Lassonde School of Engineering

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EECS 1028 E. Problem Set No1 —Solutions

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1. True or False **and Why**. (**NOTE**: NO Why – NO Points)

(a) (3 MARKS) $\{\mathbb{U}\}$ is a set.

Answer. Worse than FALSE: MEANINGLESS:

We DON'T ALLOW PROPER CLASSES like $\mathbb{U}$ as members of **ANY** Class. $\square$

(b) (2 MARKS) $1 \in 1$. Do you need set-formation-by-stages to answer this?

Answer. FALSE. No need for Principles 0–2.

1 is an *atom* and atoms have NO ELEMENTS. So the rhs CANNOT contain "1". $\square$

(c) (2 MARKS) For sets or atoms c, d: $\bigcup\{\{c\},\{d\}\} = \{c, d\}$

Proof. TRUE. We learnt in Class that $\bigcup\{A, B, C, \ldots\}$ is **the class that we obtain** by starting with an "EMPTY container" $\{\quad\}$, and then "emptying" in it **each** of the sets $A, B, C, \ldots$.

So the above is right because it does just that! $\square$

(d) (2 MARKS) $\emptyset \subseteq \emptyset$

Answer. TRUE. The assertion —by definition of "$\subseteq$"— says

$$x \in \emptyset \to x \in \emptyset$$

The above has the form of $S \to S$ ("S" for "statement") which by our truth tables is TRUE regardless of whether S is TRUE or FALSE. $\square$

(e) (2 MARKS) $\emptyset \in \{1\}$

Answer. FALSE. The object "$\emptyset$" cannot be in the rhs as it then MUST BE equal to "1".

Impossible!

$$\overbrace{\emptyset}^{a\ set} = \overbrace{1}^{an\ atom} \ ???$$

No atom is a set. No set is available at stage 0 BUT the *atom* "1" *is*! $\square$

2. (4 MARKS) Prove for any set A that $\{\{A\}\}$ is a set. No random words please! A *MATH proof* is expected that one step implies the next.

Proof. For any set A, $\{A\} = \{A, A\}$ (Def. of "="; duplicates don't matter as we know) hence $\{A\}$ is a set (FOR ANY SET A) by the theorem for the UNordered pair from class.

Did I say "ANY SET?". Well then, the underlined class $\underline{\{\overbrace{\{A\}}^{a\ set}\}}$ is a set by the above argument! $\square$

3. (3 MARKS) Let A, B, C be sets or atoms. Prove that $\{A, B, C\}$ *is a set, without* using *any* of the Principles 0, 1, 2. *Rather use results (theorems) that we already established in class/Notes. Which ones?*

Proof. By the Theorem for Pair (as above), $\{A, B\}$ and $\{C\}$ are sets.

By the Theorem of Union (class/NOTEs) $\{A, B\} \cup \{C\}$ is a set.

But $\{A, B\} \cup \{C\} = \{A, B, C\}$ since the rhs contains all members that we empty from $\{A, B\}$ and $\{C\}$ into an empty container $\{\quad\}$. $\square$

4. Prove that Principle 2 implies that we have infinitely many stages available.

Hint. Arguing by contradiction, assume instead that we only have **finitely many** stages. So repeatedly applying Principle 2 we can form a non ending sequence of stages

$$\cdots < \Sigma_r < \Sigma_{r+1} < \Sigma_{r+2} < \Sigma_{r+3} < \cdots \quad (1)$$

If the sequence (1) contains only a *finite* number of distinct Σ_i, then at least two of the Σ_i in (1) —say, Σ_t and Σ_j— are (or, stand for) the same stage.

(a) (1 MARK) Why is the red underlined statement above true?

Answer. Because we keep adding new names forever —infinitely many names $\Sigma, \Sigma_1, \ldots$ but we have **assumed** that **actual stages are finitely many**. **So there MUST be two names standing for the same stage!** $\square$

(b) (4 MARKS) Use this conclusion of stage-repetition and properties of "<" to get a contradiction.

Answer. Let stage-names Σ_k and Σ_m —where $k < m$— **stand for the same stage**.

By transitivity of "<" we have $\Sigma_k < \Sigma_{k+1} < \cdots < \Sigma_{m-1} < \Sigma_m$ implies $\Sigma_k < \Sigma_m$.

BUT also $\Sigma_k = \Sigma_m$ since the two names stand for the **same stage**. This is a contradiction as no stage is before (or after) itself.

The contradiction forces us to reverse the assumption that there are only finitely many stages. $\square$

5. (3 MARKS) Find a *one element* set A for which $\mathbb{U} - A \neq \mathbb{U}$.

Prove that indeed $\mathbb{U} - A \neq \mathbb{U}$.

Answer. $A = \{1\}$ works: $1 \notin \mathbb{U} - A$ by Def. of class-minus.

Thus, $\mathbb{U} - A \neq \mathbb{U}$ since $1 \notin lhs$ but $1 \in rhs$. The two classes are NOT equal as one contains 1 and the other does not. $\square$

6. (3 MARKS) Let $\mathbb{A}$ be any class (proper or not). Prove that

$$\mathbb{U} \cup \mathbb{A} = \mathbb{U} \tag{1}$$

Caution. "Must prove" means do not wear the wig, but argue MATHEMATICALLY that (1) is true.

Statement (1) requires TWO proofs (two directions); State WHY and proceed:

1) Let $x \in lhs$. Analyse what this means and prove $x \in rhs$.

2) Let $x \in rhs$. Analyse what this means and prove $x \in lhs$.

Proof. As the hint 1)-2) indicates, we need $\mathbb{U} \cup \mathbb{A} \subseteq \mathbb{U}$ AND $\mathbb{U} \cup \mathbb{A} \supseteq \mathbb{U}$, that is, to do 1) and 2) above!

1) Let $x \in \mathbb{U} \cup \mathbb{A}$ and prove $x \in \mathbb{U}$. The assumption ("Let") and definition of $\cup$ lead to TWO Cases, namely $x \in \mathbb{U}$ OR $x \in \mathbb{A}$.
   i. $x \in \mathbb{U}$. Well, Done! The case is the same as the sought conclusion.
   ii. $x \in \mathbb{A}$. Well, $\mathbb{A} \subseteq \mathbb{U}$ since $\mathbb{U}$ contains everything, and, in particular, *ALL the members of* $\mathbb{A}$.
   So it includes x in particular. Done one last time. □

7. (4 MARKS) Prove for any classes $\mathbb{A}, \mathbb{B}$, that

$$\mathbb{A} \cap \mathbb{B} = \mathbb{U} - \Big(\big(\mathbb{U} - \mathbb{A}\big) \cup \big(\mathbb{U} - \mathbb{B}\big)\Big)$$

Hint. This is a simple case of proving $lhs \subseteq rhs$ by doing "*Let* $x \in lhs$. BLA BLA BLA *AND* concluding $x \in rhs$", and then *ALSO* doing $rhs \subseteq lhs$ by doing "Let $x \in rhs$. BLA BLA BLA and concluding $x \in lhs$".

Proof.

Per *Hint*, we have *two* directions/tasks:

1) ($\subseteq$ Direction) Prove

$$\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{U} - \Big(\big(\mathbb{U} - \mathbb{A}\big) \cup \big(\mathbb{U} - \mathbb{B}\big)\Big) \tag{1}$$

So LET a fixed $x \in lhs$ of (1). The following are immediate consequences, in sequence:

i. By Def. of "$\cap$": $x \in \mathbb{A}$ *and* $x \in \mathbb{B}$.

ii. Thus, $x \notin \mathbb{U} - \mathbb{A}$ *and* $x \notin \mathbb{U} - \mathbb{B}$.

iii. Since $x \in \mathbb{U}$ (everything is in $\mathbb{U}$) but in *neither* $\mathbb{U} - \mathbb{A}$ *nor* in $\mathbb{U} - \mathbb{B}$ by ii, the Definition of "$-$" places x *IN* the *rhs* of (1).

2) ($\supseteq$ Direction) Prove

$$\mathbb{A} \cap \mathbb{B} \supseteq \mathbb{U} - \Big(\big(\mathbb{U} - \mathbb{A} \big) \cup \big(\mathbb{U} - \mathbb{B} \big) \Big) \tag{2}$$

So LET a fixed $x \in$ R*hs* of (2). The following are immediate consequences, in sequence:

i. By Def. of "$-$":

$$x \notin \big(\mathbb{U} - \mathbb{A} \big) \cup \big(\mathbb{U} - \mathbb{B} \big) \tag{3}$$

ii. Thus x **cannot be in either** of $\mathbb{U} - \mathbb{A}$ or $\mathbb{U} - \mathbb{B}$ (else it would be in the union (3) above).

iii. ii says that x is in **both** $\mathbb{A}$ and $\mathbb{B}$, that is, $x \in$ L*hs*. Done!

$\square$

8. Use the notation by explicitly listing **all the members** of each rhs $\{???\}$ to complete the following incomplete equalities:

(a) (2 MARKS) $2^{\emptyset} = \{\emptyset\}$

(b) (2 MARKS) $2^{\{a,b,c\}} = \{\emptyset, \{a,b,c\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$