Lassonde Faculty of Engineering
EECS
MATH1090A. Problem Set No. 3 —Solutions
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A brief but full justification of each proof step is required!
Do all the following problems; (5 Points Each).

Important Notes; Read First!

"Show that —or prove that— $\Gamma \vdash A$" means "write a Γ-proof that establishes A". The proof can be Equational or Hilbert-style. Equational is rather easier in Boolean Logic. But it is your choice, unless a problem explicitly asks for a particular proof style.

"**Required Method**" means that any other method will get a 0-grade.

It is all right to use the Cut Rule in each problem, unless specified otherwise!

Post's Theorem is NOT allowed in Problems 1–4.

1. Show that $A \equiv B \equiv C \vdash A \to B \to C$

Required Method: Use a **Hilbert style** proof and the Deduction Theorem.

Proof. By DThm (twice in a row) it suffices to prove

$$A \equiv B \equiv C, A, B \vdash C$$

instead.

1) $A \equiv B \equiv C$ $\langle$hyp$\rangle$
2) A $\langle$hyp$\rangle$
3) B $\langle$hyp$\rangle$
4) $B \equiv C$ $\langle 1 + 2 + \text{Eqn}\rangle$
5) C $\langle 3 + 4 + \text{Eqn}\rangle$

□

2. Prove **Equationally** that $\vdash (A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$.

Proof.

$$
\begin{array}{l}
\quad (A \wedge B) \wedge C \\
\Longleftrightarrow \langle \text{GR} \rangle \\
\quad (A \wedge B) \vee C \equiv C \equiv (A \wedge B) \\
\Longleftrightarrow \langle \text{GR+Leib; Denom: } \mathbf{p} \vee C \equiv C \equiv \mathbf{p} \ \rangle \\
\quad (A \vee B \equiv A \equiv B) \vee C \equiv C \equiv (A \vee B \equiv A \equiv B) \\
\Longleftrightarrow \langle \vee\!\equiv + \text{ Leib; Denom: } \mathbf{p} \equiv C \equiv A \vee B \equiv A \equiv B \ \rangle \\
\quad A \vee B \vee C \equiv A \vee C \equiv B \vee C \equiv C \equiv A \vee B \equiv A \equiv B
\end{array}
$$

Also

$$
\begin{array}{l}
\quad A \wedge (B \wedge C) \\
\Longleftrightarrow \langle \text{GR} \rangle \\
\quad A \vee (B \wedge C) \equiv A \equiv (B \wedge C) \\
\Longleftrightarrow \langle \text{GR+Leib; Denom: } A \vee \mathbf{p} \equiv A \equiv \mathbf{p} \ \rangle \\
\quad A \vee (B \vee C \equiv B \equiv C) \equiv A \equiv (B \vee C \equiv B \equiv C) \\
\Longleftrightarrow \langle \vee\!\equiv + \text{ Leib; Denom: } \mathbf{p} \equiv A \equiv B \vee C \equiv B \equiv C \ \rangle \\
\quad A \vee B \vee C \equiv A \vee B \equiv A \vee C \equiv A \equiv B \vee C \equiv B \equiv C
\end{array}
$$

The two red $\equiv$-chains are provably equivalent (being permutations of each other) and thus so are the beginnings (tops) of the two equational segments. □

3. For any formulas A, B and C, prove that

$$\vdash (A \to B) \to (B \to C) \to (A \to C)$$

Required Method: Use a **Hilbert style** proof **and the Deduction Theorem**.

Proof. By DThm applied three consecutive times it suffices to prove

$$A \to B, B \to C, A \vdash C$$

instead.

1)	$A \to B$	$\langle$hyp$\rangle$
2)	$B \to C$	$\langle$hyp$\rangle$
3)	A	$\langle$hyp$\rangle$
4)	B	$\langle 1 + 3 + \text{MP}\rangle$
5)	C	$\langle 2 + 4 + \text{MP}\rangle$

□

4. For any formulas A, B and C, prove **again** that

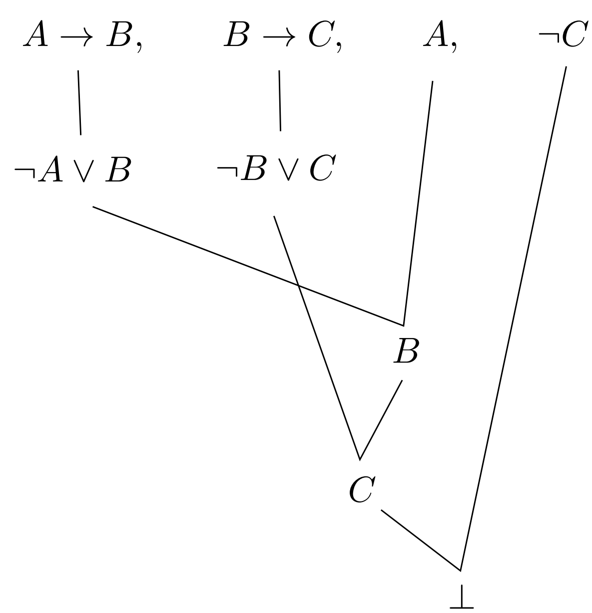
$$\vdash (A \to B) \to (B \to C) \to (A \to C)$$

Required Method: Use a a **Resolution** proof.

Proof. By DThm applied three consecutive times followed by proof by contradiction it suffices to prove

$$A \to B, B \to C, A, \neg C \vdash \bot$$

instead. We do Resolution BELOW:



5. For any formulas A and B prove **two things**:

$$\vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{y})B \to (\forall \mathbf{x})A$$

Proof: The above is a tautology of the form $X \wedge Y \to X$. All tautologies are 1st-order theorems. $\square$

and

$$\vdash (\forall \mathbf{x})(\forall \mathbf{y})(\forall \mathbf{z})\Big(A \to A \vee B\Big)$$

Proof: The above is an axiom since $A \to A \vee B$ is a tautology. $\square$

6. Let A, B be any formulas, and $\mathbf{x}$ a variable that is not free in B.

Prove in **Hilbert-style** that

$$(\forall \mathbf{x})(A \to B),\, \neg B \vdash (\forall \mathbf{x})\neg A$$

Proof.

1)	$(\forall \mathbf{x})(A \to B)$	$\langle$hyp$\rangle$
2)	$\neg B$	$\langle$hyp$\rangle$
3)	$A \to B$	$\langle$1 + Spec$\rangle$
4)	$\neg B \to \neg A$	$\langle$3 + Post$\rangle$
5)	$\neg A$	$\langle$4 + 2 + MP$\rangle$
6)	$(\forall \mathbf{x})\neg A$	$\langle$5 + Gen; OK, as no free $\mathbf{x}$ in lines 1, 2$\rangle$

$\square$

7. Use an **Equational** Proof, and the **result from class/NOTEs** that

$$\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \equiv (\forall \mathbf{y})(\forall \mathbf{x})A \tag{1}$$

and *this time show*

$$\vdash (\exists \mathbf{x})(\exists \mathbf{y})A \equiv (\exists \mathbf{y})(\exists \mathbf{x})A \tag{2}$$

Proof.

$(\exists \mathbf{x})(\exists \mathbf{y})A$

$\Longleftrightarrow\langle$remove abbreviation "$\exists$"; *this does NOT need Leib to do so!!*$\rangle$

$\underbrace{\neg(\forall \mathbf{x})\neg}_{1st\ \exists \mathbf{x}}\underbrace{\neg(\forall \mathbf{y})\neg}_{2nd\ \exists \mathbf{y}} A$

$\Longleftrightarrow\langle$double neg and WL; Denom: $\neg(\forall \mathbf{x})\mathbf{p}$; WL OK: its hyp is "$\neg\neg$"–absolute thm$\rangle$

$\neg(\forall \mathbf{x})(\forall \mathbf{y})\neg A$

$\Longleftrightarrow\langle$(1) and WL; Denom: $\neg\mathbf{p}$; WL OK: its hyp (1) is an absolute thm$\rangle$

$\neg(\forall \mathbf{y})(\forall \mathbf{x})\neg A$

$\Longleftrightarrow\langle$double-neg inserted via WL; Denom: $\neg(\forall \mathbf{y})\mathbf{p}$; WL OK; see 2nd $\Leftrightarrow$ reason$\rangle$

$\underbrace{\neg(\forall \mathbf{y})\neg}_{\exists \mathbf{y}}\underbrace{\neg(\forall \mathbf{x})\neg}_{\exists \mathbf{x}} A$

$\Longleftrightarrow\langle$using "$\exists$"-abbreviation —twice$\rangle$

$(\exists \mathbf{y})(\exists \mathbf{x})A$

□

8. Use a **Hilbert-style proof** to show

$$\vdash (\forall \mathbf{x})(A \to C) \wedge (\forall \mathbf{x})(B \to C) \to (\forall \mathbf{x})(A \vee B \to C)$$

Proof. By DThm, I prove instead

$$(\forall \mathbf{x})(A \to C) \wedge (\forall \mathbf{x})(B \to C) \vdash (\forall \mathbf{x})(A \vee B \to C)$$

1)	$(\forall \mathbf{x})(A \to C) \wedge (\forall \mathbf{x})(B \to C)$	$\langle$hyp$\rangle$
2)	$(\forall \mathbf{x})(A \to C)$	$\langle$1 + Post$\rangle$
3)	$(\forall \mathbf{x})(B \to C)$	$\langle$1 + Post$\rangle$
4)	$A \to C$	$\langle$2 + Spec$\rangle$
5)	$B \to C$	$\langle$3 + Spec$\rangle$
6)	$A \vee B \to C$	$\langle$4 +5 + Post$\rangle$
7)	$(\forall \mathbf{x})(A \vee B \to C)$	$\langle$6 + Gen; OK, no free $\mathbf{x}$ in line 1.$\rangle$

□