This page **must** be submitted as the first page of your FINAL EXAM-paper answer pages.

York University
Department of Electrical Engineering and Computer Science
Lassonde School of Engineering

MATH 1090 B. FINAL EXAM, December 22, 2021;
9:05am-11:05am

Professor George Tourlakis

This page must be submitted as the first page of your FINAL EXAM-paper answer pages.

By putting my name and student ID on this MID TERM page, I attest to the fact that my answers included here and submitted by Moodle are my own work, and that I have acted with integrity, abiding by the *Senate Policy on Academic Honesty* that the instructor discussed at the beginning of the course and *linked the full Policy to the Course Outline.*

Student NAME (**Clearly**):______________________________

Student NUMBER (**Clearly**):______________________________

DATE (**Clearly**):______________________________

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README FIRST! INSTRUCTIONS:

1. **TIME-LIMITED ON LINE *FINAL EXAM*.**

   You have **120 MIN** ***to answer the EXAM questions***.

   ABSOLUTELY last opportunity to upload to eClass is **BY 11:20am**, that is, **EXACTLY 15 min** after the Official End Time.

2. **Only ONE file can be uploaded per student.**

3. If you submit photographed copy **it still must be ONE file that you submit**. Either ZIP the JPEG/PNG images **OR** import them in **MS Word** and submit *ONE* Word *file* with the images attached.

4. *Using the time allotted for the uploading mechanisms (15 min)* for the FINAL EXAM-answering part is at your own **discretion**.

   But also at your own **risk**.

   FINAL EXAM *not* uploaded = FINAL EXAM *not* written.

5. Please write your answers by hand **as you normally do for assignments** or use a word processor that can convert to PDF. Microsoft Word is acceptable to upload as is (without conversion to PDF).

6. Whatever results were *proved* in class or appeared in the assignments you may use without proof, **unless I am asking you to prove them in this Examination**. If you are not sure whether some statement has **indeed** been proved *in class*, I recommend that you prove it in order to be "safe".

Question	MAX POINTS	MARK
1	3	
2	3	
3	5	
1	3	
2	5	
3	5	
4	5	
5	5	
TOTAL	34	

The following are the axioms of Propositional Calculus: In what follows, A, B, C stand for arbitrary formulae.

Properties of $\equiv$

Associativity of $\equiv$	$((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C))$	(1)
Symmetry of $\equiv$	$(A \equiv B) \equiv (B \equiv A)$	(2)

Properties of $\bot, \top$

$\top$ vs. $\bot$	$\top \equiv \bot \equiv \bot$	(3)

Properties of $\neg$

Introduction of $\neg$	$\neg A \equiv A \equiv \bot$	(4)

Properties of $\vee$

Associativity of $\vee$	$(A \vee B) \vee C \equiv A \vee (B \vee C)$	(5)
Symmetry of $\vee$	$A \vee B \equiv B \vee A$	(6)
Idempotency of $\vee$	$A \vee A \equiv A$	(7)
Distributivity of $\vee$ **over** $\equiv$	$A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C$	(8)
Excluded Middle	$A \vee \neg A$	(9)

Properties of $\wedge$

Golden Rule	$A \wedge B \equiv A \equiv B \equiv A \vee B$	(10)

Properties of $\rightarrow$

Implication	$A \rightarrow B \equiv A \vee B \equiv B$	(11)

The **Primary** Boolean rules are:

$$\frac{A, A \equiv B}{B} \qquad (Eqn)$$

and

$$\frac{A \equiv B}{C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]} \qquad (Leib)$$

The following are the Predicate Calculus Axioms:

Any partial generalisation of any formula in groups* Ax1–Ax6 *is an axiom for Predicate Calculus.

Groups **Ax1–Ax6** contain the following schemata:

Ax1. Every tautology.

Ax2. $(\forall \mathbf{x})A \to A[\mathbf{x} := t]$, for any term t.

Ax3. $A \to (\forall \mathbf{x})A$, provided $\mathbf{x}$ is not free in A.

Ax4. $(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\forall \mathbf{x})B$.

Ax5. For *each* object variable $\mathbf{x}$, the formula $\mathbf{x} = \mathbf{x}$.

Ax6. For any terms t, s, the schema $t = s \to (A[\mathbf{x} := t] \equiv A[\mathbf{x} := s])$.

There is ONLY ONE **Primary** First-Order rule; MODUS PONENS (*MP*)

$$\frac{A, A \to B}{B} \qquad (MP)$$

In predicate calculus the most natural proofs are Hilbert-style.

The following **metatheorems** hold for ***both*** Propositional and Predicate Calculus:

1. *Redundant* $\top$. $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$
2. *Cut Rule.* $A \vee B, \neg A \vee C \vdash B \vee C$
3. *Deduction Theorem.* If $\Gamma, A \vdash B$, then $\Gamma \vdash A \to B$
4. *Proof by contradiction.* $\Gamma, \neg A \vdash \bot$ iff $\Gamma \vdash A$
5. *Post's Theorem.* (Also called "tautology theorem", or even "completeness of Propositional Calculus theorem")

   If $\models_{\text{taut}} A$, then $\vdash A$.

   Also: If $\Gamma \models_{\text{taut}} A$ for finite Γ, then also $\Gamma \vdash A$.
6. *Proof by cases.* $A \to B, C \to D \vdash A \vee C \to B \vee D$

   Also the special case: $A \to B, C \to B \vdash A \vee C \to B$

The Existential Quantifier $\exists$

$$(\exists \mathbf{x})A \textit{ stands for } \neg(\forall \mathbf{x})\neg A$$

therefore $(\exists \mathbf{x})A \equiv \neg(\forall \mathbf{x})\neg A$ is a tautology, hence an absolute theorem.

Useful facts from Predicate Calculus (proved in class—you may use them without proof):

We ***know*** that WL (*not stated here; you should know this rule well!*) is a **derived rule** useful in *Equational proofs within predicate calculus*.

- More "rules" and (meta)theorems.

(i) *"Renaming the Bound Variable".*

If $\mathbf{z}$ does not occur in $(\forall \mathbf{x})A$ as either free or bound, then $\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})\big(A[\mathbf{x} := \mathbf{z}]\big)$

If $\mathbf{z}$ does not occur in $(\exists \mathbf{x})A$ as either free or bound, then $\vdash (\exists \mathbf{x})A \equiv (\exists \mathbf{z})\big(A[\mathbf{x} := \mathbf{z}]\big)$

(ii) $\forall$ *over* $\circ$ *distribution, where "$\circ$" is "$\vee$" or "$\rightarrow$".*

$$\vdash A \circ (\forall \mathbf{x})B \equiv (\forall \mathbf{x})(A \circ B), \textbf{ provided } \mathbf{x} \text{ is not free in } A$$

$\exists$ *over* $\wedge$ *distribution*

$$\vdash A \wedge (\exists \mathbf{x})B \equiv (\exists \mathbf{x})(A \wedge B), \textbf{ provided } \mathbf{x} \text{ is not free in } A$$

(iii) $\forall$ *over* $\wedge$ *distribution.*

$$\vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \equiv (\forall \mathbf{x})(A \wedge B)$$

$\exists$ *over* $\vee$ *distribution.*

$$\vdash (\exists \mathbf{x})A \vee (\exists \mathbf{x})B \equiv (\exists \mathbf{x})(A \vee B)$$

(iv) $\forall$ *commutativity (symmetry).*

$$\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \equiv (\forall \mathbf{y})(\forall \mathbf{x})A$$

(v) *Specialisation. "Spec"* $(\forall \mathbf{x})A \vdash A[\mathbf{x} := t]$, for any term t.

(vi) *Dual of Specialisation. "Dual Spec"* $A[\mathbf{x} := t] \vdash (\exists \mathbf{x})A$, for any term t.

(vii) *Generalisation. "Gen"* If $\Gamma \vdash A$ and if, moreover, the formulae in Γ have **no free x occurrences**, then also $\Gamma \vdash (\forall \mathbf{x})A$.

(viii) $\forall$ *Monotonicity.* If $\Gamma \vdash A \rightarrow B$ so that the formulae in Γ have **no free x occurrences**, then we can infer

$$\Gamma \vdash (\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$$

(ix) $\forall$ *Introduction; a special case of* $\forall$ *Monotonicity.* If $\Gamma \vdash A \rightarrow B$ so that neither the formulae in Γ nor A have **any free x occurrences**, then we can infer

$$\Gamma \vdash A \rightarrow (\forall \mathbf{x})B$$

(x) Finally, the *Auxiliary Hypothesis Metatheorem.* If $\Gamma \vdash (\exists \mathbf{x})A$, and if $\mathbf{y}$ is a variable that ***does not*** occur as either free or bound variable in any of A or B or the formulae of Γ —that is, it is fresh— then

$$\Gamma, A[\mathbf{x} := \mathbf{y}] \vdash B \text{ implies } \Gamma \vdash B$$

Semantics facts

Propositional Calculus	Predicate Calculus
(**Boolean Soundness**) $\vdash A$ implies $\models_{\text{taut}} A$ (Post) $\models_{\text{taut}} A$ implies $\vdash A$	$\vdash A$ does **NOT** imply $\models_{\text{taut}} A$ However, $\models_{\text{taut}} A$ implies $\vdash A$, **and** (**Pred. Calc. Soundness**) $\vdash A$ implies $\models A$

CAUTION! The above facts/tools are only *a fraction* of what we have covered in class. They are ***very important and very useful***, and that is why they are listed for your reference here.

You can also use *without proof* ***ALL*** the things we have covered (such as the absolute theorems known as "$\exists$-definition", "de Morgan's laws", etc.).

But these —the unlisted ones— are up to you to remember and to correctly state!

Whenever in doubt of whether or not a "tool" you are about to use was indeed covered in class, ***prove*** *the validity/fitness of the tool before using it!*

Boolean Logic 1. (3 MARKS) Suppose $\vdash A$ and $\vdash B$. Does it follow that $\vdash A \equiv B$?

If yes, give a proof.

If not, use soundness to justify your "NO".

Post's theorem is NOT allowed.

Boolean Logic 2. (3 MARKS) Suppose $\vdash A \equiv B$. Does it follow that $\vdash A$ and $\vdash B$?

If yes, give a proof.

If not, use soundness to justify your "NO".

Post's theorem is NOT allowed.

Boolean Logic 3. (5 Marks) Prove **by Resolution**:

$$\vdash \Big(X \to (Y \to Z)\Big) \to \Big((X \to Y) \to (X \to Z)\Big)$$

Caution: **0 Marks** gained **if any other technique is used. In particular, Post's theorem is NOT allowed.**

A proof by resolution

1) MUST use proof by contradiction, and

2) *It cannot/must not* be "preloaded" with *a long Equational or Hilbert proof* only to conclude with *ONE CUT*.

Such a proof, *if correct*, loses half the points.

Predicate Logic 1. (3 MARKS) **True** or **False** and **WHY —No correct "WHY" = 0 MARKS**:

For any formula A, we have $\vdash (\forall \mathbf{x})(\forall \mathbf{z})(A \equiv A)$.

Predicate Logic 2. (5 MARKS) Use an **Equational** proof to establish the $\exists$-version of the one-point-rule:

If $\mathbf{x}$ is not free in t then

$$\vdash (\exists \mathbf{x})(\mathbf{x} = t \wedge A) \equiv A[\mathbf{x} := t]$$

Hint. Use the $\forall$-version of the one-point rule as a given.

Predicate Logic 3. (5 MARKS) Using soundness of predicate logic show why the converse of Ax4 (p.3)

$$(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\forall \mathbf{x})B \qquad (Ax4)$$

is not a theorem.

Predicate Logic 4. (5 MARKS) Prove $\vdash (\forall \mathbf{x})A \vee (\forall \mathbf{x})B \to (\forall \mathbf{x})(A \vee B)$.

Predicate Logic 5. (5 MARKS) You must use the technique of the "auxiliary hypothesis metatheorem" in the proof that you are asked to write here. Any other proof (***even IF correct***) will *MAX* at 0 MARKS.

For any formulas A, B, and C show that

$$\vdash (\exists \mathbf{x})\big(A \vee B \to C\big) \to (\exists \mathbf{x})(A \to C) \wedge (\exists \mathbf{x})(B \to C)$$