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EECS 1028 Z. Problem Set No3 —Solutions

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1. (4 MARKS) Show that if $\mathbb{F}$ is a function and $\mathrm{dom}(\mathbb{F})$ is a set then $\mathbb{F}$ is a set.

Proof. Note that $\mathrm{ran}(\mathbb{F})$ is also a set —by the function (forward) image theorem 5.1.10— since

$$\mathrm{ran}(\mathbb{F}) = \mathbb{F}[\mathrm{dom}(\mathbb{F})]$$

We are done by the subclass theorem and $\mathbb{F} \subseteq \mathrm{dom}(\mathbb{F}) \times \mathrm{ran}(\mathbb{F})$. □

2. (3 MARKS) Show by an easy counterexample that "if $\mathbb{F}$ is a function and ran($\mathbb{F}$) is a set then $\mathbb{F}$ is a set" is false.

Answer. Define the function $\mathbb{F}$ by $\mathbb{F}(x) = 0$ for all $x \in \mathbb{U}$. Then $\mathrm{ran}(\mathbb{F}) = \{0\}$, a set, but $\mathrm{dom}(\mathbb{F}) = \mathbb{U}$, a proper class. □

3. (4 MARKS) Define a **DIFFERENT implementation**

$$\langle x, y\rangle \stackrel{Def}{=} \Big\{\{x\}, \{x, y\}\Big\}$$

for **ordered pair** where this time we denote the latter as "$\langle x, y\rangle$" (angular brackets).

Prove that

$$\langle x, y\rangle = \langle x', y'\rangle \to x = x' \wedge y = y' \tag{1}$$

Caution. This does NOT require arguments via "set formation by stages".

Proof. Assume LHS of (1). Then, applying the function "$\bigcap$" to the same input $\langle x, y\rangle = \langle x', y'\rangle$), that is,

$$\bigcap \Big\{\{x\}, \{x, y\}\Big\} = \bigcap \Big\{\{x'\}, \{x', y'\}\Big\}$$

we get $\{x\} = \{x'\}$, hence

$$x = x' \tag{2}$$

.

Next, applying the function "$\bigcup$" to the same input $\langle x, y\rangle = \langle x', y'\rangle$, that is,

$$\bigcup\Big\{\{x\}, \{x, y\}\Big\} = \bigcup\Big\{\{x'\}, \{x', y'\}\Big\}$$

we get (mindful of (2))

$$\{x, y\} = \{x, y'\} \tag{3}$$

.

We now have two cases:

- Case $x = y$. Then $y' \in \{x, y\} = \{y\}$, hence $y = y'$ which along with (2) settles RHS of (1).
- Case $x \neq y$. Now, $y \in \{x, y'\}$ by (3). Since the Case forbids $y = x$ we must have the only alternative, $y = y'$. This, along with (2), proves the RHS of (1) one last time.

$\square$

4. (a) (2 MARKS) State the definition given in Class/NOTEs/Text for

$$\boxed{A \text{ is countable}} \tag{2}$$

Definition: A is countable **iff** we have some **onto** (not necessarily total) $g : \mathbb{N} \to A$.

(b) (4 MARKS) Prove that the Definition from Class/NOTEs/Text is equivalent to

$$\boxed{A \text{ is countable } \textbf{iff}\text{, there is a } \underline{\text{total}} \text{ and } \underline{\text{1-1}}\ f : A \to \mathbb{N}} \tag{3}$$

Proof.

i. Suppose A is countable according to Definition 4a. That means some (not necessarily total) onto function $g : \mathbb{N} \to A$ exists.

By Theorem 5.1.30 of the Notes, we also have a total and 1-1 $f : A \to \mathbb{N}$. (The theorem from Notes adds: "such that $gf = 1_A$" but this part is irrelevant to this problem).

ii. Conversely, suppose that we have $f : A \to \mathbb{N}$ that is total and 1-1. By Theorem 5.1.29 of the Notes, there is an onto function $g : \mathbb{N} \to A$, so A is countable by 4a. (The quoted theorem from Notes also observes: "such that $gf = 1_A$" but this part is irrelevant to this problem). □

5. (4 MARKS) Prove transitivity of $\sim$, that is, if $A \sim B \sim C$, then $A \sim C$.

Proof. Hypothesis says that

(a) We have a total, 1-1 and onto $f : A \to B$.

(b) We have a total, 1-1 and onto $g : B \to C$.

To prove transitivity we need to prove that

$$gf : A \to C$$

is a 1-1 correspondence.

(1) gf is onto C. Indeed, let $z \in C$. By ontoness of g there is a $y \in B$ such that $g(y) = z$. By ontoness of f there is a $x \in A$ such that $f(x) = y$.

So, $(gf)(x) = g(f(x)) = g(y) = z$. So gf is onto.

(2) Prove gf is total on A. OK, show $(gf)(a)$ is defined for any $a \in A$. But $(gf)(a) = g(f(a))$. Since f is total, we have $f(a)$ is an object in B. But g is total, so $g\Big(f(a)\Big)$ is an object in C.

(3) Prove (gf) is 1-1. Let

$$(gf)(x) = (gf)(y) = z \tag{$\dagger$}$$

We shall show $x = y$.
Now, ($\dagger$) says $g(f(x)) = g(f(y)) = z$. But g is 1-1, hence

$$f(x) = f(y) = w \tag{$\ddagger$}$$

the w since f is total. Now we remember that f is 1-1 too. So $x = y$ by ($\ddagger$) and we proved 1-1ness of (gf). □