correct as we see from

$$\overbrace{\boxed{x = y}}^{p} \models_{taut} \overbrace{\boxed{x = y}}^{p} \vee \overbrace{\boxed{z = v}}^{q}$$

□

Substitutions

A substitution is a *textual substitution*: **Find and Replace**.

In $A[\mathbf{x} := t]$ we will replace all occurrences of a *free* $\mathbf{x}$ in A by the term t: *Find and replace*.

In $A[\mathbf{p} := B]$ we will replace all occurrences of a $\mathbf{p}$ in A by B: *Find and replace*.

7.1.15 Example. (What to avoid) Consider the substitution below

$$\Big((\exists x)\neg x = y\Big)[y := x]$$

If we go ahead with it *as a brute force "find and replace" asking no questions*, then we are met by a *serious problem*:

The result

$$(\exists x)\neg x = x \tag{1}$$

says *something other than* what the original formula says!

The original —$((\exists x)\neg x = y)$— says "for any choice of y-value there is a *different* x-value".

The above is true in any application of logic *where we have infinitely many objects*. For example, it is true of real numbers and natural numbers.

On the other hand, (1) though is *NEVER* true! It says that there is an object that is *different from itself*! □

7.1.16 Definition. (Substitution) Each of

1. In $A[\mathbf{x} := t]$ replace all occurrences of a free $\mathbf{x}$ in A by the term t: *Find and replace*.

2. In $A[\mathbf{p} := B]$ replace all occurrences of a $\mathbf{p}$ in A by B: *Find and replace*.

However we *abort* the substitution 1 or 2 if it so happens that going ahead with it makes a *free* variable $\mathbf{y}$ of t or B *bound* because *t or B ended up in the scope of a $(\forall \mathbf{y})$ or $(\exists \mathbf{y})$*.

We say that the substitution is undefined in such cases, and that the reason is that *we had a "free variable* capture*"*.

There is a variant of substitution 2, above:

3. In $A[\mathbf{p} \backslash B]$ replace all occurrences of a $\mathbf{p}$ in A by B: *Find and replace*.

   For technically justified reasons to be learnt later, *we never abort this one,* **capture or not**.

We call the substitutions 1. and 2. *conditional* or *constrained*, while the substitution 3. unconditional or *unconstrained*.

There is NO unconditional version of 1.

PRIORITIES (AGAIN!)

$[\mathbf{x} := t], [\mathbf{p} := B], [\mathbf{p} \setminus B]$ have higher priority than all connectives $\forall, \exists, \neg, \wedge, \vee, \rightarrow, \equiv$. They associate from LEFT to RIGHT that is $A[\mathbf{x} := t][\mathbf{p} := B]$ means

$$\Big(\Big(A[\mathbf{x} := t]\Big)[\mathbf{p} := B]\Big)$$

$\square$

7.1.17 Example. Several substitutions based on Definition 7.1.16.

(1) $(y = x)[y := x]$.

The red brackets are META brackets. I need them to show the substitution applies to the whole formula.

The result is $x = x$.

(2) $\Big((\forall x)x = y\Big)[y := x]$. By 7.1.16, this is undefined because if I go ahead then x is captured by $(\forall x)$.

(3) $(\forall x)(x = y)[y := x]$. According to priorities, this means $(\forall x)\Big\{(x = y)[y := x]\Big\}$.

That is, "apply the quantifier $(\forall x)$ to $x = x$", which is all right.

Result is $(\forall x)x = x$.

(4) $\Big((\forall x)(\forall y)\phi(x, y)\Big)[y := x]$. This says

- Do $\Bigg((\forall x)\Big((\forall y)\phi(x, y)\Big)\Bigg)[y := x]$
- This is all right since y is not free in $\Big((\forall y)\phi(x, y)\Big)$ —so *not found; no replace!*

  Result is the original formula UNCHANGED.

(5) $\Big(z = a \vee ((\forall x)x = y)\Big)[y := x]$. *Abort*: x is captured when we attempt substitution in the subformula $(\forall x)x = y$.

(6) $\Big((\forall x)p\Big)[p \setminus x = y]$ Unconditional substitution. *Just find and replace, no questions asked!*

Result: $(\forall x)x = y$.

(7) $\Big((\forall x)p\Big)[p := x = y]$ Undefined. *x in $x = y$ will get captured if you go ahead!* □

7.1.18 Definition. (Partial Generalisation) We say that B is *a partial generalisation* of A if B is formed *by adding as a PREFIX to A* zero or more strings of the form $(\forall \mathbf{x})$ for any choices whatsoever of the variable $\mathbf{x}$ —*repetitions allowed*. □

7.1.19 Example. Here is a small list of partial generalisations of the formula $x = z$:

$x = z,$

$(\forall w)x = z,$

$(\forall x)(\forall x)x = z,$

$(\forall x)(\forall z)x = z,$

$(\forall z)(\forall x)x = z,$

$(\forall z)(\forall y)(\forall z)(\forall x)(\forall u)x = z.$ □

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7.2 Axioms and Rules for Predicate Logic

7.2.1 Definition. (1st-Order Axioms) These are **all the partial generalisations** of *all the instances of the following schemata*.

1. All tautologies (e.g., $x = y \to x = y$ is here)
2. $(\forall \mathbf{x})A \to A[\mathbf{x} := t]$

Note that *we get an instance of this schema ONLY IF the substitution is not aborted*.

3. $A \to (\forall \mathbf{x})A$ —*PROVIDED* $\mathbf{x}$ *is not free in* A.
4. $(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\forall \mathbf{x})B$
5. $\mathbf{x} = \mathbf{x}$
6. $t = s \to (A[\mathbf{x} := t] \equiv A[\mathbf{x} := s])$

Note that *we get an instance of this schema ONLY IF the none of the substitutions above is aborted*.

The set of all first-order axioms is named "Λ_1" —"1" for 1st-order. □

Our only INITIAL (or *Primary* or *Primitive*) rule is **Modus Ponens**:

$$\frac{A, A \to B}{B} \qquad (MP)$$

You may think that including all tautologies as axioms is overkill. However

1. It is customary to do so in the literature ([Tou08, Sho67, End72, Man77, Tou03])

2. After Post's Theorem we do know that every tautology is a theorem of Boolean logic.

Adopting axiom 1. makes every tautology also a theorem of Predicate Logic outright!

This is the easiest way (a literature favourite) to incorporate Boolean logic as a *sublogic* of 1st-order logic.

7.3 First-order Proofs and Theorems

A Hilbert-style proof from Γ (Γ-proof) is *exactly as defined in the case of Boolean Logic*. Namely:

It is a finite sequence of *wff*

$$A_1, A_2, A_3, \ldots, A_i, \ldots, A_n$$

such that each A_i is ONE of

1. Axiom from Λ_1 OR a member of Γ

   OR

2. Is obtained by *MP* from $X \to Y$ and X that appear to the LEFT of A_i (*A_i is the same string as Y then.*)

However, here "*wff*" is *1st-order*, and Λ_1 is a DIFFERENT set of axioms than the old Λ. Moreover we have ONLY one rule up in front.

As in Boolean definitions, a 1st-order theorem from Γ (Γ-theorem) is *a formula that occurs in a 1st-order Γ-proof*.

As before we write "$\Gamma \vdash A$" to say "A is a Γ-theorem" and write "$\vdash A$" to say "A is an absolute theorem".

Hilbert proofs in 1st-order logic are written vertically as well, with line numbers and annotation.

The metatheorems about proofs and theorems

- proof tail removal,
- proof concatenation,
- a wff is a Γ-theorem iff it occurs at the end of a proof
- hypothesis strengthening,
- hypothesis splitting,
- usability of derived rules,
- usability of previously proved theorems

hold with the same metaproofs *as in the Boolean case*.

We TRIVIALLY have Post's Theorem (the weak form that we proved for Boolean logic).

7.3.1 Theorem. (Weak Post's Theorem for 1st-order logic)
If $A_1, \ldots, A_n \models_{taut} B$ *then* $A_1, \ldots, A_n \vdash B$

Proof. Exactly the same as in Boolean logic, see 5.2.4, since the assumption yields $\models_{taut} A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$ as before, and hence we have $\vdash A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$, by Axiom 1.

For the rest see 5.2.4. □

Thus we may use

$$A_1, \ldots, A_n \vdash B$$

as a *DERIVED rule in any 1st-order proof, if we know that*

$$A_1, \ldots, A_n \models_{taut} B$$

.

7.4 Deduction Theorem

This Metatheorem of First-Order Logic says:

7.4.1 Metatheorem. *If* $\Gamma, A \vdash B$*, then also* $\Gamma \vdash A \to B$

OR

7.4.2 Metatheorem. *If I want to prove* $\Gamma \vdash A \to B$ it is enough to prove $\Gamma, A \vdash B$ instead.

WAIT! Did we not already prove this for Boolean Logic? Yes, but to do so we used *Boolean Semantics*.

Boolean Semantics will *NOT help* in Predicate Logic, and First-Order Semantics are tricky and we will do them at the end of the course!

So here we use an easy proof by Induction on the *length* of First-Order Proofs from $\Gamma + A$.*

*Recall that "$\Gamma \cup \{A\}$", "Γ, A" and "$\Gamma + A$" are *alternative notations* for the *same* set of wff!

Proof. Induction on the proof length L that we used for $\Gamma, A \vdash B$:

1. Proof of length $L = 1$ (*Basis*). There is only one formula in the proof: The proof must be

$$B$$

   Only two subcases apply:

   - $B \in \Gamma$. Then $\Gamma \vdash B$. But $B \models_{taut} A \to B$, thus by 7.3.1 also $B \vdash A \to B$. We get now $\Gamma \vdash A \to B$ by 4.1.11.
   - B *IS* A. So, $A \to B$ is a tautology hence axiom hence $\Gamma \vdash A \to B$.
   - $B \in \Lambda_1$. Then $\Gamma \vdash B$. Conclude as in the *first* bullet.

2. *Assume (I.H.) the claim for all proofs of lengths $L \leq n$.*
3. *I.S.*: The proof has length $L = n + 1$:

$$\overbrace{\ldots, B}^{n+1}$$

If $B \in \Gamma \cup \{A\} \cup \Lambda_1$ then we are done by the same argument as in 1.

Assume instead that it is the result of MP on formulas to the left of B:

$$\overbrace{\underbrace{\underbrace{\ldots, X, \ldots, X \to B}_{\leq n}, \ldots,}_{n} B}^{n+1}$$

By the I.H. we have

$$\Gamma \vdash A \to X \tag{$*$}$$

and

$$\Gamma \vdash A \to (X \to B) \tag{$**$}$$

The following Hilbert proof concludes the case and the entire proof:

1)	$A \to X$	$\langle \Gamma$-thm by $(*)\rangle$
2)	$A \to (X \to B)$	$\langle \Gamma$-thm by $(**)\rangle$
3)	$A \to B$	$\langle 1 + 2 +$ taut. implication$\rangle$

The last line proves the metatheorem. □

Comment. Line 3 uses $A \to X, \quad A \to (X \to B) \models_{taut} A \to B$, which translates (by 7.3.1) into the "RULE" $A \to X, A \to (X \to B) \vdash A \to B$.

The annotation said "1 + 2 + taut. implication".

It could also have said instead "1 + 2 + Post".

7.5 Generalisation and "weak" *Leibniz* Rules

We learn here HOW exactly to handle the quantifier $\forall$.

7.5.1 Adding and Removing "$(\forall \mathbf{x})$"

7.5.1 Metatheorem. (Weak Generalisation) *Suppose that no wff in* Γ has any free occurrences *of* $\mathbf{x}$.

Then if we have $\Gamma \vdash A$*, we will also have* $\Gamma \vdash (\forall \mathbf{x})A$.

It is normally the case and of no adverse consequence that A does have free occurrences of $\mathbf{x}$, else $(\forall \mathbf{x})A$ would be trivial.

Proof. Induction on the length L of the Γ-proof used for A.

1. $L = 1$ (*Basis*). There is only one formula in the proof: The proof must be

$$A$$

Only two subcases apply:

- $A \in \Gamma$. Then A has *no free* $\mathbf{x}$, hence $A \to (\forall \mathbf{x})A$ is axiom 3. Thus, we have a Hilbert proof (written horizontally for speed),

$$\overbrace{A}^{\Gamma-proved}, \overbrace{A \to (\forall \mathbf{x})A}^{axiom}, \overbrace{(\forall \mathbf{x})A}^{\text{MP on the previous two}}$$

- $A \in \Lambda_1$. Then so is $(\forall \mathbf{x})A \in \Lambda_1$ by partial generalisation:

Note that if axiom A is $(\forall \mathbf{z})(\forall \mathbf{z}')(\forall \mathbf{z}'') \ldots (\forall \mathbf{z_1}) \underbrace{B}_{\text{axiom schema inst.}}$, then $(\forall \mathbf{x})A$ is $(\forall \mathbf{x})(\forall \mathbf{z})(\forall \mathbf{z}') \ldots (\forall \mathbf{z_1})B$. Hence an axiom too.

Hence $(\forall \mathbf{x})A$ is in Λ_1, thus $\Gamma \vdash (\forall \mathbf{x})A$ once more. (Definition of Γ-proof.)

AHA! So that’s what “partial generalisation” <u>does</u> for us!

2. *Assume (I.H.) the claim for all proofs of lengths $L \leq n$.*
3. *I.S.*: The proof has length $L = n + 1$:

$$\overbrace{\ldots, A}^{n+1}$$

If $A \in \Gamma \cup \Lambda_1$ then we are done by the argument in 1.

Assume instead that A is the result of MP on formulas to the left of it:

$$\underbrace{\overbrace{\underbrace{\ldots, X, \ldots, X \rightarrow A}_{\leq n}, \ldots, A}^{n+1}}_{n}$$

By the I.H. we have

$$\Gamma \vdash (\forall \mathbf{x})X \tag{$*$}$$

and

$$\Gamma \vdash (\forall \mathbf{x})(X \rightarrow A) \tag{$**$}$$

The following Hilbert proof concludes this case and the entire proof:

1)	$(\forall \mathbf{x})X$	$\langle \Gamma$-thm by $(*)\rangle$
2)	$(\forall \mathbf{x})(X \rightarrow A)$	$\langle \Gamma$-thm by $(**)\rangle$
3)	$(\forall \mathbf{x})(X \rightarrow A) \rightarrow (\forall \mathbf{x})X \rightarrow (\forall \mathbf{x})A$	$\langle$axiom 4$\rangle$
4)	$(\forall \mathbf{x})X \rightarrow (\forall \mathbf{x})A$	$\langle 2 + 3 +$ MP$\rangle$
5)	$(\forall \mathbf{x})A$	$\langle 1 + 4 +$ MP$\rangle$

The last line proves the metatheorem. □

7.5.2 Corollary. *If* $\vdash A$*, then* $\vdash (\forall \mathbf{x})A$*.*

Proof. The condition that no X in Γ has free $\mathbf{x}$ is met: Vacuously. Γ is empty! □

7.5.3 Remark.

1. **HOW TO USE Generalisation**: So, the Metatheorem says that if A is a Γ-theorem then so is $(\forall \mathbf{x})A$ as long as the restriction of 7.5.1 is met.

   But then, *since I can invoke* Γ*-THEOREMS* (not only axioms and hypotheses) in a proof, I can insert the Γ-theorem $(\forall \mathbf{x})A$ anywhere AFTER A in any Γ-proof of A where Γ obeys the restriction on $\mathbf{x}$.

   $$\ldots, A, \ldots, \quad \overbrace{(\forall \mathbf{x})A}^{\text{insert at any time after } A} \quad , \ldots$$

2. Why "weak"? Because I need to know how the A was obtained before I may use $(\forall \mathbf{x})A$. □

7.5.4 Metatheorem. (Specialisation Rule) $\big((\forall \mathbf{x})A\big) \vdash A[\mathbf{x} := t]$

Goes without saying that *IF* the expression $A[\mathbf{x} := t]$ is undefined (*due to "capture"*), then we have nothing to prove.

Proof.

(1)	$(\forall \mathbf{x})A$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{x})A \to A[\mathbf{x} := t]$	$\langle$axiom 2$\rangle$
(3)	$A[\mathbf{x} := t]$	$\langle 1 + 2 + \text{MP}\rangle$

□

7.5.5 Corollary. $(\forall \mathbf{x})A \vdash A$

Proof. This is the special case where t is $\mathbf{x}$. □

Specialisation removes a $(\forall \mathbf{x})$ iff the quantifier is the *very first symbol*[a] of a formula B and the entire remaining part of the formula is the scope of that leading $(\forall \mathbf{x})$:

$$\overbrace{(\forall \mathbf{x}) \underbrace{A}_{A \text{ is in the scope of } (\forall \mathbf{x})}}^{B}$$

[a] "$(\forall \mathbf{x})$" is ONE compound symbol.

The $(\forall x)$ in the following two CANNOT be removed: $(\forall x)A \vee B$, $A \vee (\forall x)B$.

Really Important! The metatheorems 7.5.5 and 7.5.1 (or 7.5.2) — *which we nickname "spec" and "gen" respectively*— are tools that make our life easy in Hilbert proofs where handling of $\forall$ is taking place.

7.5.5 *with no restrictions* allows us to REMOVE a leading "$(\forall \mathbf{x})$".

Doing so *we might uncover Boolean glue* and thus benefit from applications of "Post" (7.3.1).

If we need to *re-INSERT* $(\forall \mathbf{x})$ before the end of proof, we employ 7.5.1 to do so.

This is a good recipe for success in 1st-order proofs!

7.5.2 Examples

Ping-Pong proofs.

Hilbert proofs are not well-suited to handle equivalences.

However, trivially

$$A \to B, B \to A \models_{taut} A \equiv B$$

and —by 7.3.1—

$$A \to B, B \to A \vdash A \equiv B \tag{1}$$

Thus, *to prove* $\Gamma \vdash A \equiv B$ *in Hilbert style* it suffices —by (1), which is a derived rule!— to offer TWO Hilbert proofs:

$$\Gamma \vdash A \to B \text{ AND } \Gamma \vdash B \to A$$

This back and forth motivates the nickname "ping-pong" for this proof technique.

7.5.3 A Few Memorable Examples

7.5.6 Theorem. (Distributivity of $\forall$ over $\wedge$)
$\vdash (\forall \mathbf{x})(A \wedge B) \equiv (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$

Proof. By Ping-Pong argument.

We will show TWO things:

1. $\vdash (\forall \mathbf{x})(A \wedge B) \to (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$

and

2. $\vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \to (\forall \mathbf{x})(A \wedge B)$

$(\to)$ ("1." above)

By DThm, it suffices to prove $(\forall \mathbf{x})(A \wedge B) \vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$.

(1)	$(\forall \mathbf{x})(A \wedge B)$	$\langle$hyp$\rangle$
(2)	$A \wedge B$	$\langle$1 + spec (7.5.5)$\rangle$
(3)	A	$\langle$2 + Post$\rangle$
(4)	B	$\langle$2 + Post$\rangle$
(5)	$(\forall \mathbf{x})A$	$\langle$3 + gen; OK: hyp contains no free $\mathbf{x}\rangle$
(6)	$(\forall \mathbf{x})B$	$\langle$4 + gen; OK: hyp contains no free $\mathbf{x}\rangle$
(7)	$(\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$	$\langle$(5,6) + Post$\rangle$

NOTE. We *ABSOLUTELY MUST* acknowledge for each application of "gen" that *the restriction is met.*

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($\leftarrow$) ("2." above)

By DThm, it suffices to prove $(\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \vdash (\forall \mathbf{x})(A \wedge B)$.

(1)	$(\forall \mathbf{x})A \wedge (\forall \mathbf{x})B$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{x})A$	$\langle 1 + \text{Post}\rangle$
(3)	$(\forall \mathbf{x})B$	$\langle 1 + \text{Post}\rangle$
(4)	A	$\langle 2 + \text{spec}\rangle$
(5)	B	$\langle 3 + \text{spec}\rangle$
(6)	$A \wedge B$	$\langle (4,5) + \text{Post}\rangle$
(7)	$(\forall \mathbf{x})(A \wedge B)$	$\langle 6 + \text{gen; OK: hyp has no free } \mathbf{x}\rangle$ □

Easy and Natural! Right?

7.5.7 Theorem. $\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \equiv (\forall \mathbf{y})(\forall \mathbf{x})A$

Proof. By Ping-Pong. $\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \overset{\rightarrow}{\leftarrow} (\forall \mathbf{y})(\forall \mathbf{x})A$.

($\rightarrow$) direction.

By DThm it suffices to prove $(\forall \mathbf{x})(\forall \mathbf{y})A \vdash (\forall \mathbf{y})(\forall \mathbf{x})A$

(1)	$(\forall \mathbf{x})(\forall \mathbf{y})A$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{y})A$	$\langle 1 +$ spec$\rangle$
(3)	A	$\langle 2 +$ spec$\rangle$
(4)	$(\forall \mathbf{x})A$	$\langle 3 +$ gen; OK hyp has no free $\mathbf{x}\rangle$
(5)	$(\forall \mathbf{y})(\forall \mathbf{x})A$	$\langle 4 +$ gen; OK hyp has no free $\mathbf{y}\rangle$

($\leftarrow$)
Exercise! Justify that you can write the above proof backwards!

□

7.5.8 Example. Say <u>A has no free x</u>. Then $\vdash (\forall x)A \equiv A$. Indeed, $\vdash (\forall x)A \rightarrow A$ by ax. 2 and $\vdash A \rightarrow (\forall x)A$ by ax. 3. □

7.5.9 Metatheorem. (Monotonicity of $\forall$) *If $\Gamma \vdash A \rightarrow B$, then $\Gamma \vdash (\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$, as long as no wff in Γ has a free* **x**.

Proof.

(1)	$A \rightarrow B$	$\langle$invoking a Γ-thm$\rangle$
(2)	$(\forall \mathbf{x})(A \rightarrow B)$	$\langle$1 + gen; OK no free $\mathbf{x}$ in $\Gamma\rangle$
(3)	$(\forall \mathbf{x})(A \rightarrow B) \rightarrow (\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$	$\langle$axiom 4$\rangle$
(4)	$(\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$	$\langle(2,3) + MP\rangle$ □

We annotate an application of "Monotonicity of $\forall$" by either of "A-MON" or "$\forall$-MON".

7.5.10 Corollary. *If $\vdash A \rightarrow B$, then $\vdash (\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$.*

Proof. Case of $\Gamma = \emptyset$. The restriction is vacuously satisfied. □

7.5.11 Corollary. *If* $\Gamma \vdash A \equiv B$*, then also* $\Gamma \vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{x})B$*,* as long as Γ *does not contain wff with* **x** *free.*

Proof.

(1)	$A \equiv B$	$\langle\Gamma$-theorem$\rangle$
(2)	$A \to B$	$\langle 1 + \text{Post}\rangle$
(3)	$B \to A$	$\langle 1 + \text{Post}\rangle$
(4)	$(\forall \mathbf{x})A \to (\forall \mathbf{x})B$	$\langle 2 + \forall\text{-MON (7.5.9)}\rangle$
(5)	$(\forall \mathbf{x})B \to (\forall \mathbf{x})A$	$\langle 3 + \forall\text{-MON (7.5.9)}\rangle$
(6)	$(\forall \mathbf{x})A \equiv (\forall \mathbf{x})B$	$\langle (4,5) + \text{Post}\rangle$

□

7.5.12 Corollary. *If* $\vdash A \equiv B$*, then also* $\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{x})B$*.*

Proof. Take $\Gamma = \emptyset$. □

7.6 Weak Leibniz

Note that since Post's theorem holds in first-order logic, we have that the *Boolean* primary rules (and all Boolean derived rules; WHY?) hold in predicate logic.

For example, the Boolean Leibniz rule

$$A \equiv B \vdash C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$$

holds since we have

$$A \equiv B \models_{taut} C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$$

What makes the rule "Boolean" is that we look at all of A, B, C and $\mathbf{p}$ from the Boolean "citizen's" point of view (Boolean abstractions). In particular, $\mathbf{p}$ is NOT in the scope of any quantifier!

It is *visible* to a Boolean citizen!

Hmmm. Can I do Leibniz with a $\mathbf{p}$ that is IN the scope of a quantifier? You bet!!

7.6.1 Metatheorem. (Weak (1st-order) Leibniz —Acronym "WL")

If $\vdash A \equiv B$, *then also* $\vdash C[\mathbf{p} \setminus A] \equiv C[\mathbf{p} \setminus B]$.

Proof. This generalises 7.5.12.

The metatheorem is proved by *Induction on the (formation of) wff* C.

Basis. Atomic case for C:

(1) C is $\mathbf{p}$. The metatheorem boils down to "if $\vdash A \equiv B$, then $\vdash A \equiv B$", which trivially holds!

(2) C is NOT $\mathbf{p}$ —that is, it is $\mathbf{q}$ (other than $\mathbf{p}$), or is $\bot$ or $\top$, or is $t = s$, or it is $\phi(t_1, \ldots, t_n)$. That is, C does not contain the "text" $\mathbf{p}$.

Then our Metatheorem statement becomes "*if* $\vdash A \equiv B$, *then* $\vdash C \equiv C$".

Given that $\vdash C \equiv C$ is indeed the case by Axiom 1, the "if" part is irrelevant. Done.

The complex cases.

(i) C is $\neg D$. From the I.H. we have $\vdash D[\mathbf{p} \setminus A] \equiv D[\mathbf{p} \setminus B]$,

hence $\vdash \neg D[\mathbf{p} \setminus A] \equiv \neg D[\mathbf{p} \setminus B]$ by Post.

But

$$\text{I } \underline{\textbf{want}} \text{ and have } \vdash \overbrace{(\neg D)}^{C}[\mathbf{p} \setminus A] \equiv \overbrace{(\neg D)}^{C}[\mathbf{p} \setminus B]$$

because

$$(\neg D)[\mathbf{p} \setminus A] \text{ is the same wff as } \neg D[\mathbf{p} \setminus A] \qquad \text{WHY?}$$

(ii) C is $D \circ E$, where $\circ \in \{\wedge, \vee, \rightarrow, \equiv\}$.

The I.H. yields $\vdash D[\mathbf{p} \setminus A] \equiv D[\mathbf{p} \setminus B]$ and $\vdash E[\mathbf{p} \setminus A] \equiv E[\mathbf{p} \setminus B]$ hence $\vdash D[\mathbf{p} \setminus A] \circ E[\mathbf{p} \setminus A] \equiv D[\mathbf{p} \setminus B] \circ E[\mathbf{p} \setminus B]$ by Post.

I want and have

$$\vdash \overbrace{(D \circ E)}^{C}[\mathbf{p} \setminus A] \equiv \overbrace{(D \circ E)}^{C}[\mathbf{p} \setminus B]$$

due to the way substitution works, namely,

$$(D \circ E)[\mathbf{p} \setminus A] \text{ is the same wff as } D[\mathbf{p} \setminus A] \circ E[\mathbf{p} \setminus A] \quad \text{WHY?}$$

(iii) C is $(\forall \mathbf{x})D$. This is the "*interesting case*".
From the I.H. follows $\vdash D[\mathbf{p} \setminus A] \equiv D[\mathbf{p} \setminus B]$.

From 7.5.12 we get $\vdash (\forall \mathbf{x})D[\mathbf{p} \setminus A] \equiv (\forall \mathbf{x})D[\mathbf{p} \setminus B]$, also written as

$$\vdash \overbrace{\big((\forall \mathbf{x})D\big)}^{C}[\mathbf{p} \setminus A] \equiv \overbrace{\big((\forall \mathbf{x})D\big)}^{C}[\mathbf{p} \setminus B]$$

because

$$\big((\forall \mathbf{x})D\big)[\mathbf{p} \setminus A] \text{ is the same wff as } (\forall \mathbf{x})D[\mathbf{p} \setminus A] \qquad \square$$

WL is the only "Leibniz" we will ever need (practically) in our use of 1st-order logic in these lectures.

Why "weak"? Because of the restriction on the Rule's Hypothesis: $A \equiv B$ must be an absolute theorem. (Recall that the Boolean Leibniz was not so restricted).

Why not IGNORE the restriction and "adopt" the strong rule (i) below?

Well, in logic you do *NOT* arbitrarily "*adopt*" derived rules; you prove them.

BUT, CAN I prove (i) *below* then?

NO, our logic does not allow it; here is why: If I can prove (i) then I can also prove STRONG generalisation (ii) from (i).

$$A \equiv B \vdash C[\mathbf{p} \setminus A] \equiv C[\mathbf{p} \setminus B] \qquad (i)$$

$$\text{strong generalisation: } A \vdash (\forall \mathbf{x})A \qquad (ii)$$

Here is why $(i) \Rightarrow (ii)$:

So, assume I have "Rule" (i). THEN (towards proving (ii))

(1)	A	$\langle$hyp$\rangle$
(2)	$A \equiv \top$	$\langle$(1) + Post$\rangle$
(3)	$(\forall \mathbf{x})A \equiv (\forall \mathbf{x})\top$	$\langle$(2) + (i); "Denom:" $(\forall \mathbf{x})\mathbf{p}\rangle$
(4)	$(\forall \mathbf{x})\top \equiv \top$	$\langle$Ax2 +Ax3 + ping-pong; cf. Example 7.5.8$\rangle$
(5)	$(\forall \mathbf{x})A \equiv \top$	$\langle$(3, 4) + Post$\rangle$
(6)	$(\forall \mathbf{x})A$	$\langle$(5) + Post$\rangle$

So if I have (i) I have (ii) too.

BUT: Here is *an informal reason* that I cannot possibly have (ii).

It is a provable fact —this is *1st-order Soundness*[†]— that *all absolute theorems* of 1st-order logic are true *in every informal interpretation I build for them*.

So *IF I have* (ii), then by the DThm I also have

$$\vdash A \to (\forall \mathbf{x})A \tag{1}$$

Interpret the above over the natural numbers as the specific

$$\vdash \overbrace{x = 0}^{A} \to (\forall x)\overbrace{x = 0}^{A} \tag{2}$$

By 1st-order Soundness, IF I have (1), *then (2) is true for all values of (the free)* x.

Well, try $x = 0$. We get $0 = 0 \to (\forall x)x = 0$. The lhs of "$\to$" is true but the rhs is false.

So I cannot have (ii) —nor (i), because this implies (ii)!

[†]For a proof wait until the near-end of the course

▶ SKIP Strong Leibniz in Predicate Logic, but here it is for the curious!

We *CAN* have a MODIFIED (i) where the substitution into **p** is restricted.

7.6.2 Metatheorem. (Strong Leibniz —Acronym "SL") $A \equiv B \vdash C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$

Goes without saying that if the rhs of $\vdash$ *is NOT* defined, then there is nothing to prove since the expresion "$C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$" represents no wff.

Remember this comment during the proof!

Proof. As we did for WL, the proof is an induction on the definition/formation of C.

Basis. C is atomic:

subcases

- C is **p**. We need to prove $A \equiv B \vdash A \equiv B$, which is the familiar $X \vdash X$.
- C is not **p**. The metatheorem now claims $A \equiv B \vdash C \equiv C$ which is correct since $C \equiv C$ is an axiom.

The complex cases.

(i) C is $\neg D$. By the I.H. we have $A \equiv B \vdash D[\mathbf{p} := A] \equiv D[\mathbf{p} := B]$, thus, $A \equiv B \vdash \neg D[\mathbf{p} := A] \equiv \neg D[\mathbf{p} := B]$ by Post.

We can rewrite the above as $A \equiv B \vdash (\neg D)[\mathbf{p} := A] \equiv (\neg D)[\mathbf{p} := B]$ since when substitution is allowed

$$\overbrace{(\neg D)}^{C}[\mathbf{p} := A] \text{ is the same as } \neg D[\mathbf{p} := A], \text{ etc.}$$

(ii) C is $D \circ E$. By the I.H. we get $A \equiv B \vdash D[\mathbf{p} := A] \equiv D[\mathbf{p} := B]$

and

$A \equiv B \vdash E[\mathbf{p} := A] \equiv E[\mathbf{p} := B]$.

Thus, by Post,

$$A \equiv B \vdash D[\mathbf{p} := A] \circ E[\mathbf{p} := A] \equiv D[\mathbf{p} := B] \circ E[\mathbf{p} := B]$$

The way substitution works (when defined), the above says

$$A \equiv B \vdash \overbrace{(D \circ E)}^{C}[\mathbf{p} := A] \equiv \overbrace{(D \circ E)}^{C}[\mathbf{p} := B]$$

(iii) *C is $(\forall \mathbf{x})D$*. This is the "interesting case".
From the I.H. we get

$$A \equiv B \vdash D[\mathbf{p} := A] \equiv D[\mathbf{p} := B]$$

Now, since the expressions $C[\mathbf{p} := A]$ and $C[\mathbf{p} := B]$ *ARE* defined —else we wouldn't be doing all this— the definition of *conditional* (restricted) substitution implies that neither A nor B have any free occurrences of $\mathbf{x}$.

Then $\mathbf{x}$ does not occur free in $A \equiv B$ either.

From 7.5.11 we get

$$A \equiv B \vdash (\forall \mathbf{x})D[\mathbf{p} := A] \equiv (\forall \mathbf{x})D[\mathbf{p} := B]$$

which —the way substitution works— *is the same as*

$$A \equiv B \vdash \overbrace{\big((\forall \mathbf{x})D\big)}^{C}[\mathbf{p} := A] \equiv \overbrace{\big((\forall \mathbf{x})D\big)}^{C}[\mathbf{p} := B]$$

□

Nov. 14, 2022

More Memorable Examples and "Techniques".

7.6.3 Theorem. $\vdash (\forall \mathbf{x})(A \to B) \equiv (A \to (\forall \mathbf{x})B)$, as long as $\mathbf{x}$ has no free occurrences in A.

Proof.
Ping-Pong using DThm.

($\to$) I want

$$\vdash (\forall \mathbf{x})(A \to B) \to (A \to (\forall \mathbf{x})B)$$

Better still, let me do (DThm)

$$(\forall \mathbf{x})(A \to B) \vdash A \to (\forall \mathbf{x})B$$

and, even better, (DThm!) I will do

$$(\forall \mathbf{x})(A \to B), A \vdash (\forall \mathbf{x})B$$

(1)	$(\forall \mathbf{x})(A \to B)$	$\langle$hyp$\rangle$
(2)	A	$\langle$hyp$\rangle$
(3)	$A \to B$	$\langle$(1) + spec$\rangle$
(4)	B	$\langle$(2, 3) + MP$\rangle$
(5)	$(\forall \mathbf{x})B$	$\langle$(4) + gen; OK: no free $\mathbf{x}$ in (1) or (2)$\rangle$

(←) I want

$$\vdash (A \to (\forall \mathbf{x})B) \to (\forall \mathbf{x})(A \to B)$$

or better still (DThm)

$$A \to (\forall \mathbf{x})B \vdash (\forall \mathbf{x})(A \to B) \tag{1}$$

Seeing that $A \to (\forall \mathbf{x})B$ has no free $\mathbf{x}$, I can prove the even easier

$$A \to (\forall \mathbf{x})B \vdash A \to B \tag{2}$$

and *after* this proof is done, then I can apply gen to $A \to B$ to get $(\forall \mathbf{x})(A \to B)$.

OK! By DThm I can prove the even simpler than (2)

$$A \to (\forall \mathbf{x})B, A \vdash B \tag{3}$$

Here it is:

(1)	$A \to (\forall \mathbf{x})B$	⟨hyp⟩
(2)	A	⟨hyp⟩
(3)	$(\forall \mathbf{x})B$	⟨(1, 2) + MP⟩
(4)	B	⟨(3) + spec ⟩

□

7.6.4 Corollary. $\vdash (\forall \mathbf{x})(A \lor B) \equiv A \lor (\forall \mathbf{x})B$, as long as $\mathbf{x}$ does not occur free in A.

Proof.

$$
\begin{aligned}
& (\forall \mathbf{x})(A \lor B) \\
\Leftrightarrow\ & \langle \text{WL} + \neg\lor \text{ (axiom, so abs. thm!); ``Denom:'' } (\forall \mathbf{x})\mathbf{p} \rangle \\
& (\forall \mathbf{x})(\neg A \to B) \\
\Leftrightarrow\ & \langle \text{``}\forall \to\text{'' (7.6.3)} \rangle \\
& \neg A \to (\forall \mathbf{x})B \\
\Leftrightarrow\ & \langle \text{tautology, hence axiom} \rangle \\
& A \lor (\forall \mathbf{x})B
\end{aligned}
$$

□

Most of the statements we prove in what follows have *Dual* counterparts obtained by swapping $\forall$ and $\exists$ and $\vee$ and $\wedge$.

Let us give a theorem version of the definition of $\exists$. This is useful in *Equational* proofs in Predicate Logic.

Definition (Recall):

$$(\exists \mathbf{x})A \text{ is short name for } \neg(\forall \mathbf{x})\neg A \tag{1}$$

Next consider the axiom

$$\neg(\forall \mathbf{x})\neg A \equiv \neg(\forall \mathbf{x})\neg A \tag{2}$$

Let me use the *ABBREVIATION* (1) ONLY on *ONE* side of "$\equiv$" in (2). I get the theorem

$$(\exists \mathbf{x})A \equiv \neg(\forall \mathbf{x})\neg A$$

So I can write the theorem without words like this:

$$\vdash (\exists \mathbf{x})A \equiv \neg(\forall \mathbf{x})\neg A \tag{3}$$

HEY! I can apply (3) in Equational proofs —via WL— easily!

I will still refer to (3) in proofs as "Def of E".

Here's something useful AND good practise too! It is the *Dual* of 7.6.4.

7.6.5 Corollary. $\vdash (\exists \mathbf{x})(A \wedge B) \equiv A \wedge (\exists \mathbf{x})B$, as long as $\mathbf{x}$ does not occur free in A.

In annotation we may call the above the "$\exists\wedge$ theorem".

Proof.

$$
\begin{aligned}
& (\exists \mathbf{x})(A \wedge B) \\
\Leftrightarrow\ & \langle \text{Def of E} \rangle \\
& \neg(\forall \mathbf{x})\neg(A \wedge B) \\
\Leftrightarrow\ & \langle \text{WL + axiom 1 (deM); "Denom:" } \neg(\forall \mathbf{x})\mathbf{p} \rangle \\
& \neg(\forall \mathbf{x})(\neg A \vee \neg B) \\
\Leftrightarrow\ & \langle \text{WL + } \forall \text{ over } \vee \text{ (7.6.4) —no free } \mathbf{x} \text{ in } \neg A\text{; "Denom:" } \neg\mathbf{p} \rangle \\
& \neg(\neg A \vee (\forall \mathbf{x})\neg B) \\
\Leftrightarrow\ & \langle \textbf{Ax1 (deM)} \rangle \\
& A \wedge \neg(\forall \mathbf{x})\neg B \\
\Leftrightarrow\ & \langle \text{WL + Def of E; "Denom:" } A \wedge \mathbf{p} \rangle \\
& A \wedge (\exists \mathbf{x})B
\end{aligned}
$$

□

7.7 Ad hoc Memorable Examples

1. While the following theorem —nicknamed "One-point rule"— will not play a big role in our lectures, still, on one hand it gives us an example of how we *use the axioms of equality* (Axioms 5 and 6) and on the other hand every mathematician uses it without even thinking about it, in the form, for example,

"$A(3)$ is the same as $(\exists x)(x = 3 \wedge A(x))$"

7.7.1 Theorem. (One point rule —$\forall$ version) *On the condition that $\mathbf{x}$ does not occur in t,*[†] *we have* $\vdash (\forall \mathbf{x})(\mathbf{x} = t \to A) \equiv A[\mathbf{x} := t]$.[‡]

Proof. By Ping-Pong.

($\to$) Note that since $\mathbf{x}$ does not occur in t, we have

$$\boxed{(\mathbf{x} = t \to A)[\mathbf{x} := t] \quad \text{means the same thing as} \quad t = t \to A[\mathbf{x} := t]}$$

Thus,

(1)	$(\forall \mathbf{x})\overbrace{(\mathbf{x} = t \to A)}^{B} \to \overbrace{t = t \to A[\mathbf{x} := t]}^{B[\mathbf{x}:=t]}$	$\langle \mathbf{Ax2} \rangle$
(2)	$(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$	$\langle$**Ax5** —partial gen. of $\mathbf{x} = \mathbf{x}\rangle$
(3)	$t = t$	$\langle (2) + \text{spec} \rangle$
(4)	$(\forall \mathbf{x})(\mathbf{x} = t \to A) \to A[\mathbf{x} := t]$	$\langle (1, 3) + \text{Post} \rangle$

($\leftarrow$) Recall the **General form of Axiom 6**: $\boxed{s = t \to (A[\mathbf{x} := s] \equiv A[\mathbf{x} := t])}$

(1)	$\mathbf{x} = t \to (A \equiv A[\mathbf{x} := t])$	$\langle \mathbf{Ax6} \rangle$
(2)	$A[\mathbf{x} := t] \to \mathbf{x} = t \to A$	$\langle (1) + \text{Post} \rangle$

Re above step note: $p \to (q \equiv r) \models_{taut} r \to p \to q$

[†] We can also say "does not occur free in t", but that is an overkill: A term t has NO bound variables.
[‡] Of course, if $A[\mathbf{x} := t]$ is *undefined*, then there is nothing to prove!

(3) $(\forall \mathbf{x})A[\mathbf{x} := t] \to (\forall \mathbf{x})(\mathbf{x} = t \to A)$ ⟨(2) + ∀-MON —(2) is an abs. thm⟩
(4) $A[\mathbf{x} := t] \to (\forall \mathbf{x})A[\mathbf{x} := t]$ ⟨**Ax3**⟩
(5) $A[\mathbf{x} := t] \to (\forall \mathbf{x})(\mathbf{x} = t \to A)$ ⟨(3, 4) + Post⟩

Note that Ax3 is applicable in (4) since $\mathbf{x}$ is not free in $A[\mathbf{x} := t]$

2. **7.7.2 Corollary. (One point rule —∃ version)** *On the condition that* $\mathbf{x}$ *does not occur in* t*, we have* $\vdash (\exists \mathbf{x})(\mathbf{x} = t \land A) \equiv A[\mathbf{x} := t]$.

Proof. Exercise! (*Hint.* Use the ∀ version and an Equational proof to prove the ∃ version (use the "Def of E" Theorem).) □

7.7.3 Theorem. (Bound variable renaming ($\forall$)) *IF* $\mathbf{z}$ *is fresh for* A *—that is, does not occur* as either free or bound *in* A*— then*
$\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$. $\leftarrow$ *Read this right: "$(\forall \mathbf{z})A(\mathbf{z})$"*

"Everyday mathematician's" notation is $\vdash (\forall \mathbf{x})A(\mathbf{x}) \equiv (\forall \mathbf{z})A(\mathbf{z})$.
But NOT our notation!

Proof. Ping-Pong.

($\rightarrow$)

(1) $(\forall \mathbf{x})A \rightarrow A[\mathbf{x} := \mathbf{z}]$ $\langle$**Ax2** —fresh $\mathbf{z}$; *no capture*: no "$(\forall \mathbf{z})(\ldots, \mathbf{x}, \ldots)$" in $A\rangle$

(2) $(\forall \mathbf{z})(\forall \mathbf{x})A \rightarrow (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$ $\langle$(1) + $\forall$-mon; OK: (1) is abs. thm$\rangle$

(3) $(\forall \mathbf{x})A \rightarrow (\forall \mathbf{z})(\forall \mathbf{x})A$ $\langle$**Ax3**$\rangle$

(4) $(\forall \mathbf{x})A \rightarrow (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$ $\langle$(2, 3) + Post$\rangle$

($\leftarrow$) Let us first settle a useful "lemma" for the proof below:

7.7.4 Lemma. *Under the assumptions about* $\mathbf{z}$ *(freshness), we have that* $A[\mathbf{x} := \mathbf{z}][\mathbf{z} := \mathbf{x}]$ *is just the original* A*.*

Proof. Now, $\mathbf{z}$ is *neither*

- *Bound* in A. That is, there is NO "$(\forall \mathbf{z})(\ldots)$" in A. So the substitution $A[\mathbf{x} := \mathbf{z}]$ *GOES THROUGH, AND* "flags" (and replaces) all FREE $\mathbf{x}$ in A as $\mathbf{z}$.

  nor is

- *Free* in A. So NO FREE $\mathbf{z}$ *pre-existed* in A before doing $A[\mathbf{x} := \mathbf{z}]$. That is, ALL FREE $\mathbf{z}$ in $A[\mathbf{x} := \mathbf{z}]$ are EXACTLY the $\mathbf{x}$ that became $\mathbf{z}$. *These* $\mathbf{z}$ *are PLACEHOLDERS for THE ORIGINAL FREE* $\mathbf{x}$ *in* A.

BUT then! Doing now $[\mathbf{z} := \mathbf{x}]$ changes ALL $\mathbf{z}$ in $A[\mathbf{x} := \mathbf{z}]$ back to $\mathbf{x}$.

We are back to the original A!

□

(1)	$(\forall \mathbf{z})\overbrace{A[\mathbf{x} := \mathbf{z}]}^{B} \to \overbrace{A[\mathbf{x} := \mathbf{z}]}^{B}[\mathbf{z} := \mathbf{x}]$	$\langle$**Ax2** —$A[\mathbf{x} := \mathbf{z}][\mathbf{z} := \mathbf{x}]$ OK by lemma$\rangle$
(2)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to A$	$\langle$same as (1) —see lemma$\rangle$
(3)	$(\forall \mathbf{x})(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to (\forall \mathbf{x})A$	$\langle$abs. thm (2) + $\forall$ MON$\rangle$
(4)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to (\forall \mathbf{x})(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}]$	$\langle$Ax3; no free $\mathbf{x}$ in lhs$\rangle$
(5)	$(\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to (\forall \mathbf{x})A$	$\langle$(3, 4) + Post$\rangle$ □

7.8 Adding and Removing the Quantifier "$(\exists x)$"

Nov. 16, 2022

First, introducing (adding) $\exists$ is easy via the following tools:

7.8.1 Theorem. (Dual of Ax2) $\vdash A[\mathbf{x} := t] \to (\exists \mathbf{x})A$

Proof.

$$
\begin{aligned}
& A[\mathbf{x} := t] \to (\exists \mathbf{x})A \\
\Leftrightarrow\ & \langle \text{WL + "Def of E" (this is an abs. thm); "Denom:" } A[\mathbf{x} := t] \to \mathbf{p} \rangle \\
& A[\mathbf{x} := t] \to \neg(\forall \mathbf{x})\neg A \\
\Leftrightarrow\ & \langle \text{tautology} \rangle \\
& (\forall \mathbf{x})\neg A \to \neg A[\mathbf{x} := t] \qquad \text{Bingo!}
\end{aligned}
$$

□

7.8.2 Corollary. (The Dual of Specialisation) $A[\mathbf{x} := t] \vdash (\exists \mathbf{x})A$

Proof. 7.8.1 and MP. □

7.8.3 Corollary. $A \vdash (\exists \mathbf{x})A$

Proof. 7.8.2, taking $\mathbf{x}$ as t. □

Either corollaries above we call "*Dual Spec*" in annotating proofs.

But how can I remove a leading (the entire formula) $\exists$?

We need two preliminary results to answer this.

7.8.4 Metatheorem. ($\forall$ Introduction) *If* $\mathbf{x}$ *does not occur free in* Γ *nor in* A*, then* $\Gamma \vdash A \to B$ *iff* $\Gamma \vdash A \to (\forall \mathbf{x})B$*.*

Proof. of the "iff".

($\to$) direction.

Assumption gives $\Gamma \vdash (\forall \mathbf{x})(A \to B)$ by valid generalisation.

But we have

$$\begin{aligned} & (\forall \mathbf{x})(A \to B) \\ \Leftrightarrow\; & \langle 7.6.3 \rangle \\ & A \to (\forall \mathbf{x})B \end{aligned}$$

So the bottom formula is a Γ-theorem.

($\leftarrow$) direction.

This time we know the bottom of the above short Equational proof is a Γ-theorem.

Then so is the top. But from the latter I get $\Gamma \vdash A \to B$ by spec.

□

7.8.5 Corollary. **($\exists$ Introduction)** *IF* $\mathbf{x}$ *does not occur free in* Γ *nor in* B*, then* $\Gamma \vdash A \to B$ *iff* $\Gamma \vdash (\exists \mathbf{x})A \to B$*.*

Note how we shifted the condition for $\mathbf{x}$ from A to B.

Proof. of the "*iff*". Well,

$$\Gamma \vdash A \to B \overset{Post}{\textbf{iff}} \Gamma \vdash \neg B \to \neg A \overset{7.8.4}{\textbf{iff}} \Gamma \vdash \neg B \to (\forall \mathbf{x})\neg A$$
$$\overset{Post}{\textbf{iff}} \Gamma \vdash \neg(\forall \mathbf{x})\neg A \to B$$

□

You already know that removing a leading $\forall$ "uncovers" (in general[†]) "Boolean structure" which is amenable to proofs "by Post".

It would be a shame if we did not have techniques to remove a leading $\exists$.

We DO have such a technique! Read on.

[†]Clearly, removing $\forall$ from $(\forall x)x = y$ uncovers $x = y$. But that has no Boolean structure —no glue. Hence I said "in general".

7.8.6 Metatheorem. (Aux. Hypothesis Metatheorem) *Suppose that $\Gamma \vdash (\exists \mathbf{x})A$.*

Moreover, suppose that we know that $\Gamma, A[\mathbf{x} := \mathbf{z}] \vdash B$, where $\mathbf{z}$ is fresh for ALL of Γ, $(\exists \mathbf{x})A$, and B.

Then we have $\Gamma \vdash B$.

In our annotation we call $A[\mathbf{x} := \mathbf{z}]$ an "**auxiliary hypothesis associated with $(\exists \mathbf{x})A$**". $\mathbf{z}$ is called the auxiliary variable that we chose.

Essentially the fact that we proved $(\exists \mathbf{x})A$ allows us to adopt $A[\mathbf{x} := \mathbf{z}]$ as a *NEW AUXILIARY HYPOTHESIS* to help in the proof of B.

▶ How does it help? (1) I have a new hypothesis to work with; (2) $A[\mathbf{x} := \mathbf{z}]$ has *NO LEADING QUANTIFIER*.

(2), in general, results in uncovering the Boolean structure of $A[\mathbf{x} := \mathbf{z}]$ to enable proof by "Post"!

Halt-and-Take-Notice: Important! $A[\mathbf{x} := \mathbf{z}]$ is an *ADDED HYPOTHESIS!*

▶ It is *NOT TRUE* that either $(\exists \mathbf{x})A \vdash A[\mathbf{x} := \mathbf{z}]$ or that $\Gamma \vdash A[\mathbf{x} := \mathbf{z}]$.◀

WE WILL PROVE LATER IN THE COURSE THAT SUCH A THING IS NOT TRUE!

Proof. of the Metatheorem.

By the DThm, the metatheorem assumption yields

$$\Gamma \vdash A[\mathbf{x} := \mathbf{z}] \to B$$

Thus, by $\exists$-Intro (7.8.5) we get

$$\Gamma \vdash (\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to B \tag{1}$$

We now can prove $\Gamma \vdash B$ as follows:

1) $(\exists \mathbf{x})A$ $\langle \Gamma\text{-thm} \rangle$
2) $(\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \to B$ $\langle \Gamma\text{-thm; (1) above} \rangle$
3) $(\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \equiv (\exists \mathbf{x})A$ $\langle$Bound var. renaming since $\mathbf{z}$ fresh$\rangle$
4) $(\exists \mathbf{x})A \to B$ $\langle (2, 3) + \text{Post} \rangle$
5) B $\langle (1, 4) + \text{MP} \rangle$

□

The most frequent form encountered in using Metatheorem 7.8.6 is the following corollary.

7.8.7 Corollary. *To prove* $(\exists \mathbf{x})A \vdash B$ IT SUFFICES *to*

- *pick a* $\mathbf{z}$ *that is* FRESH *for* $(\exists \mathbf{x})A$ *and* B *and*

- PROVE INSTEAD $(\exists \mathbf{x})A, A[\mathbf{x} := \mathbf{z}] \vdash B$.

Proof. Take $\Gamma = \{(\exists x)A\}$ and invoke Metatheorem 7.8.6. □

Some folks believe that the most important thing in logic is to know that the following is provable but the converse is not.

True, it is important.

But so are so many other things in logic, like Metatheorem 7.8.6, *precisely and correctly formulated* AND proved in our earlier pages.

7.8.8 Example. $\vdash (\exists \mathbf{x})(\forall \mathbf{y})A \to (\forall \mathbf{y})(\exists \mathbf{x})A$.

Let us share two proofs!

First *Proof.* By DThm it suffices to prove instead:

$(\exists \mathbf{x})(\forall \mathbf{y})A \vdash (\forall \mathbf{y})(\exists \mathbf{x})A$

(1)	$(\exists \mathbf{x})(\forall \mathbf{y})A$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{y})A[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$(2) + spec$\rangle$
(4)	$(\exists \mathbf{x})A$	$\langle$(3) + Dual spec: $B[\mathbf{x} := t] \vdash (\exists \mathbf{x})B\rangle$
(5)	$(\forall \mathbf{y})(\exists \mathbf{x})A$	$\langle$(4) + gen; OK, *all hyp lines*, (1,2), have no free $\mathbf{y}\rangle$

We used the Corollary 7.8.7 of Metatheorem 7.8.6.

Second *Proof.* $\vdash A \rightarrow (\exists \mathbf{x})A$ (that is, the Dual of Ax2) we get $\vdash (\forall \mathbf{y})A \rightarrow (\forall \mathbf{y})(\exists \mathbf{x})A$ by $\forall$-mon.

Applying $\exists$-intro (7.8.5) we get

$$\vdash (\exists \mathbf{x})(\forall \mathbf{y})A \rightarrow (\forall \mathbf{y})(\exists \mathbf{x})A$$

□

7.8.9 Example. We prove $(\exists \mathbf{x})(A \to B), (\forall \mathbf{x})A \vdash (\exists \mathbf{x})B$.

(1)	$(\exists \mathbf{x})(A \to B)$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{x})A$	$\langle$hyp$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}] \to B[\mathbf{x} := \mathbf{z}]$	$\big\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\big\rangle$
(4)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$(2) + spec$\rangle$
(5)	$B[\mathbf{x} := \mathbf{z}]$	$\langle$(3, 4) + $MP\rangle$
(6)	$(\exists \mathbf{x})B$	$\langle$(5) + Dual spec$\rangle$

Remark. The above proves the conclusion using 7.8.6 and $\Gamma = \{(\exists \mathbf{x})(A \to B), (\forall \mathbf{x})A\}$. Of course, this Γ proves $(\exists \mathbf{x})(A \to B)$. □

7.8.10 Example. We prove $(\forall \mathbf{x})(A \to B), (\exists \mathbf{x})A \vdash (\exists \mathbf{x})B$.

(1)	$(\forall \mathbf{x})(A \to B)$	$\langle$hyp$\rangle$
(2)	$(\exists \mathbf{x})A$	$\langle$hyp$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (2); $\mathbf{z}$ fresh$\rangle$
(4)	$A[\mathbf{x} := \mathbf{z}] \to B[\mathbf{x} := \mathbf{z}]$	$\langle$(1) + spec$\rangle$
(5)	$B[\mathbf{x} := \mathbf{z}]$	$\langle$(3, 4) + $MP\rangle$
(6)	$(\exists \mathbf{x})B$	$\langle$(5) + Dual spec$\rangle$ $\Box$

7.8.11 Example. Here is a common mistake people make when arguing informally.

Let us prove the following informally.

$\vdash (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \rightarrow (\exists \mathbf{x})(A \wedge B)$.

So let $(\exists \mathbf{x})A(\mathbf{x})$ *and* $(\exists \mathbf{x})B(\mathbf{x})$ *be true.*[†]

Thus, for some value c *of* $\mathbf{x}$ *we have that* $A(c)$ *and* $B(c)$ *are true.*

But then so is $A(c) \wedge B(c)$.

The latter implies the truth of $(\exists \mathbf{x})\big(A(\mathbf{x}) \wedge B(\mathbf{x})\big)$.

Nice, crisp and short.

And very, very wrong as we will see once we have **1st-order Soundness** in hand. Namely, we will show in the near future that $(\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \rightarrow (\exists \mathbf{x})(A \wedge B)$ *is NOT* a theorem schema. It is NOT provable.

[†]The experienced mathematician considers self-evident and unworthy of mention at least two things:
(1) The deduction theorem, and

(2) The Split Hypothesis metatheorem.

What went wrong above?

We said

"Thus, for some value c of $\mathbf{x}$ we have that $A(c)$ and $B(c)$ are true".

The blunder was to assume that THE SAME c verified BOTH $A(x)$ and $B(x)$.

Let us see that formalism protects even the inexperienced from such blunders.

Here are the first few steps of a(n attempted) FORMAL proof via the Deduction theorem:

(1)	$(\exists \mathbf{x})A \wedge (\exists \mathbf{x})B$	$\langle$hyp$\rangle$
(2)	$(\exists \mathbf{x})A$	$\langle$(1) + Post$\rangle$
(3)	$(\exists \mathbf{x})B$	$\langle$(1) + Post$\rangle$
(4)	$A[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (2); $\mathbf{z}$ fresh$\rangle$
(5)	$B[\mathbf{x} := \mathbf{w}]$	$\langle$aux. hyp for (3); $\mathbf{w}$ fresh$\rangle$

The requirement of freshness makes $\mathbf{w}$ DIFFERENT from $\mathbf{z}$. These variables play the role of two distinct c and c'. Thus the proof cannot be continued. Saved by freshness! □

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7.8.12 Example. The last Example in this section makes clear that the Russell Paradox was the result of applying *bad Logic*, not just *bad Set Theory*!

I will prove that for any binary predicate ϕ we have

$$\vdash \neg(\exists \mathbf{y})(\forall \mathbf{x})(\phi(\mathbf{x}, \mathbf{y}) \equiv \neg\phi(\mathbf{x}, \mathbf{x})) \tag{R}$$

By the Metatheorem "Proof by Contradiction" I can show

$$(\exists \mathbf{y})(\forall \mathbf{x})(\phi(\mathbf{x}, \mathbf{y}) \equiv \neg\phi(\mathbf{x}, \mathbf{x})) \vdash \bot$$

instead. Here it is

(1)	$(\exists \mathbf{y})(\forall \mathbf{x})(\phi(\mathbf{x}, \mathbf{y}) \equiv \neg\phi(\mathbf{x}, \mathbf{x}))$	$\langle$hyp$\rangle$
(2)	$(\forall \mathbf{x})(\phi(\mathbf{x}, \mathbf{z}) \equiv \neg\phi(\mathbf{x}, \mathbf{x}))$	$\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\rangle$
(3)	$\phi(\mathbf{z}, \mathbf{z}) \equiv \neg\phi(\mathbf{z}, \mathbf{z})$	$\langle$(2) + spec$\rangle$
(4)	$\bot$	$\langle$(3) + Post$\rangle$

If we let the atomic formula $\phi(\mathbf{x}, \mathbf{y})$ be Set Theory's "$\mathbf{x} \in \mathbf{y}$" then ($R$) that we just proved (in fact *for ANY* binary predicate ϕ not just $\in$) morphs into

$$\vdash \neg(\exists \mathbf{y})(\forall \mathbf{x})(\mathbf{x} \in \mathbf{y} \equiv \mathbf{x} \notin \mathbf{x}) \tag{R'}$$

In plain English (R') says that *there is NO set* $\mathbf{y}$ *that contains ALL x satisfying $x \notin x$*.

This theorem was proved without using even a single axiom of set theory, indeed not even using "$\{\ldots\}$-notation" for sets, or any other symbols from set theory.

After all we proved (R') generally and abstractly in the form (R) and that expression and its proof has NO SYMBOLS from set theory!

In short, Russell's Paradox can be expressed AND demonstrated in PURE LOGIC.

It is remarkable that Pure Logic can tell us that NOT ALL COLLECTIONS are SETS, a fact that escaped Cantor! $\square$

Semantics of First-Order Languages —Simplified

7.9 Interpretations

Systematically translate an *abstract* formula —symbol by symbol— until it becomes a *concrete* mathematical formula, preferably familiar to you.

In this translation ensure that there are no free variables, so the mathematical formula is *exactly ONE* of true or false.

An *interpretation* of *ONE* wff —and of *THE ENTIRE language*, that is, the set of *ALL* Terms and wff— is INHERITED from an **interpretation of all symbols of the Alphabet**.

This tool —*the Interpretation*— Translates *each wff* to some formula of a familiar branch of mathematics that *we choose*, and thus questions such as "is the translated formula true?" can in principle be dealt with (see 7.9.2 below for details).

An interpretation is totally up to us, just as states were in Boolean logic.

The process is only slightly more complex.
Here we need to interpret not only wff but also terms as well.

The latter requires that we *choose a* *NONEMPTY* *set of objects to begin with*. We call this set the *Domain* of our Interpretation and *generically* call it "D" but in specific cases it could be $D = \mathbb{N}$ or $D = \mathbb{R}$ (the *reals*) or even something "small" like $D = \{0, 5\}$.

An *Interpretation* of a 1st-order language consists of a *PAIR* of *two things*:

The aforementioned domain D and a translation *mapping* M — the latter translates the abstract symbols of the Alphabet of logic to concrete *mathematical* symbols.

► This translation of the ALPHABET INDUCES a translation for each term and wff of the language; thus of ALL THE LANGUAGE. ◄

We denote the interpretation "package" as $\mathfrak{D} = (D, M)$ displaying the two ingredients D and M in round brackets.

The unusual calligraphy here is German capital letter calligraphy that is usual in the printed literature *to name an interpretation package*.

On the chalk board I would use ordinary calligraphy, like "$\mathcal{D}$".

The package name chosen is usually the same as that of the Domain.

Let me repeat that both D and M are OUR choice.

7.9.1 Definition. (Translating the Alphabet $\mathcal{V}_1$)
An *Interpretation* $\mathfrak{D} = (D, M)$ gives concrete *counterparts* (translations) to ALL elements of the *Alphabet* as follows:

In the listed cases below we may use notation $M(X)$ to indicate the concrete translation (mapping) of an abstract *linguistic object* X.

We also may use $X^{\mathfrak{D}}$ as an alternative notation for $M(X)$.

The literature favours $X^{\mathfrak{D}}$ and so will we.

Here are the actual translation RULES:

(1) For each *FREE* variable (of a wff) $\mathbf{x}$, $\mathbf{x}^{\mathfrak{D}}$ —that is, the translation $M(\mathbf{x})$— is some *chosen* (BY US!) *FIXED* member of D.

BOUND variables are NOT translated! They stay AS IS.

(2) For each Boolean variable $\mathbf{p}$, $\mathbf{p}^{\mathfrak{D}}$ is a member of $\{\mathbf{t}, \mathbf{f}\}$ that *WE CHOOSE*!

(3) $\top^{\mathfrak{D}} = \mathbf{t}$ and $\bot^{\mathfrak{D}} = \mathbf{f}$.

This is just as we did —via states— in the Boolean case. As was the case there we choose the value $\mathbf{p}^{\mathfrak{D}}$ anyway we please, but for $\top$ and $\bot$ we follow the fixed (Boolean) rule.

(4) For any (object) *constant* of the alphabet, say, c, we choose a *FIXED* $c^{\mathfrak{D}}$, as we wish, in D.

(5) For every *function* symbol f of the alphabet, the translation $f^{\mathfrak{D}}$ is a *mathematical function* of the "real" or "concrete" MATH. It has the same arity as f.

$f^{\mathfrak{D}}$ —*which WE choose!*— *takes inputs* from D and *gives outputs* in D.

(6) For every predicate ϕ of the alphabet OTHER THAN "=", our CHOSEN translation $\phi^{\mathfrak{D}}$ is a mathematical RELATION of the metatheory with the same arity as ϕ. It takes its inputs from D while its outputs are one or the other of the truth values $\mathbf{t}$ or $\mathbf{f}$.

► **NOTE THAT ALL the Boolean glue as well as the equality symbol translate exactly as THEMSELVES: "=" for "equals", $\vee$ for "OR", etc.**

Finally, brackets translate as the SAME TYPE of bracket (left or right).

□

We have all we need now to translate wff, terms and thus the entire Language:

7.9.2 Definition. (The Translation of wff)
Consider a wff A in a[†] first-order language.

Suppose we have chosen an interpretation $\mathfrak{D} = (D, M)$ of the alphabet.

The interpretation or translation of A via $\mathfrak{D}$ *is a mathematical ("concrete") formula of the metatheory* or a concrete object of the metatheory that we will denote by

$$A^{\mathfrak{D}}$$

It is constructed as follows one symbol at a time, **scanning A from left to right until no symbol is left**:

[†] A, not THE. For every choice of constant, predicate and function symbols we get a different alphabet, as we know, hence a different first-order language. Remember the examples of Set Theory vs. Peano Arithmetic!

(i) *We replace* every occurrence of $\perp, \top$ in A by $\perp^{\mathfrak{D}}, \top^{\mathfrak{D}}$ —that is, by $\mathbf{f}, \mathbf{t}$— respectively.

(ii) *We replace* every occurrence of $\mathbf{p}$ in A by $\mathbf{p}^{\mathfrak{D}}$ —this is an *assigned by US TRUTH VALUE*; we assigned it *when we translated the alphabet*.

(iii) *We replace* each *FREE* occurrence of an object variable $\mathbf{x}$ of A by the value $\mathbf{x}^{\mathfrak{D}}$ from D that we assigned *when we translated the alphabet*.

(iv) *We replace* every occurrence of $(\forall \mathbf{x})$ in A by $(\forall \mathbf{x} \in D)$, which means *AND is read* "for all values of $\mathbf{x}$ in D".

(iv′) *We replace* every occurrence of $(\exists \mathbf{x})$ in A by $(\exists \mathbf{x} \in D)$, which means *AND is read* "for some value of $\mathbf{x}$ in D".

(v) We emphasise again that Boolean connectives (glue) *translate as themselves*, and so do "=" and the brackets "(" and ")".

Theory-specific symbols in A:

(vi) *We replace* every occurrence of a(n object) constant c in A by the specific fixed $c^{\mathfrak{D}}$ from D —which we chose when translating the alphabet.

(vii) *We replace* every occurrence of a function f in A by the specific fixed $f^{\mathfrak{D}}$ —which we chose when translating the alphabet.

(viii) *We replace* every occurrence of a predicate ϕ in A by the specific fixed $\phi^{\mathfrak{D}}$ —which we chose when translating the alphabet. □

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7.9.3 Definition. (Partial Translation of a wff) Given a wff A in a first-order language and an interpretation $\mathfrak{D}$ of the alphabet.

Sometimes *we do NOT wish to translate a FREE variable* $\mathbf{x}$ of A. Then the result of the translation that *leaves* $\mathbf{x}$ *as is* is denoted by $A^{\mathfrak{D}}_{\mathbf{x}}$.

Similarly, if we choose *NOT* to translate *ANY* of

$$\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}, \ldots$$

that (may) occur FREE in A, then we show the result of such "*partial*" translation as

$$A^{\mathfrak{D}}_{\mathbf{x_1},\ldots,\mathbf{x_n},\ldots}$$

Thus $A^{\mathfrak{D}}$ has no free variables, but $A^{\mathfrak{D}}_{\mathbf{x}}$ will have $\mathbf{x}$ free IF $\mathbf{x}$ actually DID occur free in A —the notation guarantees that if $\mathbf{x}$ so occurred, then we left it alone.

□

7.9.4 Remark. What is the need for the concept and notation "$A_{\mathbf{x}}^{\mathfrak{D}}$"?

Well, for one, note that when we translate $(\forall \mathbf{x})A$ *FROM LEFT TO RIGHT*, we get "$(\forall \mathbf{x} \in D)$" followed by the translation of A.

However, ANY $\mathbf{x}$ that occur *free* IN A *BELONG* to $(\forall \mathbf{x})$ in the wff $(\forall \mathbf{x})A$ thus are *NOT FREE* in the latter and hence are NOT translated!

Therefore, "$(\forall \mathbf{x} \in D)$" concatenated with "$A_{\mathbf{x}}^{\mathfrak{D}}$" is what we get: "$(\forall \mathbf{x} \in D)A_{\mathbf{x}}^{\mathfrak{D}}$". □

Study ALL Examples!

7.9.5 Example. Consider the AF $\phi(x, x)$, ϕ is a binary predicate. Here are some possible interpretations:

(a) $D = \mathbb{N}$, $\phi^{\mathfrak{D}} =<$.

Here "<" is the "less than" relation on natural numbers.

So $\big(\phi(x, x)\big)^{\mathfrak{D}}$, which is the same as $\phi^{\mathfrak{D}}(x^{\mathfrak{D}}, x^{\mathfrak{D}})$ —in familiar notation is the formula over $\mathbb{N}$:

$$x^{\mathfrak{D}} < x^{\mathfrak{D}}$$

More specifically, if we took $x^{\mathfrak{D}} = 42$, then $\big(\phi(x, x)\big)^{\mathfrak{D}}$ is specifically "$42 < 42$".

Incidentally, $\big(\phi(x, x)\big)^{\mathfrak{D}}$ is false for ANY choice of $x^{\mathfrak{D}}$.

We will write $\big(\phi(x, x)\big)^{\mathfrak{D}} = \mathbf{f}$ to denote the above sentence symbolically.

I would have preferred to write something like "$V\Big(\big(\phi(x, x)\big)^{\mathfrak{D}}\Big) = \mathbf{t}$ —"V" for value— but it is *so much easier to agree that writing the above I mean the same thing!* :)

For the sake of practice, here are two *partial interpretations*.

In the first we exempt the variables y, z. In the second we exempt x:

(i) $\Big(\phi(x,x)\Big)^{\mathfrak{D}}_{y,z}$ is $x^{\mathfrak{D}} < x^{\mathfrak{D}}$. WHY?

(ii) $\Big(\phi(x,x)\Big)^{\mathfrak{D}}_{x}$ is $x < x$.

(b) $D = \mathbb{N}$, $\phi^{\mathfrak{D}} = \leq$ (the "less than or equal" relation on $\mathbb{N}$).
So, $\big(\phi(x,x)\big)^{\mathfrak{D}}$ is the concrete $x^{\mathfrak{D}} \leq x^{\mathfrak{D}}$ on $\mathbb{N}$.

Clearly, independently of the choice of $x^{\mathfrak{D}}$, we have

$$\big(\phi(x,x)\big)^{\mathfrak{D}} = \mathbf{t}$$

□

7.9.6 Example. Consider next the wff

$$f(x) = f(y) \to x = y \tag{1}$$

where f is a unary function.

Here are some interpretetions:

1. $D = \mathbb{N}$ and $f^{\mathfrak{D}}$ is chosen to be $f^{\mathfrak{D}}(x) = x{+}1$, for all values of x in D.

   Thus $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}$ translates as this formula over $\mathbb{N}$:

   $$f^{\mathfrak{D}}(x^{\mathfrak{D}}) = f^{\mathfrak{D}}(y^{\mathfrak{D}}) \to x^{\mathfrak{D}} = y^{\mathfrak{D}}$$

   $$x^{\mathfrak{D}} + 1 = y^{\mathfrak{D}} + 1 \to x^{\mathfrak{D}} = y^{\mathfrak{D}}$$

   Note that *every* choice of $x^{\mathfrak{D}}$ and $y^{\mathfrak{D}}$ makes the above true.

2. $D = \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

   Take $f^{\mathfrak{D}}(x) = x^2$, for all x in $\mathbb{Z}$.
   Then, $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}$ is, more concretely, the following formula over $\mathbb{Z}$:

   $$(x^{\mathfrak{D}})^2 = (y^{\mathfrak{D}})^2 \to x^{\mathfrak{D}} = y^{\mathfrak{D}}$$

   The above is true for some choices of $x^{\mathfrak{D}}$ and $y^{\mathfrak{D}}$ but not for others:

   E.g., it is false if we took $x^{\mathfrak{D}} = -2$ and $y^{\mathfrak{D}} = 2$.

Finally here are two *partial interpretations* of (1) at the beginning of this example:

(i) $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}_{x}$ is $x^2 = (y^{\mathfrak{D}})^2 \to x = y^{\mathfrak{D}}$.

(ii) $\big(f(x) = f(y) \to x = y\big)^{\mathfrak{D}}_{x,y}$ is $x^2 = y^2 \to x = y$. □

7.9.7 Example. (Important!) Consider the wff

$$x = y \to (\forall x)x = y \tag{1}$$

Here are a few interpretations:

1. $D = \{3\}$, $x^{\mathfrak{D}} = 3$, $y^{\mathfrak{D}} = 3$.
   Since D contains one element only the above "choice" *is ALL we HAVE*, being unique.
   Thus (1) translates as
   $$3 = 3 \to (\forall x \in D)x = 3 \tag{2}$$
   Incidentally, (2) is TRUE.

2. This time I take
   $D = \{3, 5\}$, and again $x^{\mathfrak{D}} = 3$ and $y^{\mathfrak{D}} = 3$.
   Thus (1) translates as:
   $$3 = 3 \to (\forall x \in D)x = 3 \tag{3}$$
   This time (3) is FALSE since "$3 = 3$" is TRUE as before, BUT
   $$\text{"}(\forall x \in D)x = 3\text{" is FALSE.}$$

□

7.9.8 Example. Let's interpret the following in a few different ways:

$$(\forall x)(x \in y \equiv x \in z) \to y = z \tag{1}$$

1. First this is true if we really are talking about sets as "$\in$" compels us to think, being THE predicate of set theory that says "*is a member of*".

   Incidentally, (1) if interpreted in Set Theory, says that any two sets y and z are equal if they happen to have the same elements (x is in y iff x is in z). *Hence is true, as I noted*.

2. Let us now interpret in number theory (of $\mathbb{N}$).

   Take $D = \mathbb{N}$ and $\in^{\mathfrak{D}} = <$, where "$<$" is the relation "*less than*" on $\mathbb{N}$.

Wait a minute! Can I do that?! Can I interpret "$\in$" as something OTHER than "*is a member of*"?

Of course you can!

Only "$=, (,), \neg, \vee, \wedge, \to, \equiv, \forall, \exists$" translate as themselves!

EVERYTHING ELSE is fair game to translate as you please!

So (1) translates as:

$$(\forall x \in \mathbb{N})(x < y^{\mathfrak{D}} \equiv x < z^{\mathfrak{D}}) \rightarrow y^{\mathfrak{D}} = z^{\mathfrak{D}}$$

which is TRUE no matter how we choose $y^{\mathfrak{D}}$ and $z^{\mathfrak{D}}$.

3. Next, let $D = \mathbb{N}$ and $\in^{\mathfrak{D}} = |$, where "$|$" indicates the relation "*divides*" (with remainder zero).

   E.g., $2\,|\,3$ and $2\,|\,1$ are FALSE but $2\,|\,4$ and $2\,|\,0$ are TRUE. Then (1) translates as:

   $$(\forall x \in \mathbb{N})(x\,|\,y^{\mathfrak{D}} \equiv x\,|\,z^{\mathfrak{D}}) \rightarrow y^{\mathfrak{D}} = z^{\mathfrak{D}}$$

   which is also TRUE for all choices of $y^{\mathfrak{D}}, z^{\mathfrak{D}}$.

   It says: "*Two natural numbers, $y^{\mathfrak{D}}$ and $z^{\mathfrak{D}}$, are EQUAL if they have exactly the same divisors*".

4. But consider something slightly different now: Take $D = \mathbb{Z}$ —the set of all integers— and $\in^{\mathfrak{D}} = |$. Take also $y^{\mathfrak{D}} = 2$ and $z^{\mathfrak{D}} = -2$.

   Then (1) translates as

   $$(\forall x \in \mathbb{Z})(x\,|\,2 \equiv x\,|\,-2) \rightarrow 2 = -2$$

   This is FALSE, for 2 and -2 have the same divisors, but $2 \neq -2$.

   So (1) is NOT TRUE IN ALL INTERPRETATIONS. □

7.10 Soundness in Predicate Logic

7.10.1 Definition. (Universally Valid wff)

Suppose that $A^{\mathfrak{D}} = \mathbf{t}$ for some A and $\mathfrak{D}$.

$A^{\mathfrak{D}} \equiv \mathbf{t}$ is more correct, but the habit of writing "$\ldots = \mathbf{t}$" for the sentence "$\ldots$ is true" deeply embedded in the mathematical culture and even if we agree to write "$\equiv \mathbf{t}$" I will many times forget and write "$= \mathbf{t}$" :)

We say that A *is true in the interpretation* $\mathfrak{D}$ or that $\mathfrak{D}$ *is a model of A.*

We write this thus:

$$\models_{\mathfrak{D}} A \tag{1}$$

A 1st-order wff, A, is *universally valid* —or just "valid"— iff *EVERY interpretation of the wff is a model of it*, that is, we have that (1) *holds for every interpretation* $\mathfrak{D}$ of the language of A.

In symbols,

$$A \text{ is valid iff, for all } \mathfrak{D}, \text{ we have } \models_{\mathfrak{D}} A \tag{2}$$

(2) has the short expression (3) below:

$$\models A \tag{3}$$

A formula A that satisfies (3) is sometimes also called Logically or Absolutely valid. □

7.10.2 Remark. NOTE the absence of the subscript "*taut*" in the notation (3) above.

The symbols $\models$ and $\models_{taut}$ are NOT the same!

For example, $\mathbf{x} = \mathbf{x}$ translates as

$$\mathbf{x}^{\mathfrak{D}} = \mathbf{x}^{\mathfrak{D}} \tag{4}$$

in EVERY interpretation $\mathfrak{D}$, and is thus true in every interpretation, since it is a self-evident philosophical truth that every object is equal to itself!

Thus, we have $\models \mathbf{x} = \mathbf{x}$.

On the other hand, $\models_{\text{taut}} \mathbf{x} = \mathbf{x}$ is NOT a TRUE meta statement.

$\mathbf{x} = \mathbf{x}$ is NOT a tautology! It is a prime formula (WHY?) hence a Boolean variable!

NO Boolean variable is a tautology as I can assign to it the VALUE FALSE. □

Valid Axioms 1. Ax1. Every axiom here is a tautology A. Thus $\models_{taut} A$.

> This means that for all values that WE assign to all the $\mathbf{p}, \mathbf{q}, \ldots$ that occur in A, and for all values that WE assign to all prime formulas — RECALL: these behave as *Boolean variables*— we get the truth value of A come out TRUE.

Well, when we interpret A in some Interpretation $\mathfrak{D}$ we actually COMPUTE the values of the prime formulas in this interpretation (rather than assign them).

However, **the BOXED paragraph above makes clear**, that whether we *COMPUTE* OR **ARBITRARILY ASSIGN** values to *the prime formulas* of A, the final value will be TRUE.

► A tautology does NOT CARE how the values of its variables are obtained!◄

So, $\models_{\mathfrak{D}} A$. As $\mathfrak{D}$ was arbitrary, I got

$$\models A$$

Valid Axioms 2. Ax2. $(\forall \mathbf{x})A \to A[\mathbf{x} := t]$ is valid.

Indeed, take a $\mathfrak{D}$, for the language of $A, \mathbf{x}, t$.

Now $\Big((\forall \mathbf{x})A \to A[\mathbf{x} := t]\Big)^{\mathfrak{D}}$ is

$$(\forall \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}} \to \Big(A[\mathbf{x} := t]\Big)^{\mathfrak{D}} \tag{1}$$

To the left of $\to$ we explained the translation of $(\forall \mathbf{x})A$ in Remark 7.9.4.

Let's make the rhs of $\to$ more useable:

Claim: $\Big(A[\mathbf{x} := t]\Big)^{\mathfrak{D}}$ is the same as $A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := t^{\mathfrak{D}}]$.

Indeed, start with the wff A depicted as a box below.

$$A: \quad \boxed{\ldots\boxed{\mathbf{x}}\ldots\boxed{\mathbf{x}}\ldots}$$

Thus

$$A[\mathbf{x} := t]: \quad \boxed{\ldots\boxed{t}\ldots\boxed{t}\ldots} \tag{3}$$

Hence

$$\Big(A[\mathbf{x} := t]\Big)^{\mathfrak{D}}: \quad \boxed{(\ldots)^{\mathfrak{D}}\boxed{t^{\mathfrak{D}}}(\ldots)^{\mathfrak{D}}\boxed{t^{\mathfrak{D}}}(\ldots)^{\mathfrak{D}}} \tag{4}$$

But (4) is the result of applying "$[\mathbf{x} := t^{\mathfrak{D}}]$" to

$$A^{\mathfrak{D}}_{\mathbf{x}}: \quad (\ldots)^{\mathfrak{D}}\boxed{\mathbf{x}}(\ldots)^{\mathfrak{D}}\boxed{\mathbf{x}}(\ldots)^{\mathfrak{D}}$$

that is, it is the same as

$$A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := t^{\mathfrak{D}}]$$

With *the claim verified*, (1) is now TRUE:

Here is why: Assume the lhs of $\rightarrow$ in (1). That is, suppose $A^{\mathfrak{D}}_{\mathbf{x}}$ is true for all values $i \in D$. But *then it is true IN PARTICULAR* for the value $i = t^{\mathfrak{D}}$.

Valid Axioms 3. **Ax6**. $t = s \rightarrow (A[\mathbf{x} := t] \equiv A[\mathbf{x} := s])$. The translation of this in $\mathfrak{D}$ is —see the work we did for **Ax2**!)

$$t^{\mathfrak{D}} = s^{\mathfrak{D}} \rightarrow \\ (A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := t^{\mathfrak{D}}] \equiv A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := s^{\mathfrak{D}}]) \tag{1}$$

Assume the lhs of "$\rightarrow$" in (1). Thus $t^{\mathfrak{D}} = s^{\mathfrak{D}} = k \in D$.

The rhs of (1) becomes

$$A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := k] \equiv A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := k]$$

which is trivially true.

Valid Axioms 4. For the remaining axioms there is nothing new to learn; see the text for proofs of their validity. Incidentally, the axiom $\mathbf{x} = \mathbf{x}$ has already been shown to be valid (7.10.2).

7.10.3 Metatheorem. (Soundness of Predicate Logic)
If $\vdash A$*, then* $\models A$*.*

We omit the trivial proof by induction on proof length (we saw two such proofs already).

For length one we NOTE that the ONLY formula that appears in the proof is an axiom. But that is valid!

The induction step notes that our *ONLY PRIMARY*[†] *rule*, MP, preserves truth.

[†]Given up in front.

7.10.4 Example. (Strong Gen; Again!) Can our logic **prove** *strong generalisation* as a "derived rule"?

Namely, can we have

If $\Gamma \vdash A$, then $\Gamma \vdash (\forall \mathbf{x})A$, ***with NO restriction on*** $\mathbf{x}$?

If yes, take $\Gamma = \{A\}$.† We get

$$A \vdash (\forall \mathbf{x})A \tag{1}$$

By the DThm, (1) allows this:

$$\vdash A \to (\forall \mathbf{x})A \tag{2}$$

Soundness OBJECTS to (2):

If we got (2) then, by Soundness, we get

$$\models A \to (\forall \mathbf{x})A \tag{3}$$

I will contradict (3) showing

$$\not\models A \to (\forall \mathbf{x})A \tag{4}$$

The Definition of "$\models$" (7.10.1) (4) dictates that I find ONE $\mathfrak{D}$ such that

$$(A \to (\forall \mathbf{x})A)^{\mathfrak{D}} = \mathbf{f} \tag{5}$$

This $\mathfrak{D}$ is called a countermodel of (2).

†Then $A \vdash A$, hence $A \vdash (\forall \mathbf{x})A$.

PRACTICAL ADVISE: It is hopeless to search for a $\mathfrak{D}$ *FOR A GENERAL A*.

For a *countermodel* I ONLY need a SPECIFIC A (a countermodel is a counterexample!)

▶ Always work with an *atomic* formula in place of A.

Now then! Take A to be atomic, for example, take A to be "$x = y$" If (3) works, it should work with *this* special case of A!

DOES IT?

NO. We saw in Example 7.9.7(2.) (cf. Definition 7.10.1)

$$\not\models x = y \to (\forall x)x = y$$

So (2) is wrong and so is (1). □

7.10.5 Example. We have proved in class/NOTES/Text

$$\vdash (\exists \mathbf{y})(\forall \mathbf{x})A \to (\forall \mathbf{x})(\exists \mathbf{y})A$$

We hinted in class that we cannot also prove

$$\vdash (\forall \mathbf{x})(\exists \mathbf{y})A \to (\exists \mathbf{y})(\forall \mathbf{x})A \tag{1}$$

To show that (1) is unprovable I pick a countermodel (=an interpretation that makes the wff in it false).

Pick A to be something simple. Atomic is best!

I take $D = \mathbb{N}$ and $\mathbf{x} = \mathbf{y}$ for A. Translating the wff in (1) I note

$$\overbrace{(\forall \mathbf{x} \in \mathbb{N})(\exists \mathbf{y} \in \mathbb{N})\mathbf{x} = \mathbf{y}}^{\mathbf{t}} \to \overbrace{(\exists \mathbf{y} \in \mathbb{N})(\forall \mathbf{x} \in \mathbb{N})\mathbf{x} = \mathbf{y}}^{\mathbf{f}}$$

Since the interpretation falsifies a special case of (1) the latter is not provable (by soundness). □

7.10.6 Example. We noted in class/NOTES/Text that we cannot prove

$$\vdash (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \rightarrow (\exists \mathbf{x})(A \wedge B) \tag{1}$$

To demonstrate this fact now we use Soundness and countermodels.

So, I pick a countermodel.

Pick A and B to be something simple. Atomic is best!

I take $D = \mathbb{N}$ and "$\mathbf{x} < 42$" for A while I take "$\mathbf{x} > 42$" for B. Translating the wff in (1) I note

$$\overbrace{(\exists \mathbf{x} \in \mathbb{N})\mathbf{x} < 42 \wedge (\exists \mathbf{x} \in \mathbb{N})\mathbf{x} > 42}^{\mathbf{t}} \rightarrow \overbrace{(\exists \mathbf{x} \in \mathbb{N})(\mathbf{x} < 42 \wedge \mathbf{x} > 42)}^{\mathbf{f}}$$

Since the interpretation falsifies a special case of (1) the latter is not provable (by soundness). □

7.10.7 Exercise. On the other hand, do prove by ∃-elimination the other direction: We DO have

$$\vdash (\exists \mathbf{x})(A \wedge B) \to (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B$$

□

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7.10.8 Example. (Important!) Why is $D \neq \emptyset$ important?

Well let us start by proving

$$\vdash (\forall \mathbf{x})A \to (\exists \mathbf{x})A \tag{1}$$

Use DThm to prove instead

$$(\forall \mathbf{x})A \vdash (\exists \mathbf{x})A$$

1)	$(\forall \mathbf{x})A$	$\langle$hyp$\rangle$
2)	A	$\langle 1 +$ spec$\rangle$
3)	$(\exists \mathbf{x})A$	$\langle 2 +$ Dual spec$\rangle$

However, if I took $\mathfrak{D} = (D, M)$ with $D = \emptyset$ then look at the transaltion of the formula in (1):

$$\overbrace{(\forall \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}}^{\textbf{t vacuously}}{}^{\dagger} \to \overbrace{(\exists \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}}^{\textbf{f}} \tag{2}$$

Soundness fails for the formula in (1). *We DON'T like this! So we NEVER allow $D = \emptyset$.* □

[†] Do not forget that "$(\forall \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}$" means "$(\forall \mathbf{x})(\mathbf{x} \in D \to A^{\mathfrak{D}}_{\mathbf{x}})$", while "$(\exists \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}$" means "$(\exists \mathbf{x})(\mathbf{x} \in D \land A^{\mathfrak{D}}_{\mathbf{x}})$".

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