# Lassonde School of Engineering

## Dept. of EECS

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## MATH1090 A. Problem Set No2 —Solutions

**Posted: Due: Oct. 31, 2025**

In this problem set and onwards, $\mathbf{p}, \mathbf{q}, \mathbf{r}'$ etc., are *metavariables* that stand for *actual* Boolean variables. As such, it is possible that, say, $\mathbf{p}$ and $\mathbf{q}$ stand for the same actual variable in some line of reasoning.

**1.** (5 MARKS)

Prove that

$$\vdash A \equiv A$$

in **ONE DIFFERENT way**, which does **not** use the "trick" of a Leibniz variable $\mathbf{p}$ that does not appear in $A$.

**Proof**.

1) $A \equiv A \vee A$ $\langle$axiom$\rangle$
2) $A \vee A \equiv A$ $\langle$1 + Commutativity of $\equiv$ (derived) Rule from clss/NOTEs$\rangle$
3) $A \equiv A$ $\langle$1 + 2 + Trans. of "$\equiv$"$\rangle$

Or we can say —for (2)— "axiom" since we can scramble axioms. BUT this is *not* so appropriate (1 point lost) since at the time we originally proved "$A \equiv A$" we did not know about scrambling a $\equiv$-chain.

$\square$

**2.** (5 MARKS) *True* or *False* **and WHY**?

The following two statements —(1) and (2)— are equivalent

$$\Gamma \vdash A \text{ iff } \Gamma \vdash B \tag{1}$$

$$\Gamma \vdash A \equiv B \tag{2}$$

**Answer**. *False*. (2) does *not* imply (1).

Indeed, if (2) was to so imply, this would work with $\Gamma = \emptyset$.

So, let $\vdash A \equiv B$ and let us take the formula $p$ as being both $A$ and $B$.

Now $\vdash p \equiv p$. *But* $\nvdash p$ —this by (Boolean) soundness that *requires* $\vdash p$ imply $\models_{taut} p$. □

**3.** (5 MARKS) We have learnt that $\Gamma \vdash A \wedge B$ implies that $\Gamma \vdash A$ **AND** $\Gamma \vdash B$.

Is (1) below *True* or *False* **and WHY**?

$$\Gamma \vdash A \vee B \text{ implies that } \Gamma \vdash A \textbf{ OR } \Gamma \vdash B \tag{1}$$

**Answer**. FALSE! The why: Take $A$ to be the variable $p$ and $B$ to be $\neg p$.

So, $\vdash p \vee \neg p$ (by axiom $A \vee \neg A$).

However, neither $\models_{taut} p$ nor $\models_{taut} \neg p$. Hence, by *soundness*, both of $\vdash p$ and $\vdash \neg p$ are impossible. (1) is *FALSE*! □

**Caution**. If a proof style is explicitly **required** in what follows, then any other style used gets 0 marks regardless of its correctness.

**4.** (5 MARKS) Prove **Hilbert-Style** that

$$A, \neg A \vdash \bot$$

**directly**, **without** using the derived rule $CUT$ in **any** of its special forms.

**Proof**.

| | | |
|---|---|---|
| 1) | $A$ | $\langle$hyp$\rangle$ |
| 2) | $\neg A$ | $\langle$hyp$\rangle$ |
| 3) | $\neg A \equiv A \equiv \bot$ | $\langle$axiom$\rangle$ |
| 4) | $A \equiv \bot$ | $\langle 2 + 3 + \text{Eqn}\rangle$ |
| 5) | $\bot$ | $\langle 1 + 4 + \text{Eqn}\rangle$ |

□

**5.** (5 MARKS) Give a **Hilbert-Style proof** of $\vdash A \wedge \top \equiv A$.

**Proof**.

| | | |
|---|---|---|
| 1) | $A \vee \top \equiv A \wedge \top \equiv A \equiv \top$ | $\langle$axiom$\rangle$ |
| 2) | $A \vee \top$ | $\langle$thm (from NOTES/Class)$\rangle$ |
| 3) | $A \wedge \top \equiv A \equiv \top$ | $\langle 1 + 2 + \text{Eqn}\rangle$ |
| 4) | $A \wedge \top \equiv A$ | $\langle 3 + \text{Red. } \top$ META$\rangle$ |

□

**6.** (4 MARKS) Prove **Equationally** that $A \vdash B \to A$.

**Proof**. From hypothesis $\Gamma = \{A\}$ we have (by Red. $\top$ META)

$$A \vdash A \equiv \top \qquad (\dagger)$$

Thus,

$$\begin{array}{l} \quad B \to A \\ \Leftrightarrow \langle \text{thm} \rangle \\ \quad \neg B \vee A \\ \Leftrightarrow \langle (\dagger) + \text{Leib.; Denom: } \neg B \vee \mathbf{p} \rangle \\ \quad \neg B \vee \top \qquad\qquad\qquad \text{bingo!} \end{array}$$

□

**7.** (4 MARKS) Prove **Equationally** that $A \vee B \vdash \neg B \to A$.

**Proof**.

$$
\begin{aligned}
& \neg B \to A \\
\Leftrightarrow & \langle \text{thm} \rangle \\
& \neg\neg B \vee A \\
\Leftrightarrow & \langle \text{thm + Leib. Denom: } \mathbf{p} \vee A \rangle \\
& B \vee A \qquad \text{bingo1}
\end{aligned}
$$

□

**8.** Prove that $A \to B, A \to C \vdash A \to B \wedge C$.

Do **two** proofs:

- (5 MARKS) One **with** the Deduction theorem (and a Hilbert-style proof).

  **Proof**.

  > Recall derived rule MP from class:
  >
  > $A, A \to B \vdash B$.

  By one application of DThm, suffices to prove instead:

  $$A \to B, A \to C, A \vdash B \wedge C$$

  Here it goes:

  1) $A \to B$ $\langle \text{hyp} \rangle$
  2) $A \to C$ $\langle \text{hyp} \rangle$
  3) $A$ $\langle \text{hyp} \rangle$
  4) $B$ $\langle 1 + 3 + \text{MP} \rangle$
  5) $C$ $\langle 3 + 2 + \text{MP} \rangle$
  6) $B \wedge C$ $\langle 4 + 5 + \text{class/NOTEs: } B, C \vdash B \wedge C \rangle$

  □

- (5 MARKS) One Equational, **WITHOUT** using the Deduction theorem.

**Proof**. Here is an Equational proof: Note that on hypothesis $\Gamma = \{A \to B, A \to C\}$ we have (Red. $\top$ *META*)

$$\vdash A \to B \equiv \top \tag{$*$}$$

and

$$\vdash A \to C \equiv \top \tag{$**$}$$

Then, (next page)

$$
\begin{aligned}
& A \to B \wedge C \\
\Leftrightarrow & \langle \text{from class/NOTES} \rangle \\
& \neg A \vee B \wedge C \\
\Leftrightarrow & \langle \text{from NOTES (distr. } \vee \text{ over } \wedge \rangle \\
& (\neg A \vee B) \wedge (\neg A \vee C) \\
\Leftrightarrow & \langle \text{Leib} + \neg\vee \text{ vs} \to \text{thm; Denom } \mathbf{p} \wedge (\neg A \vee C) \rangle \\
& (A \to B) \wedge (\neg A \vee C) \\
\Leftrightarrow & \langle \text{Leib} + \neg\vee \text{ vs} \to \text{thm; Denom } (A \to B) \wedge \mathbf{p} \rangle \\
& (A \to B) \wedge (A \to C) \\
\Leftrightarrow & \langle \text{Leib} + (*)\text{; Denom } \mathbf{p} \wedge (A \to C) \rangle \\
& \top \wedge (A \to C) \\
\Leftrightarrow & \langle \text{Leib} + (**)\text{; Denom } \top \wedge \mathbf{p} \rangle \\
& \top \wedge \top \\
\Leftrightarrow & \langle \text{Idempotent for } \wedge \rangle \\
& \top \qquad\qquad \text{bingo!}
\end{aligned}
$$