Posted: Nov. 15, 2019
**Due: Dec. 4, 2019 by 3:00pm in the course box.**

# Problem Set No. 3

**NB.** *All problems are equally weighted and will be assigned a letter grade; an overall letter grade for the paper will be computed using York's 0–9 gpa scale.*

This is not a course on *formal* recursion theory. Your proofs should be informal (but **not** sloppy), correct, and informative (and if possible short). Please do not trade length for correctness or readability.

All problems are from "Theory of Computation".

(1) (**Grad**)From Section 2.12 do 67 , 73.

(2) Prove that there is a function $f \in \mathcal{P}$, such that for any $W_x$ we have

$$\text{if } W_x \neq \emptyset \text{, then } f(x)\downarrow \text{ and } f(x) \in W_x$$

(3) From Section 5.3: Do 23.

*Hint.* Show that "this is in $\mathcal{PR}$ or in $\mathcal{PR}_*$" can be replaced by "this is in $\mathcal{E}^3$ or in $\mathcal{E}^3_*$" throughout the proof that we gave for $T \in \mathcal{PR}_*$. In particular, we know that $\mathcal{E}^3$ is closed under simultaneous bounded recursion, *and you can show* that $\lambda xy.x^y \in \mathcal{E}^3$, $\lambda n.p_n \in \mathcal{E}^3$, and thus $\lambda \vec{x}.\langle \vec{x} \rangle \in \mathcal{E}^3$ as well as the $\lambda iz.(z)_i \in \mathcal{E}^3$.

(4) From the previous, conclude that the correctness problem for loop-programs in $L_2$ is not semi-recursive.

(5) Prove that

$$\lambda x.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\} \; x \text{ 2s}$$

is in $\mathcal{E}^4 - \mathcal{E}^3$.

(6) Prove that there is no $g \in \mathcal{R}^{(1)}$ such for all $f \in \mathcal{R}^{(1)}$ we have the "normal form"

if $f \in \mathcal{R}^{(1)}$, then, for some $i$, $f(x) = out\Big((\mu y)_{\leq g(x)} T(i, x, y), i, x\Big)$, for all $x$.

(7) Prove that *if* Gödel's incompleteness theorem is *false*, then the halting problem is *solvable*. This contradiction gives yet another proof of the incompleteness theorem.

*Hint.* Start by stating the negation of Gödel's theorem symbolically, using $CA$ and $\Theta$ that we introduced in class. Then devise two parallel processes that, for any given $a \in \mathbb{N}$, one trying to verify that $\phi_a(a)\uparrow$ holds, while the other is trying to verify that $\phi_a(a)\downarrow$ holds.