York University
Department of Electrical Engineering and Computer Science
Lassonde School of Engineering

MATH 1028Z. MID TERM —SOLUTIONS, March 3, 2025; 13:30-14:30
Professor George Tourlakis

Question 1. (4 MARKS)

Suppose we know NOTHING about stages, but know two FACTS instead:

FACT 1. For any set A, $\{A\}$ is also a set.

FACT 2. Let $B \neq \emptyset$ be a *set*.
THEN

There is a $C \in B$, such that NO $D \in B$ is also in C.

From these two (FACT 1 and FACT 2) PROVE that for any set A, $A \in A$ is false.

Hint. By FACT 1, $\{A\}$ is a set. Use "FACT 2" where $B = \{A\}$. Now, $B \neq \emptyset$, ETC., ETC.

Proof. Pick ANY set A.

Following the *Hint*.

So, we want to prove for the arbitrary A, that $A \notin A$.

Let us take $B = \{A\}$, a SET by FACT 1.

As $A \in B$ trivially, $B \neq \emptyset$. Thus by FACT 2 there is a C in B

$$\boxed{\text{that "}C\text{" can only be } C = A} \quad (2)$$

such that

$$\boxed{\text{NO } D \in B \text{ is also in } C}$$

Now, the only "D in B" is A, so $A \notin C$. By (2), $C = A$, so we got $\boxed{A \notin A}$. □

Question 2. (a) (2 MARKS) Give the MATHEMATICAL definition of $\bigcup F$ where F is a set family of sets.

Answer. The definition of $\bigcup F$ (class/NOTEs/Text) is the definition of membership in $\bigcup F$:

$$x \in \bigcup F \text{ iff } \overbrace{(\exists A)}^{for\ some\ A} \Big(A \in F \wedge x \in A\Big) \tag{1}$$

□

(b) (4 MARKS) Calculate $\bigcup \emptyset$.

Caution. Be sure to follow the definition of $\bigcup F$.

Answer. To calculate $\bigcup \emptyset$ we just apply (1) from subquestion 2a.

$$x \in \bigcup \emptyset \text{ iff } \underbrace{\overbrace{(\exists A)}^{for\ some\ A} \underbrace{\Big(\overbrace{A \in \emptyset}^{\mathbf{f}} \wedge x \in A\Big)}_{\mathbf{f\ by}\ \text{“}\wedge\text{”}}}_{\mathbf{f}} \tag{2}$$

Thus, $x \in \bigcup \emptyset$ is false for any x and therefore $(\bigcup \emptyset) = \emptyset$. □

Question 3. (4 MARKS) Prove that the relation $\mathbb{P}$ is a set iff $\mathbb{P}^{-1}$ is a set.

Caution. Be mathematically precise!

Hint. There are two directions in "iff".

Proof.

(a) *Assume* $\mathbb{P}$ is a set and prove $\mathbb{P}^{-1}$ is a set.
By Assumption and *thm* from class/NOTEs/text, we know that $\text{dom}(\mathbb{P})$ and $\text{ran}(\mathbb{P})$ are sets.
But (class/NOTEs/text) we also know that $\text{dom}(\mathbb{P}^{-1}) = \text{ran}(\mathbb{P})$ and $\text{ran}(\mathbb{P}^{-1}) = \text{dom}(\mathbb{P})$.
Thus $\mathbb{P}^{-1} \subseteq \text{dom}(\mathbb{P}^{-1}) \times \text{ran}(\mathbb{P}^{-1}) = \text{ran}(\mathbb{P}) \times \text{dom}(\mathbb{P})$ and we are done by the subclass theorem.

(b) *Assume* $\mathbb{P}^{-1}$ is a set and prove $\mathbb{P}$ is a set. We prove this by *using* rather than *aping* the proof of 3a.
Indeed this case is true because $\left(\mathbb{P}^{-1}\right)^{-1} = \mathbb{P}$. Here is why:

$$x\left(\mathbb{P}^{-1}\right)^{-1} y \overset{Def}{\iff} y\mathbb{P}^{-1}x \overset{Def}{\iff} x\mathbb{P}y$$

using the conjunctional equivalence "$\iff$". □

Question 4. (5 MARKS) Let P and R be (set) relations.

Prove that $(P \circ R)^{-1} = R^{-1} \circ P^{-1}$.

Caution. An "=" between sets requires two directions to prove, or a *conjunctional* chain of correct equivalences.

Proof. $x(P \circ R)^{-1}y \Leftrightarrow y(P \circ R)x \Leftrightarrow yP \overset{\overset{some}{\downarrow}}{z} Rx \Leftrightarrow xR^{-1} \overset{\overset{same\ z}{\downarrow}}{z} P^{-1}y \Leftrightarrow xR^{-1} \circ P^{-1}y.$ □