Posted: Nov. 17, 2018
Due: Dec. 4, 2018 by 3:00pm in the course box.

Problem Set No. 3

NB. *All problems are equally weighted and will be assigned a letter grade; an overall letter grade for the paper will be computed using York's 0–9 gpa scale.*

This is not a course on *formal* recursion theory. Your proofs should be informal (but **not** sloppy), correct, and informative (and if possible short). Please do not trade length for correctness or readability.

All problems are from "Theory of Computation".

(1) From Section 2.12 do 53 , 61 (the Hint helps!)

For #53, please do not use *either* of the two Rice's Lemmata!

On the positive side, each subproblem in #53 asks *also* "is it c.e.?" If you happen to answer this *first* (in the negative), then it cuts your work in half!

(2) From Section 5.3: Do 23, 32.

(3)

The answer to this problem is way shorter than the question!

In class we proved Gödel's (first) Incompleteness theorem by showing that CA —the set of all true *sentences*[*] of Arithmetic— is not c.e., that is, it is "bigger" than (hence not equal to!) the set of theorems of Arithmetic, which *is* c.e.

Ignore that methodology and prove instead *in outline* —exactly as Gödel did— that there is a *sentence* of Peano Arithmetic (PA) S such that PA proves neither S nor $\neg S$.

Here is what you should recall/do to realise this plan:

- We proved in class that if $Q \in \mathscr{PR}_*$ then Q is arithmetical, i.e., it can be expressed as a formula of (Peano) arithmetic, that is, it is obtained from initial predicates $y = x + 1, z = x + y, z = xy$ and $z = x^y$, using *closure* under Boolean operations, quantifiers, and *term substitutions*.
- For convenience, use the notation $\ulcorner A \urcorner$ or $gn(A)$ to denote the Gödel number of formula A under some coding.
- *Take as a given* that Peano Arithmetic is *sound*, that is, all its theorems that are *sentences* are true.[†]

[*] Recall that a "sentence" of first-order logic is a formula with no free variables.

[†] Of course, the concept of truth for a non-sentence depends on variables; that is why we restrict our soundness statement to sentences.

- *Take as a given* that we have a *proof predicate* $P(y, x) \in \mathscr{PR}_*$ for Peano Arithmetic that is true for any specific values y and x precisely when the formula with Gödel number x has a proof coded by y.[‡]
- Thus, the predicate Θ given by

$$\Theta(x) \stackrel{Def}{\equiv} (\exists y) P(y, x)$$

  is true iff the formula with Gödel number x is a theorem.
- Fix the order (and actual names) of object variables of our fisrt-order Logic to

$$v_0, v_1, v_2, \ldots$$

  Take as a given that we have a ("Gödel's") "substitution" function $\lambda xy.s(x, y)$ in $\mathscr{PR}$ such that $s(x, y)$ is the Gödel number of the formula we obtain from the one of Gödel number x, after we replace v_0 in that formula by the number y.[§]
- As Gödel did, **prove** his "fix point theorem", which is an *identical* precursor to Kleene's recursion theorem, in statement *and* proof (hint, hint! :-) This states: For any one variable formula $A(v_0)$ there is a natural number e such that

$$e = \ulcorner A(e) \urcorner \tag{1}$$

  Hint. Immitate the proof of the recursion theorem: Start with formula $A(s(v_0, v_0))$.
- As Gödel did, use $\Theta(x)$ and (1) to express the sentence

$$\text{“I am not a theorem”} \tag{2}$$

  as a formula of PA.
- Using one of our bulleted facts above, prove that PA cannot prove (2), hence sentence (2) is true! This is your S!
- Argue that $\neg S$ cannot be proved either!

[‡] Gödel proved this in his paper.

[§] Gödel proved this. $s(x, y)$ is his version of the Smn function! Indeed, what does Kleene's $S^1_1(x, y)$ denote? It denotes the program code (read: Gödel number) of a program obtained from program (of Gödel number) x, after some designated input variable $\mathbf{z}$ was replaced by y (by doing $\mathbf{z} \leftarrow y$).