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EECS 1028Z. Problem Set No4 —Solutions

Posted: Apr. 4, 2025

1. (1 MARK) Define: "A is uncountable".

Answer. "A is *uncountable* means that A is *NOT* countable!" □

2. (5 MARKS) Prove that if A is uncountable and $A \subseteq B$, then B is also uncountable.

Proof. By contradiction: If B is *NOT* uncountable then it *is countable*. We proved in class that **a subset of a countable set is countable**, so A is countable. *Contradiction!*

You do not have to prove what we showed in class, but here it is anyway:

Say B is countable. That means it gets a labelling ("stickers") via an onto (not necessarily total) function $f : \mathbb{N} \to B$. But then, after we remove ALL stickers from the members of $B - A$ and just leave ON those on members of A, we get a labelling of A. So A is *countable*! □

3. (5 MARKS) Prove that $A \to B \vdash (\exists x)A \to (\exists x)B$.

Proof. Done in Apr. 2 class. Here it is again.

By DThm prove this instead:

$$A \to B, (\exists x)A \vdash (\exists x)B$$

Here it goes AGAIN!

1) $A \to B$ $\langle$hyp$\rangle$
2) $(\exists x)A$ $\langle$DThm. hyp$\rangle$
3) $A[c]$ $\langle$aux. hyp for 2; c fresh$\rangle$
4) $A[c] \to B[c]$ $\langle$1 + Subst. OK as lines 2 (DThm) and 3 (Aux. hyp) have no free x$\rangle$

5)	$B[c]$	$\langle$(3, 4) + Post (or (3, 4) + MP)$\rangle$
6)	$(\exists x)B[x]$	$\langle$5 + Dual Spec$\rangle$

□

4. (5 MARKS) Let

$$\begin{aligned} &b_1 = 3,\, b_2 = 6 \\ &b_k = b_{k-1} + b_{k-2}, \text{ for } k \geq 3 \end{aligned} \tag{1}$$

Prove that 3 divides b_k, for all $k \geq 1$.

Proof. This is one of the obvious cases for CVI: You see, you cannot go from an I.H. with argument $k-1$ to the I.S. with argument k. The 2nd equation makes clear that you need to include —and use— case $k-2$ in the I.H.

Let's do it:

Basis: Obvious from "Prove that 3 divides b_k, for all $k \geq 1$". So, verify for $k = 1$: $b_1 = 3$, so divisible by 3.

I.H. Fix k and Assume for all m such that $1 \leq m < k$.

I.S. Do the case for k:

Well, since $k-1$ and $k-2$ are $< k$ (they are "m-values" as named in I.H.) the I.H. says b_{k-1} and b_{k-2} have 3 as a factor. So does b_k, their sum (2nd equation in (1)).

Almost, *but not so fast!*

The I.S. should be valid for all $k \geq 2$ ($k = 1$ was checked as basis). But for $k = 2$ the above reasoning does not work: We used b_{k-1} and b_{k-2} —that is, b_1 and b_0. But the b_0 term **does not exist!** The sequence starts with b_1. NO value for b_0 was given.

What do I do? I prove the "boundary case for $k = 2$ WITHOUT using I.H." OK: $b_2 = 6$, and this IS divided by 3. **I am done with this forgotten case and the proof is now COMPLETE.** □

5. (3 MARKS) Consider the statement (formula)

$$(\exists x)A(x) \to A(c) \tag{1}$$

where c is a *constant*.

Find now a *specific, very simple,* example of $A(x)$ over the set $\mathbb{N}$ and choose a specific value of $c \in \mathbb{N}$ so that (1) becomes **false** (meaning we cannot prove it, since proofs start from true axioms and preserve truth at every step).

Proof. Done in class April 2.

Regardless: Here it is AGAIN.

If (1) is *a theorem* then all its special cases are theorems too.

Well, here is a special case that is NOT true in $\mathbb{N}$, hence **NOT a theorem**. So neither is the original in (1):

Take for A the formula $x = 0$ and for c take 5. So (1) becomes

$$\overbrace{(\exists x)x = 0}^{\mathbf{t}} \rightarrow \overbrace{5 = 0}^{\mathbf{f}}$$

□

6. (5 MARKS) Using simple induction prove that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

for $n \geq 1$.

Proof. We use Simple Induction.

Basis. $n = 1$. Then $lhs = 1^2 = 1$ and $rhs = 1(1+1)(2 \cdot 1 + 1)/6 = 1$. OK!

I.H. Fix $n \geq 1$ (unspecified) and take (1) as I.H.

I.S.

$$\begin{aligned} 1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &\overset{I.H.}{=} (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\ &= (n+1)\left(\frac{6n+6}{6} + \frac{n(2n+1)}{6}\right) \\ &= (n+1)\left(\frac{2n^2+7n+6}{6}\right) \end{aligned} \tag{2}$$

To make sense of the last fraction, we factor the numerator.

(a) Find roots:

$$\frac{-7+/-\sqrt{49-48}}{4} = \frac{-7+/-1}{4} = \begin{cases} -2 \\ -3/2 \end{cases}$$

(b) Factor (as we learnt in High School):

$$2(n+2)(n+3/2) = (n+2)(2n+3)$$

(c) Resume (2)

$$\begin{aligned}
&=(n+1)\left(\frac{2(n+2)(n+3/2)}{6}\right)\\
&=(n+1)\left(\frac{(n+2)(2n+3)}{6}\right)\\
&=\frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

The last fraction is the same as $\frac{n(n+1)(2n+1)}{6}$ but with n everywhere replaced by $n+1$. Done. □