

Department of EECS

MATH1090 A. Problem Set No1 —Solutions

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1. (2 MARKS) Prove that $(\perp)$ is *NOT* a wff.

Hint. One way to *prove* this (in the metatheory) is to analyse formula-constructions/calculations. The other is to look at the inductive definition of formulas: can it be applied to define “ $(\perp)$ ” as a formula? Why?

Proof. A wff is ONE of:

- (a) *Atomic*. That is $\perp, \top, \mathbf{p}$. The given “ $(\perp)$ ” is **NONE** of these three cases (NOTE BRACKETS; WRONG!).
- (b) *A negation* $(\neg B)$. CANNOT be! NOTE that this case requires glue. The given “ $(\perp)$ ” has NONE.
- (c) *A conjunction, disjunction, implication or equivalence, all denoted by “ $\circ$ ”* $(B \circ C)$. CANNOT be! NOTE that this case requires glue. The given “ $(\perp)$ ” has NONE.

□

2. (3 MARKS) Let Q, P, R be wff's. Prove that so is

$$\left((P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R)) \right)$$

In so doing you MUST use “Hilbert style layout” in an appropriate *formula calculation*, that is, AT EACH STEP, *ONE ONLY string* —a **wff**— *is written* at the end of a growing VERTICAL list of wff, with a number at its left (its position/row number) and required annotation to the right. **Just like the example from class below.**

- | | | |
|-----|-----------------------|--|
| (1) | p | $\langle \text{atomic} \rangle$ |
| (2) | q | $\langle \text{atomic} \rangle$ |
| (3) | $(p \vee q)$ | $\langle 1 + 2 + \vee\text{-glue} \rangle$ |
| (4) | r | $\langle \text{atomic} \rangle$ |
| (5) | $((p \vee q) \vee r)$ | $\langle 4 + 3 + \vee\text{-glue} \rangle$ |

BE MINDFUL that the CAPITAL P, Q, R are *NOT necessarily* variables or constants! They are “general” iff we obtained earlier and we do not care of their details!

To depict that a formula P was constructed *earlier* you may write

$\vdots$
(k) P $\langle$ obtained earlier $\rangle$

Your construction takes place BELOW line (k) (and also below the construction of Q, R which you present similarly to P in the overall construction).

Answer.

$\vdots$	
(1) P	$\langle$ obtained earlier $\rangle$
$\vdots$	
(2) Q	$\langle$ obtained earlier $\rangle$
$\vdots$	
(3) R	$\langle$ obtained earlier $\rangle$
(4) $(Q \wedge R)$	$\langle 2+3+\wedge \rangle$
(5) $(P \vee Q)$	$\langle 1+2+\vee \rangle$
(6) $(P \vee R)$	$\langle 1+3+\vee \rangle$
(7) $(P \vee (Q \wedge R))$	$\langle 1+4+\vee \rangle$
(8) $((P \vee Q) \wedge (P \vee R))$	$\langle 5+6+\wedge \rangle$
(9) $((P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R)))$	$\langle 7+8+\equiv \rangle$

□

3. (6 MARKS) Recall that a **schema** is a tautology iff *all* its *instances* are tautologies.

Which of the following six *schemata* are tautologies? Show the *whole process* that led to your answers, *including truth tables or equivalent short cuts, if you used one or the other, and words of explanation if needed.*



*I note that in the six sub-questions below I am **NOT** using all the formally necessary brackets. You need to *reinsert* missing brackets to answer *correctly*.*



- $A \rightarrow B \equiv \neg A \vee B$

Answer. A tautology! The above says $(A \rightarrow B) \equiv (\neg A \vee B)$ (I did NOT need to put ALL brackets back).

I use truth table.

A	B	(A	$\rightarrow$	B)	$\equiv$	($\neg$	A	$\vee$	B)
f	f	(	t	)	t	(t		t	)
f	t	(	t	)	t	(t		t	)
t	f	(	f	)	t	(f		f	)
t	t	(	t	)	t	(f		t	)

□

- $((A \rightarrow B) \rightarrow C) \equiv (A \rightarrow (B \rightarrow C))$

Answer. Not a tautology! Here is a **counterexample** to tautology status:

Take the special case where all A, B, C are the wff $\perp$. Then the rhs of “ $\equiv$ ” evaluates as **t** since the A is **f**.

BUT the lhs evaluates as

$$\overbrace{(\mathbf{f} \rightarrow \mathbf{f})}^{\mathbf{t}} \rightarrow \mathbf{f}$$

□

- $A \wedge B \equiv \neg(\neg A \vee \neg B)$

Answer. A tautology! Indeed I apply a shortcut:

I note

- (1) From truth tables I NOTE that the lhs is true **IFF** BOTH A and B are true (**t**).
- (2) Here is why the rhs is true EXACTLY under the same conditions:

Indeed

rhs is true IFF $\neg A \vee \neg B$ is false IFF BOTH $\neg A$ AND $\neg B$ are FALSE

BUT “BOTH $\neg A$ AND $\neg B$ are FALSE” says that **BOTH A and B are TRUE**. Just as in CASE (1)! □

- $A \rightarrow B \rightarrow C \equiv B \rightarrow A \rightarrow C$

Answer. Of course truth tables always work for pain of boredom.

So let me use short cuts instead. Naturally

- The Lhs says (partial recovery of brackets!) $A \rightarrow (B \rightarrow C)$.
- Correspondingly, the Rhs says (partial recovery of brackets!) $B \rightarrow (A \rightarrow C)$.

The shortcut will show that the two have the same truth values for any values of A, B, C .

Here it goes:

Case 1. A is **false**. Then Lhs of “ $\equiv$ ” is TRUE (check the basic [truth table](#) from class/NOTES) REGARDLESS of the value of $(B \rightarrow C)$.

What about Rhs of “ $\equiv$ ”? Well, $(A \rightarrow C)$ is TRUE, so by truth table we DON’T care about the value of B (“anything $\rightarrow$ t” is TRUE).

Case 2. B is **false**. This is case 1 with the roles of A and B swapped.

Case 3. A AND B are TRUE. Then what are the truth values of

i. Lhs: $TRUE \rightarrow (TRUE \rightarrow C)$?

and

ii. Rhs: $TRUE \rightarrow (TRUE \rightarrow C)$?

iii. Evidently these (Lhs and Rhs) have the same values for the same value of C !!

We have equivalence again!

□

- $A \rightarrow B \equiv \neg B \rightarrow \neg A$

Answer. By shortcuts.

(a) **Case:** A is FALSE (**f**). Then Lhs is **t** and so is Rhs because $\neg A$ is.

(b) **Case:** A is TRUE (**t**). We have subcases:

- B is **f**. Then the left (of $\equiv$) arrow is **t** $\rightarrow$ **f** and so is the right. OK!
- B is **t**. Then the left (of $\equiv$) arrow is **DC**[†] $\rightarrow$ **t** and the right is **f** $\rightarrow$ **DC**. OK!

[†] “Don’t Care”.

□

- $\top \equiv \perp \equiv \perp$

Answer. A tautology! With brackets restored (not the outermost though; not needed) the formula says

$$\top \equiv (\perp \equiv \perp)$$

The truth value of the above is easy to compute: Just replace $\top$ by **t** and $\perp$ by **f** and compute!

$$\mathbf{t} \equiv (\overbrace{\mathbf{f} \equiv \mathbf{f}}^{\mathbf{t}})$$

□

4. (2 MARKS) Prove for all wff A, B that we have $A \equiv \neg A \models_{\text{taut}} B$.

Answer. No matter when A is false or true, $\neg A$ is the opposite so the “ $\equiv$ ” is false. The tautological implication goes through vacuously as we cannot make left hand side true. □

5. (6 MARKS) By using truth tables, or using related shortcuts, examine whether or not the following tautological implications are correct.



In order to show that a tautological implication that involves *meta*-variables for formulas (capital latin letters) —i.e., it is a schema— is *incorrect* you *must* consider a *special case* that *is* incorrect (since some other special cases might *work*).



Show the whole process that led to each of your answers.

- $A \vee \neg A \models_{\text{taut}} \top$

Answer. Correct. LHS is **t** regardless of value of A , but so is the RHS. □

- $A \vee \neg A \models_{\text{taut}} A \wedge B$

Answer. INcorrect. For example take A to be p and B to be q .

In the state s where $s(p) = \mathbf{t}$ and $s(q) = \mathbf{f}$, we have LHS of $\models_{\text{taut}}$ TRUE (**t**) but RHS FALSE (**f**).

- $A \wedge \neg A \models_{\text{taut}} A \wedge B$

Answer. Correct. LHS cannot be satisfied (cannot be **t** so there is nothing to prove): VACUOUSLY TRUE. $\square$

- $A, A \rightarrow B \models_{\text{taut}} B$

Answer. Correct. Assume LHS TRUE. This means A TRUE *and* $A \rightarrow B$ TRUE. By truth tables, under these conditions the truth of $A \rightarrow B$ forces the truth of B (sub table for “ $\rightarrow$ ”). $\square$

- $A \equiv B \models_{\text{taut}} \neg B \rightarrow \neg A$

Answer. Correct. By shortcuts: NOTE that LHS is **t** in exactly TWO cases:

- Case A and B are **f**. Then $\neg A$ and $\neg B$ are **t** and hence RHS is **t**.
- Case A and B are **t**. Then $\neg A$ and $\neg B$ are **f** and hence RHS is **t**. $\square$

- $A \wedge B \models_{\text{taut}} B \vee A \equiv A \equiv B$

Answer. Correct. By shortcuts: NOTE that LHS is **t** in exactly ONE case: A and B are BOTH **t**.

But then the RHS is **t** too:

- (a) $B \vee A$ is **t** by what we just said above.
- (b) A and B being both **t** so is $A \equiv B$ and hence the LAST $\equiv$ is **t**.

NOTE. The RHS with ALL brackets in is

$$\left(\underbrace{(B \vee A)}_{\text{t}} \stackrel{\text{last glue}}{\equiv} \underbrace{(A \equiv B)}_{\text{t}} \right)$$

6. (6 MARKS) Write down the most simplified result of the following substitutions, *whenever the requested substitution makes sense*. Whenever a requested substitution does not make sense, explain exactly why it does not.

Show the whole process that led to each of your answers in each case.



Remember the priorities of the various connectives as well as that of the meta-expression “[**p** := ...]”! The following formulas have not been written with all the formally required brackets.



- $(q \rightarrow r)[r := \mathbf{f}]$

Answer. Illegal. To the right of “:=” we can only place a **wff**! $\square$

- $(q \rightarrow p)[p := \top]$

Answer. $(q \rightarrow \top)$. $\square$

- $\top \rightarrow \top[\top := \mathbf{t}]$

Answer. Illegal. To the right of “:=” we can only place a **wff**! $\square$

- $p \rightarrow \top[(p \rightarrow \top) := \perp]$

Answer. Illegal. To the left of “:=” we can only place a **SINGLE Boolean variable**! $\square$

- $(\perp \rightarrow r \rightarrow q)[(q \wedge r) := p]$

Answer. Illegal. To the left of “:=” we can only place a **SINGLE Boolean variable**! $\square$

- $(\perp \rightarrow r \rightarrow q)[r := p][p := r]$

Answer. $[r := p]$ acts FIRST. So we have $(\perp \rightarrow r \rightarrow q)$; the original wff. $\square$

7. (2 MARKS) Prove by induction (on length of formula construction or on formulas) or directly via the recursive definition of wff that no wff is the empty string.

Proof. A wff A can be one of three expressions

- (a) Atomic: $\mathbf{p}, \perp$ or $\top$. In all three cases it is $\neq \lambda$.
- (b) Negation: $(\neg B)$. Clearly $\neq \lambda$. Note “(” in it!
- (c) Binary case: $(B \circ C)$. Clearly $\neq \lambda$. Note “(” in it!

NOTE. We could overkill it and do induction on wff (this proves more than we want): For step 1 rephrase and notice that a wff $\neq \lambda$ BECAUSE it contains an Atomic. For step 2 rephrase and notice that B in it contains an Atomic (by I.H. on i.p.), thus so does A . For step 3 rephrase and notice that B and C in them contain an Atomic (by I.H. on i.p.), thus so does A . $\square$