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EECS 1028 Z. Problem Set No2 —Solutions

Posted: Feb. 26, 2025

1. (3 MARKS) Find the *equivalence class* —**IDENTIFIED BY THE SMALLEST NON-NEGATIVE integer** possible— for $\equiv_3$ where the integer -1010546 belongs to.

Answer. By "long division of 1010546 by 3" we get the quotient $q = 336848$ and the remainder $r = 2$. Thus (accounting for the *leading minus* in "-1010546")

$$-1010546 = -336848 \times 3 - 2 \tag{1}$$

(1) means

$$-1010546 \equiv_3 -2 \tag{2}$$

But 3 divides $3 = 1 - (-2)$. Thus,

$$-2 \equiv_3 1$$

which along with (2) yields

$$-1010546 \equiv_3 1 \tag{3}$$

In other words $-1010546 \in [1]_3$, hence $[-1010546]_3 = [1]_3$. □

2. (2 MARKS) TRUE or FALSE and *WHY*? (No correct "WHY" yields 0 MARKS)

"If the range of a relation $\mathbb{R}$ is a set, then $\mathbb{R}$ is a set."

Answer. **FALSE**. Consider the relation

$$\mathbb{R} \stackrel{Def}{=} \{(x, 0) : x \in \mathbb{U}\}$$

Indeed $\mathrm{ran}(\mathbb{R}) = \{0\}$, a set. Now, if $\mathbb{R}$ is a *set*, then so is its domain. NOT SO! $\mathrm{dom}(\mathbb{R}) = \mathbb{U}$. □

3. (3 MARKS) Show that the relation $\subseteq$ —where NO left/right fields are chosen *a priori*— is a *proper class*.

Proof. $\emptyset \subseteq A$, for *ANY* set A. Why? Because simply,

$$\overbrace{x \in \emptyset}^{\mathbf{f}} \to x \in A$$

But this says that

$$(\emptyset) \subseteq = \mathbb{V} \tag{1}$$

where

(a) we know $\mathbb{V} = \{x : x \text{ is a set}\}$ is a *proper class*.

(b) we know $\mathrm{ran}(\subseteq) = \mathbb{V}$ because **every member of $\mathbb{V}$ is an output of the relation $\subseteq$ for SOME INPUT.** WHICH input? But look at (1) above! Input $\emptyset$ causes as outputs all the members of $\mathbb{V}$.

Hence the Relation $\subseteq$ is a proper class, else its range would be a set.

□

4. (2 MARKS) Show for a relation $\mathbb{S}$ that if both the range and the domain are sets, then $\mathbb{S}$ is a set.

Proof. From our NOTES: Indeed, $\mathbb{S} \subseteq \mathrm{dom}(\mathbb{S}) \times \mathrm{ran}(\mathbb{S})$. By the Cartesian Product theorem, $\mathrm{dom}(\mathbb{S}) \times \mathrm{ran}(\mathbb{S})$ is a set, thus so in $\mathbb{S}$ by the subclass theorem. □

5. (3 MARKS) Prove that $\mathbb{N}^2$ is an equivalence relation on $\mathbb{N}$.

Proof. (I learnt in the fall term that **ChatGPT** does not know how to solve this problem! :)

N^2 is a set of pairs. I must *verify three properties* of

$$\mathbb{N}^2 = \{(x, y) : x \in \mathbb{N} \wedge y \in \mathbb{N}\} \tag{1}$$

(a) (**Reflexive**). Indeed, since (x, y) is in $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ for *any* x and y in $\mathbb{N}$, in particular all $(x, x) \in \mathbb{N}^2$, for all $x \in \mathbb{N}$.

(b) (**Symmetric**). By definition (1), **ANY** (x, y) is in $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ as long as x and y are both in $\mathbb{N}$. For any such (x, y), WHAT ABOUT (y, x)? This TOO is in $\mathbb{N}^2$ since y and x are in $\mathbb{N}$.

(c) (**Transitive**). We want $(x, y) \in \mathbb{N}^2$ and $(y, z) \in \mathbb{N}^2$ *to imply* $(x, z) \in \mathbb{N}^2$.

But we have that! WHY? Because the assumption requires x, y in $\mathbb{N}$ and y, z in $\mathbb{N}$ —see definition of $\mathbb{N}^2$ in (1) above!

But then all three, x, y, z, are in $\mathbb{N}$. In particular, x, z are. But that says that $(x, z) \in \mathbb{N}^2$. □

6. (4 MARKS) Let R be symmetric. Show that so is R^+.

Hint. Is the same true if we replace "R^+" in the statement with "R^n", for $n \geq 1$?

Proof. I am taking the *Hint*. Indeed, if R is symmetric, then so is R^n for all $n \geq 1$.

Here is why:

Let xR^ny. Thus (class and NOTEs)

$$x \overbrace{Ra_1Ra_2Ra_3 \cdots Ra_{n-2}Ra_{n-1}R}^{n\ R\ copies\ and\ n-1\ a_i\ copies} y \tag{1}$$

hence —because of the symmetry of R, I can swap the a_i/a_{i+1} in each a_iRa_{i+1} of (1) above and also swap the xRa_1 and $a_{n-1}Ry$ at the two ends— and I obtain the true statement

$$y \overbrace{Ra_{n-1}Ra_{n-2}Ra_{n-3} \cdots Ra_2Ra_1R}^{n\ R\ copies\ and\ n-1\ a_i\ copies} x \tag{2}$$

The display (2) says yR^nx, so indeed R^n is symmetric if R is.

Back to R^+. This is symmetric too.

So let xR^+y. Then (by $R^+ = \bigcup_{i=1}^{\infty} R^i$) we have $x \left(\bigcup_{i=1}^{\infty} R^i \right) y$. By definition of $\bigcup$ the last box says xR^iy for some value of $i \geq 1$, say $i = n$, so it says

xR^ny. By what we showed at the onset of this proof, we also have $\boxed{yR^nx}$ which implies $\boxed{y\left(\bigcup_{i=1}^{\infty} R^i\right) x}$ —that is, yR^+x. Done! □

7. (3 MARKS) Show that a relation $\mathbb{R}$ is symmetric iff, for all x, y,

$$xRy \equiv y\mathbb{R}x$$

Caution 1. Be sure (by consulting the NOTEs, not any other "authority"; that we start this problem on the same page as to what "symmetric relation" is defined as.

Caution 2. There are two directions in "iff".

Proof. We are asked to prove that a relation R must satisfy

$$xRy \text{ iff } y\mathbb{R}x \tag{1}$$

(a) ($\rightarrow$) This direction of (1) is the usual definition of "R is symmetric". So what DOES the definition SAY in plain words?

> To say "R is symmetric" is the same as, for any letters x, y that stand for elements in the field of R, we have xRy implies yRx

But there is no a priori meaning or values in the arbitrarily chosen letters x, y. So, the boxed statement below is equally valid to the boxed statement above. I can choose other letters to say the same thing!

So, I can prove the same thing that I did above, but here with different letters!

(b) ($\leftarrow$)

> To say "R is symmetric" is the same as, for any letters x, y that stand for elements in the field of R, we have yRx implies xRy. This is (1) right-to-left!

$\square$

8. (3 MARKS) Show that a relation S is transitive iff $S = S^+$.

Hint. There are two directions in "iff".

Proof.

(a) $(\rightarrow)$ Let S be transitive. The definition of "S^+" requires **ALL** of i.–iii. below

i. S^+ is transitive

ii. $S \subseteq S^+$

iii. If T is transitive and $S \subseteq T$, then $S^+ \subseteq T$.

Now take T to be S. This is legitimate by $S \subseteq S$ and by the underlined, red, hypothesis above. Thus, $S^+ \subseteq S$ by iii. and hence $S = S^+$ by ii.

(b) $(\leftarrow)$ Let $S = S^+$. But S^+ is transitive by-Def, and thus so is its equal, S. $\square$

9. (4 MARKS) Let R on A be reflexive. Prove that R^+ is also reflexive.

Proof. Hypothesis means "for all $x \in A$, $(x, x) \in R$" or $\boxed{\Delta_A \subseteq R}$.

Now,

$$R^+ = \bigcup_{i=1}^{\infty} R^i = R \cup \left(\bigcup_{i=2}^{\infty} R^i \right) \tag{2}$$

By (2) and the "boxed" observation above we have $\Delta_A \subseteq R^+$, thus R^+ is reflexive. $\square$