York University
Department of Electrical Engineering and Computer Science
Lassonde School of Engineering

MATH 1090A. MID TERM, October 22, 2025; SOLUTIONS
Professor George Tourlakis

Question 1. (4 MARKS) Prove that every nonempty proper **suffix** of a wff has an excess of **right** brackets ")".

Suggestion. Do induction on formulas A, so your I.H. will assume the theorem true for the i.p. of A. Imitate the proof we gave for the "nonempty proper **prefix**" result.

Proof. Induction on formulas —based on the Inductive Definition of A:

- **Case** where A is **atomic**. Then it has NO nonempty proper suffix. So statement CANNOT be FALSE!!
- **Case** where A is $(\neg B)$. We have the following subcases of *ALL possible nonempty proper suffixes*.
  (a) (Suffixes are in red type): $)$. Trivial.
  (b) $B')$, where B' is a **nonempty proper suffix** of B. By I.H. B' has an excess of ")". Trivially, so does $B')$.
  (c) $B\Big)$. B is a wff so it has balanced *bracket numbers* (left/right). So the bracket we have typeset "big" (the rightmost) is in excess.
  (d) $\neg B\Big)$. Just as in the previous case, since "$\neg$" cannot balance the rightmost "$\Big)$"

  Case where A is $(C \diamond B)$ and $\diamond$ is any one among $\wedge, \vee, \rightarrow, \equiv$.
  We have the following subcases of *ALL possible nonempty proper suffixes*.
  (a) $)$. Trivial.
  (b) $B')$, where B' is a **nonempty proper suffix** of B. By I.H. B' has an excess of ")". Trivially, so does $B')$.
  (c) $B\Big)$. B is a wff so it has balanced *bracket numbers* (left/right). So the bracket we have typeset "big" (the rightmost) is in excess.
  (d) $\diamond B\Big)$. Just as in the previous case, since "$\diamond$" cannot balance the rightmost "$\Big)$"
  (e) $C' \diamond B\Big)$ where C' is a proper nonempty suffix of C. By I.H. C' has an excess of ")". Trivially, so does $C' \diamond B\Big)$ since B is balanced and $\Big)$ is an extra right bracket.

  Lastly

  (f) $C \diamond B\Big)$. Here C and B are balanced (wffs) and $\Big)$ breaks the balance in favour of the rights.

$\square$

Question 2. (3 MARKS) Prove that every wff has a number of brackets that is **twice** its number of pieces of "glue" —repetitions counted!

Hint. Use either "formula constructions" or do induction of the formula structure (shape).

Proof. By *formula construction*, at every step where we add GLUE we DO ALSO add exactly **TWO** brackets: ONE left, ONE right.

Thus, by the time we are done building a wff A, we have added TWICE as many brackets as we added GLUE.

□

Question 3. (4 MARKS) Insert missing brackets **correctly** and prove **by truth tables** that the following is a tautology.
$A \vee B \equiv \neg A \to B$.

Proof. The formula says

$$(A \vee B) \equiv (\neg A \to B)$$

Cases according to the (truth) value of A:

(1) A is $\mathtt{t}$. Then $\neg A$ is $\mathtt{f}$.
Thus both LHS and RHS of $\equiv$ are $\mathtt{t}$. Good!

(2) A is $\mathtt{f}$. Then $\neg A$ is $\mathtt{t}$.
SUBCases according to B:

i. B is $\mathtt{t}$. Thus both LHS and RHS of $\equiv$ are $\mathtt{t}$. Good!
ii. B is $\mathtt{f}$. Thus both LHS and RHS of $\equiv$ are $\mathtt{f}$. Good!

$\square$

Question 4. (3 MARKS) Consider the formula

$$\top \equiv \bot \equiv \bot \tag{1}$$

Show via the **Truth Table Technique** that (1) and (2) do **not** have the same truth value.

$$(\top \equiv \bot) \wedge (\bot \equiv \bot) \tag{2}$$

Proof. By the priorities of operations, (1) says

$$\underbrace{\overbrace{\top}^{\mathbf{t}} \equiv (\overbrace{\bot \equiv \bot}^{\mathbf{t}})}_{\mathbf{t}}$$

The above computes as true ($\mathbf{t}$).

By the presence of brackets, (2) says

$$\underbrace{\overbrace{(\top \equiv \bot)}^{\mathbf{f}} \wedge (\overbrace{\bot \equiv \bot}^{\mathbf{t}})}_{\mathbf{f}}$$

(2) computes as false ($\mathbf{f}$): (1) and (2) do NOT say the same thing. □

Question 5. (5 MARKS) For ANY A, B, C, D, give a **Hilbert-Style proof** of the following:

$$A \equiv B, B \equiv C, C \equiv D \vdash A \equiv D$$

Caution. Proofs of the wrong FORMAT and without annotation on **each** line are not acceptable. Proving is a "programming"-like activity

No other style of proof —such as "proof by truth tables"— is accepted (0 marks).

Proof. We know from class/NOTEs that **we have the (derived) Rule Transitivity**:

$$A \equiv B, B \equiv C \vdash A \equiv C \qquad (*)$$

We now have a proof that uses $(*)$:

1) $A \equiv B$ ⟨hyp⟩
2) $B \equiv C$ ⟨hyp⟩
3) $A \equiv C$ ⟨1 + 2 + $(*)$⟩
4) $C \equiv D$ ⟨hyp⟩
5) $A \equiv D$ ⟨3 + 4 + $(*)$⟩

□

Question 6. (5 MARKS) For any A give a **Hilbert-Style proof** of the following:

$$\vdash A \vee \top$$

Hint. Prove **first** $\vdash A \vee \bot \equiv A \vee \bot$ (so easy!) and then derive your proof of $\vdash A \vee \top$ from this result.

Caution. Proofs of the wrong FORMAT and without annotation on **each** line are not acceptable. Proving is a “programming”-like activity and needs the same precision.

No other style of proof —such as “proof by truth tables”— is accepted (0 marks).

Proof.

1)	$A \vee \bot \equiv A \vee \bot$	$\langle$thm from class/NOTEs$\rangle$
2)	$A \vee (\bot \equiv \bot) \equiv (A \vee \bot \equiv A \vee \bot)$	$\langle$axiom$\rangle$
3)	$A \vee (\bot \equiv \bot)$	$\langle$1 + 2 + Eqn$\rangle$
4)	$A \vee (\bot \equiv \bot) \equiv A \vee \top$	$\langle$Axiom “$(\bot \equiv \bot) \equiv \top$” + Leib; Denom: $A \vee \mathbf{p}$, $\mathbf{p}$ fresh$\rangle$
5)	$A \vee \top$	$\langle$3 + 4 + Eqn$\rangle$

$\square$

Extra blank "overflow" page for answers