York University
Department of Electrical Engineering and Computer Science
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EEECS1028 Z April 9 FINAL EXAM, SOLUTIONS
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Questions Start HERE!

Question 1. (a) (1 MARK) For any relation $\mathbb{P}$ define the notation $(a)\mathbb{P}$. Please do so *NOT* with words! Rather, do so with a formula!

Definition. $(a)\mathbb{P} \stackrel{Def}{=} \{y : a\mathbb{P}y\}$.

(b) (2 MARKS) Introduce an EXAMPLE of a relation $\mathbb{P}$ such that for some input a, $(a)\mathbb{P}$ is demonstrably a proper class.

Caution. Please act on "demonstrably".

Here is an Example. Take $\mathbb{P}$ to be $\subseteq$ defined everywhere in $\mathbb{V}$: Then $\emptyset \subseteq A$ for each $A \in \mathbb{V}$, or, using notation from 1a, if I take $a = \emptyset$, then

$$(a)\mathbb{P} = (\emptyset) \subseteq = \mathbb{V}$$

We learnt from class early on that $\mathbb{V}$ is a proper class: NOT a set. □

Question 2. (a) (1 MARK) **Define**: "Set A is uncountable".

Definition: "Set A is *NOT* countable".

(b) (3 MARKS) Prove that an *uncountable* set is *infinite*.

Proof. The alternative to "infinite" is "finite". But "finite" is **countable** while our set is *uncountable*. Cannot be.

Question 3. (1) (1 MARK) What does the acronym "MC" stand for?

Answer. "Minimal Condition". □

(2) (3 MARKS) Define MC. (Not in words! Use an appropriately short formula, as in Class/NOTEs).

An order relation $\mathbb{P}$ has the minimal condition means that for all $\emptyset \neq \mathbb{A}$, we have some $a \in \mathbb{A}$ such that $\mathbb{A} \cap \{y : y\mathbb{P}a\} = \emptyset$.

With fewer words MC is the condition of an order relation $\mathbb{P}$ described by the formula

$$\emptyset \neq \mathbb{A} \to (\exists a)(a \in \mathbb{A} \land \mathbb{A} \cap \{y : y\mathbb{P}a\} = \emptyset)$$

Also with fewer symbols

$$\emptyset \neq \mathbb{A} \to (\exists a \in \mathbb{A})\mathbb{A} \cap \{y : y\mathbb{P}a\} = \emptyset$$

Question 4. (a) (2 MARKS) Define "Set A is enumerable" (NOT in words. Use a simple short formula that uses "$\sim$")

Definition. A is enumerable means $A \sim \mathbb{N}$.

(b) (4 MARKS) Prove that the set $\{7^m : m \geq 0\}$ is **enumerable**.

Proof. I need to exhibit a 1-1 correspondence $f : \mathbb{N} \to \{7^m : m \geq 0\}$.

Define $f(m) = 7^m$ for each $m \in \mathbb{N}$. **Prove**:

(a) f is *total*: Yes: For every $m \in \mathbb{N}$, f gives output 7^m, a number in $\mathbb{N}$.

(b) f is *1-1*. Yes: *Distinct inputs* $m \neq n$ give *distinct outputs*:

$$7^m \neq 7^n \tag{2}$$

Why is (2) true? By *unique factorisation* theorem: If (2) is false then $7^m = 7^n$and since $m \neq n$ we have two different prime factorisations of the number 7^m.

(c) f is onto. Well, pick any entry in $\{7^m : m \geq 0\}$, say, 7^j. Is there an input that maps to 7^j?

You bet: The way we defined f, $f(j) = 7^j$. We proved $\mathbb{N} \sim \{7^m : m \geq 0\}$ via f.

So $\{7^m : m \geq 0\}$ is enumerable. □

Question 5. (4 MARKS) Prove, using techniques of predicate logic and a *Hilbert Proof*, that if A has no free occurrences of x, then

$$A \to B \vdash A \to (\forall x)B$$

Proof.

By DThm prove

$$A \to B, A \vdash (\forall x)B$$

1)	$A \to B$	$\langle hyp \rangle$
2)	A	$\langle Dthm.\ hyp \rangle$
3)	B	$\langle 1 + 2\ Post \rangle$
4)	$(\forall x)B$	$\langle 3 + Gen\ OK\ no\ free\ x\ in\ 2 \rangle$

□

Question 6. (4 MARKS) What *EXACTLY* is wrong with the following "proof" that

$$\vdash x = 0 \to (\forall x)x = 0?$$

Recall that "$\vdash$" says "is a theorem" .

"**Proof**" By DThm, suffices to prove instead $x = 0 \vdash (\forall x)x = 0$. Here it goes!

1) $x = 0$ $\langle$hyp$\rangle$
2) $(\forall x)x = 0$ $\langle$1 + Gen$\rangle$

□

Answer. Step 2 is illegal since I cannot do Gen on x if I am below line 1: Line 1 is a DThm hyp *WITH* a free x! □

Question 7. (4 MARKS) Use Induction to prove that

$$2^{2n+1} + 5^{2n+1} \text{ is divisible by 7 for all } n \geq 0. \quad (1)$$

Proof We do SI.

Basis: $n = 0$. Verify: $2^{2\cdot 0+1} + 5^{2\cdot 0+1} = 7$. This is divisible by 7.

I.H. *Fix* n and assume (1). $2^{2n+1} + 5^{2n+1} = 7k$ (some k).

I.S. Prove case for $n + 1$ (same *fixed* n as above). From I.H., we have $2^{2n+1} + 5^{2n+1} = 7k$ (some k). Thus, (replacing n by $n + 1$ for I.S.): $2^{2n+3} + 5^{2n+3} = 4(7k - 5^{2n+1}) + 5^{2n+3} = 28k - 4 \cdot 5^{2n+1} + 5^{2n+3} = 28k - 4 \cdot 5^{2n+1} + 25 \cdot 5^{2n+1} = 28k + 21 \cdot 5^{2n+1}$.

First red term is the case for $n + 1$ and the last red term has 7 as a factor. Done. □

Question 8. (4 MARKS) Use CVI-type induction to prove that *every* number $n \in \mathbb{N}$ can be expressed in decimal notation as below:

$$n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0, \text{ where } 0 \le a_i \le 9, \text{ for } i = 0, \ldots, k \tag{1}$$

Proof. We do CVI:

Basis. $n = 0$, OK! "0" has the form (1) above.

I.H. Fix n. Now **Assume** the statement (1) for all $m < n$.

I.S. Prove for n. By Euclid's division theorem, let

$$n = 10q + r, \text{ were } 0 \le r < 10 \tag{†}$$

We assumed $n > 0$ in the I.H. Thus $q < n$,

Else $n \le q$ and thus $10n \le 10q \le 10q + r = n$. The last inequality is incompatible with $n > 0$.

So apply I.H. to q to get

$$q = b_t 10^t + b_{t-1} 10^{t-1} + \cdots + b_1 10 + b_0, \text{ where } 0 \le b_i < 10, \text{ for } i = 0, 1, 2, \ldots, t \tag{2}$$

Substituting (2) into (†) we get

$$n = b_t 10^{t+1} + b_{t-1} 10^t + \cdots + b_1 10^2 + b_0 10 + r$$

which has the form (1) since $0 \le r < 10$. □