York University
Department of Electrical Engineering and Computer Science
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MATH 1090 A. FINAL EXAM, December 11, 2024; 19:00-21:00

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Boolean Logic 1. (3 MARKS) Can I *prove* $A \vee B \vdash A$?

If YES, give an Equational Proof (required).

ELSE pick a TOOL, NAME it, and *USE* it to prove that the above has *NO proof*.

Answer. NO. If I can prove the above for any A I can also prove

$$p \vee q \vdash p \tag{1}$$

Now, if (1) is a theorem, then by soundness I should have

$$p \vee q \models_{taut} p \tag{2}$$

Not so, as using the state s with $s(p) = \mathbf{f}$ and $s(q) = \mathbf{t}$, shows. $\square$

Boolean Logic 2. (2 MARKS) Is $(\top)$ a wff? *WHY YES*, or *WHY NOT?*

Prove the correctness of your answer using *formula calculations OR the recursive definition of wff*.

Proof.

A formula A has *ONE* of the FORMs

(a) $\bot, \top, \mathbf{p}$

OR

(b) $(\neg B)$

OR

(c) $(B \circ C)$, where $\circ$ is one of $\wedge, \vee, \rightarrow, \equiv$.

"$(\top)$" is not a wff because, having brackets, **it can only fit cases (b) or (c)**.

BUT it has *NO* "*glue*".

So it does not fit these cases either! *NOT* a wff.

$\square$

Boolean Logic 3. (5 Marks) Prove **by Resolution**:

$$\vdash \Big(X \to (Y \to Z)\Big) \to \Big((X \to Y) \to (X \to Z)\Big)$$

Caution: **0 Marks** gained **if any other technique is used. In particular, Post's theorem is NOT allowed.**

A proof by resolution,

1) MUST use a **graphical** proof by contradiction, and

2) *It cannot/must not* be "preloaded" with *a long Equational or Hilbert proof* only to conclude with *just ONE CUT*.

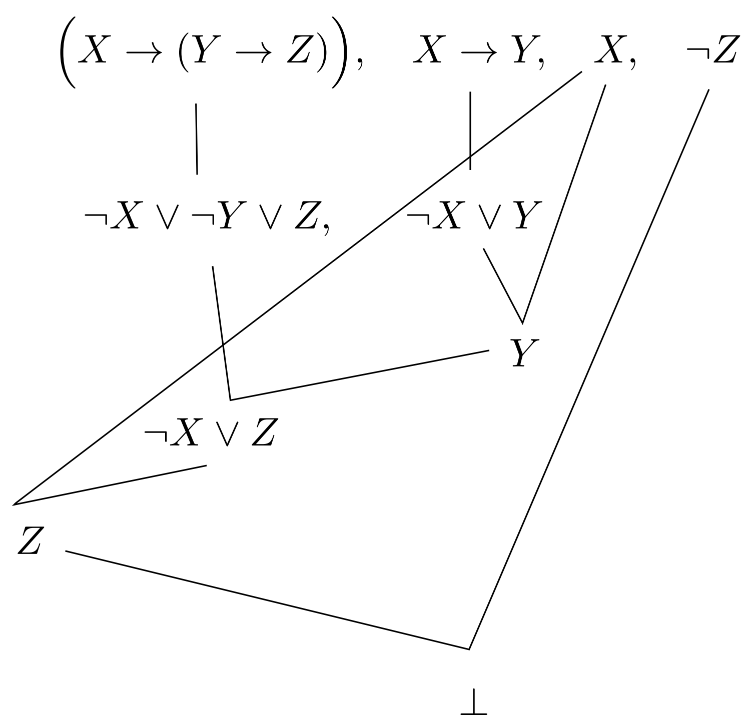
Such a proof, **IF correct**, loses half the points.

Proof. By DThm (3 times) I prove instead

$$\Big(X \to (Y \to Z)\Big), X \to Y, X \vdash Z$$

By **Proof By Contradiction** I will do instead

$$\Big(X \to (Y \to Z)\Big), X \to Y, X, \neg Z \vdash \bot$$



Predicate Logic 1. (3 MARKS) **True** or **False** and **WHY —In the absence of a correct "WHY" the answer gets 0 MARKS**:

For any formulas A, B, we have

$$\vdash (\forall \mathbf{z})(\forall \mathbf{x})(\forall \mathbf{w})(A \to B \lor A)$$

Answer. **True**. It *IS* a theorem, because it *IS* an *AXIOM*: It is a **partial generalisation of the tautology** $A \to B \lor A$. □

Predicate Logic 2. (5 MARKS) Assume that $\mathbf{z}$ is fresh for $(\exists \mathbf{x})A$.

We proved the $\forall$-version of the *bound variable renaming* Metatheorem in class/NOTEs.

Use only the statement of the $\forall$-version Metatheorem (NOT its proof!!) and **prove Equationally** the $\exists$-version of the "**renaming of bound variable metatheorem**", namely, prove:

$$\vdash (\exists \mathbf{x})A \equiv (\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}]$$

Limitations:

- A **non-Equational** proof **will *Max 0* points**.
- Equational proofs **will *Max 3* points if the "$\Leftrightarrow$" symbol is omitted in its proper placement**.
- Properly annotate WL, if used: In particular, *you* ***must*** *check and* **acknowledge** that the hypothesis of the rule is an absolute theorem.

Proof. We start from what we KNOW from class: Under the same assumptions on z, we have

$$\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})A[\mathbf{x} := \mathbf{z}] \tag{1}$$

$$
\begin{aligned}
& (\exists \mathbf{x})A \\
\Leftrightarrow & \langle \text{Def. of } \exists \rangle \\
& \neg(\forall \mathbf{x})\neg A \\
\Leftrightarrow & \langle \text{Abs. thm (1) + WL; Denom: } \neg \mathbf{p} \rangle \\
& \neg(\forall \mathbf{z})\neg A[\mathbf{x} := \mathbf{z}] \\
\Leftrightarrow & \langle \text{Def. of } \exists \rangle \\
& (\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}]
\end{aligned}
$$

$\square$

Predicate Logic 3. (5 MARKS) Prove, for any formulas A, B, that

$$\vdash (\forall x)A \vee (\forall x)B \rightarrow (\forall x)(A \vee B) \tag{1}$$

Required Method (**MAX 2 MARKS for *any* Alternative Correct Solution**)

Use an **Equational Proof** starting from the knowledge of the theorem below (Class/NOTEs).

$$\vdash (\exists x)(A \wedge B) \rightarrow (\exists x)A \wedge (\exists x)B \tag{2}$$

Prove (1); **do NOT prove (2)!**

Proof.

$$
\begin{aligned}
& (\forall x)A \vee (\forall x)B \\
\Leftrightarrow & \langle \text{Def. of } \exists + \text{WL; Denom: } \mathbf{p} \vee (\forall x)B \rangle \\
& \neg(\exists x)\neg A \vee (\forall x)B \\
\Leftrightarrow & \langle \text{Def. of } \exists + \text{WL; Denom: } \neg(\exists x)\neg A \vee \mathbf{q} \rangle \\
& \neg(\exists x)\neg A \vee \neg(\exists x)\neg B \\
\Leftrightarrow & \langle \text{tautology, hence Axiom} \rangle \\
& \neg\Big((\exists x)\neg A \wedge (\exists x)\neg B\Big) \\
\Leftrightarrow & \langle \text{by abs. thm (2) + WL; denom: } \neg\mathbf{p} \rangle \\
& \neg(\exists x)\Big(\neg A \wedge \neg B\Big) \\
\Leftrightarrow & \langle \text{Def. of } \exists \rangle \\
& (\forall x)\neg\Big(\neg A \wedge \neg B\Big) \\
\Leftrightarrow & \langle \text{WL + deM Tautology (so, } \underline{\text{Axiom}}\text{); Denom: } (\forall x)\mathbf{p} \rangle \\
& (\forall x)\Big(A \vee B\Big)
\end{aligned}
$$

$\square$

Predicate Logic 4. (5 MARKS) Use 1st-Order Soundness to prove that

$$\nvdash (\exists x)A \wedge (\exists x)B \to (\exists x)(A \wedge B) \qquad (1)$$

that is, $(\exists x)A \wedge (\exists x)B \to (\exists x)(A \wedge B)$ **is NOT a theorem of predicate logic**.

Hint. Use a **countermodel** for very simple instants of the wffs in (1), where you chose appropriate **atomic** A and B.

Proof. Counter model in $\mathfrak{N} = (\mathbb{N}, M)$ taking

A: x is even

B: x is odd

I Interpret:

$$\overbrace{(\exists x \in \mathbb{N})x \textit{ is even}}^{\mathbf{t}} \wedge \overbrace{(\exists x \in \mathbb{N})x \textit{ is odd}}^{\mathbf{t}} \to \overbrace{(\exists x \in \mathbb{N})(x \textit{ is even} \wedge x \textit{ is odd})}^{\mathbf{f}} \qquad (2)$$

We see that (2) is false, so (1) contains an NONtheorem.

□

Predicate Logic 5. (5 MARKS) Use the auxiliary hypothesis method (Any other CORRECT method Maxes at 0 points) to give a Hilbert proof of

$$\vdash (\exists x)A \to (\exists x)(B \to A \wedge B)$$

Proof. By DThm, instead of the given we prove

$$(\exists x)A \vdash (\exists x)(B \to A \wedge B)$$

1)	$(\exists x)A$	$\langle$hyp$\rangle$
2)	$A[x := z]$	$\langle$aux. hyp for 1; z fresh$\rangle$
3)	$(B \to A \wedge B)[x := z]$	$\langle 2$ + Post$\rangle$
4)	$(\exists x)(B \to A \wedge B)$	$\langle 3$ + Dual Spec$\rangle$

$\square$