3.3. Finite and Infinite Sets

Broadly speaking (that is, with very little detail contained in what I will say next) we have sets that are *finite* —intuitively meaning that we can count *all* their elements in a finite amount of time (but see the -remark 3.3.3 below)— and those that are not, naturally called *infinite*!

What is a mathematical way to say all this?

Any counting process of the elements of a finite set A will have us say out loud —every time we pick or point at an element of A— "0th", "1st", "2nd", etc., and, once we reach and pick the last element of the set, we finally pronounce "nth", for some appropriate n that we reached in our counting (Again, see 3.3.3.)

Thus, mathematically, we are pairing each member of the set with a member from $\{0, \ldots, n\}$.

So we propose,

3.3.1 Definition. (Finite and infinite sets) A set A is *finite* iff it is either empty, or is in 1-1 correspondence with $\{x \in \mathbb{N} : x \leq n\}$. This "normalized" small set of natural numbers we usually denote by $\{0, 1, 2, \ldots, n\}$.

If a set is *not* finite, then it is *infinite*. □

3.3.2 Example. For any n, $\{0, \ldots, n\}$ is finite since, trivially, $\{0, \ldots, n\} \sim \{0, \ldots, n\}$ using the identity (Δ) function on the set $\{0, \ldots, n\}$. □

3.3.3 Remark. One must be careful when one attempts to explain finiteness via counting by a human.

For example, Achilles[†] could count *infinitely many objects* by constantly accelerating his counting process as follows:

He procrastinated for a *full second*, and then counted the first element. Then, he counted the second object *exactly after* $1/2$ a second from the first. Then he got to the third element $1/2^2$ seconds after the previous, ..., he counted the n th item at exactly $1/2^{n-1}$ seconds after the previous, and so on *forever*.

Hmm! It was *not* "forever", was it? After a total of 2 seconds he was done!

You see (as you can easily verify from your calculus knowledge (limits)),[‡]

$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots = \frac{1}{1 - 1/2} = 2$$

So "time" is not a good determinant of finiteness! □

3.3.4 Theorem. *If $X \subset \{0, \ldots, n\}$, then there is no onto function $f : X \to \{0, \ldots, n\}$.*

I am saying, no such f, whether total or not; totalness is immaterial.

[†]OK, he was a demigod; but only "demi".

[‡]$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = \frac{1-1/2^n}{1-1/2}$. Now let n go to infinity at the limit.

Proof. First off, the claim holds if $X = \emptyset$, since then any such f equals $\emptyset$ and its range is empty.

Let us otherwise proceed by way of contradiction, and assume that the theorem is *wrong*: That is, **assume that** it *is* possible to have such onto functions, for some n and well chosen X.

Since I assume there are such $n > 0$ values, suppose then that the *smallest* n that allows this to happen is, say, n_0, and let X_0 be a *corresponding* set "X" that works, that is,

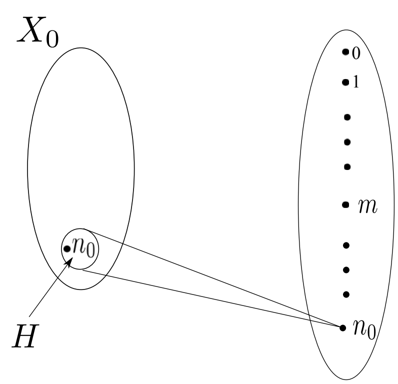
$$\text{Assume that we have an onto } f : X_0 \to \{0, \ldots, n_0\} \tag{1}$$

Thus $X_0 \neq \emptyset$, by the preceding remark, and therefore $n_0 > 0$, since otherwise $X_0 = \emptyset$.

Let us call H be the set of all x such that $f(x) = n_0$, for short, $H = f^{-1}(\{n_0\})$. $\emptyset \neq H \subseteq X_0$; the $\neq$ by ontoness.

Case 1. $n_0 \in H$. Then removing all pairs (a, n_0) from f —all these have $a \in H$— we get a new function $f' : X_0 - H \to \{0, 1, \ldots, n_0 - 1\}$, which *is still onto* as we only removed inputs that cause output n_0.

This contradicts minimality of n_0 since $n_0 - 1$ works too!



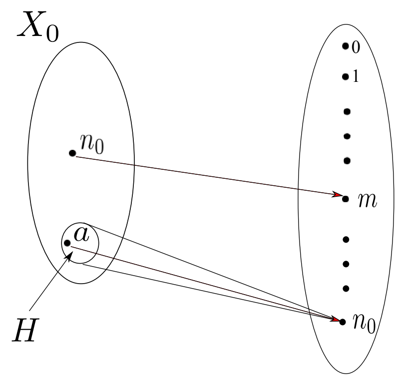
Case 2. $n_0 \notin H$.

If $n_0 \notin X_0$, then we argue exactly as in Case 1 and we just remove the base "H" of the cone (in the picture) from X_0.

Otherwise, we have two subcases:

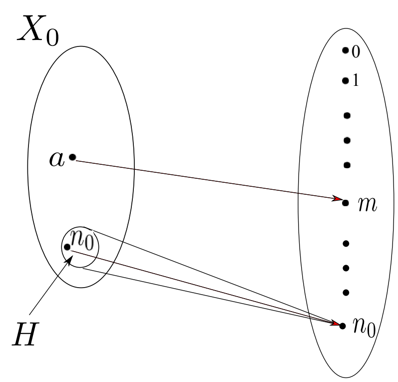
- $f(n_0) \uparrow$. Then (almost) we act as in Case 1: The new "X_0" is $(X_0 - H) - \{n_0\}$, since if we leave n_0 in, then the new "X_0" will not be a subset of $\{0, 1, \ldots, n_0 - 1\}$. We get a contradiction per Case 1.

- The picture below —that is, $f(n_0) = m$ for some m.



We simply transform the picture to the one below, "correcting" f to have $f(a) = m$ and $f(n_0) = n_0$, that is defining a new "f" that we will call f' by

$$f' = \Big(f - \{(n_0, m), (a, n_0)\}\Big) \cup \{(n_0, n_0), (a, m)\}$$



We get a contradiction per Case 1. □

3.3.5 Corollary. (Pigeon-Hole Principle) *If* $m < n$*, then* $\{0, \ldots, m\} \not\sim \{0, \ldots, n\}$.

Proof. If the conclusion fails then we have an onto $f : \{0, \ldots, m\} \to \{0, \ldots, n\}$, contradicting 3.3.4. □

Important!

3.3.6 Theorem. *If* A *is finite due to* $A \sim \{0, 1, 2, \ldots n\}$ *then there is* ***no justification of finiteness via another canonical set*** $\{0, 1, 2, \ldots m\}$ *with* $n \neq m$.

Proof. If $\{0, 1, 2, \ldots n\} \sim A \sim \{0, 1, 2, \ldots m\}$, then $\{0, 1, 2, \ldots n\} \sim \{0, 1, 2, \ldots m\}$ by 3.2.13, hence $n = m$, otherwise we contradict 3.3.5. □

3.3.7 Definition. Let $A \sim \{0,\ldots,n\}$. Since n is uniquely determined by A we say that A has $n+1$ elements and write $|A| = n+1$. □

3.3.8 Corollary. *There is no onto function from* $\{0,\ldots,n\}$ *to* $\mathbb{N}$.

"For all $n \in \mathbb{N}$, there is no..." is, of course, implied.

Proof. Fix an n. By way of contradiction, let $g : \{0,\ldots,n\} \to \mathbb{N}$ be onto. Let

$$Y \stackrel{Def}{=} \{x \le n : g(x) > n+1\}$$

Now let

$$X \stackrel{Def}{=} \{0,\ldots,n\} - Y$$

and

$$g' \stackrel{Def}{=} g - Y \times \mathbb{N}$$

The "$g - Y \times \mathbb{N}$" above is an easy way to say "remove all pairs from g that have their first component in Y".

Thus, $g' : X \to \{0,\ldots,n,n+1\}$ is onto, contradicting 3.3.4 because $X \subseteq \{0,\ldots,n\} \subset \{0,\ldots,n,n+1\}$. □

3.3.9 Corollary. $\mathbb{N}$ *is infinite.*

Proof. By 3.3.1 the opposite case requires that there is an n and a function $f : \{0,1,2,\ldots,n\} \to \mathbb{N}$ that is a 1-1 correspondence. Impossible, since any such an f will fail to be *onto.* □

Our mathematical definitions have led to what we hoped they would: That $\mathbb{N}$ *is* infinite as we intuitively understand, notwithstanding Achilles's accelerated counting!

$\mathbb{N}$ is a "canonical" infinite set that we can use to index the members of many infinite sets. Sets that can be indexed using natural number indices

$$a_0, a_1, \ldots$$

are called *countable.*

In the interest of technical flexibility, *we do not insist* that *all* members of $\mathbb{N}$ be used as indices. We might enumerate with gaps:

$$b_5, b_9, b_{13}, b_{42}, \ldots$$

Thus, informally, a set A is *countable* if it is empty or (in the opposite case) if there is a way to index, hence enumerate, all its members in an array, utilizing indices from $\mathbb{N}$. Cf. 3.1.40.

It *is* allowed to repeatedly list any element of A, so that finite sets are countable. For example, the set $\{42\}$:

$$42, 42, 42, \overbrace{\ldots}^{\text{42 forever}}$$

We may think that the enumeration above is done by assigning to "42" *all* of the members of $\mathbb{N}$ as indices, in other words, the enumeration is effected, for example, by the constant function $f : \mathbb{N} \to \{42\}$ given by $f(n) = 42$ for all $n \in \mathbb{N}$. This is consistent with our earlier definition of indexing (3.1.40).

Now, mathematically,

3.3.10 Definition. (Countable Sets) We call a set A *countable* if $A = \emptyset$, or there is an *onto* function $f : \mathbb{N} \to A$. We do NOT require f to be total. This means that some or many indices from $\mathbb{N}$ need not be used in the enumeration If $f(n) \downarrow$, then we say that $f(n)$ is the nth element of A in the enumeration f. We often write f_n instead of $f(n)$ and then call n a "subscript" or "index". □

Thus a nonempty set is countable iff it is the *range* of some function that has $\mathbb{N}$ as its *left field*.

BTW, since we allow f to be non total, the hedging "nonempty" is unnecessary: $\emptyset$ is the range of the empty function that has $\mathbb{N}$ as its left field.

We said that the f that proves countability of a set A need not be total. But such an f can always be "completed", by adding pairs to it, to get an f' such that $f' : \mathbb{N} \to A$ is onto *and* total. Here is how:

3.3.11 Proposition. *Let $f : \mathbb{N} \to A \neq \emptyset$*[†] *be onto. Then we can extend f to f' so that $f' : \mathbb{N} \to A$ is onto* and *total.*

Proof. Pick an $a \in A$ —possible since $A \neq \emptyset$— and keep it fixed. Now, our sought f' is given for all $n \in \mathbb{N}$ by cases as below:

$$f'(n) = \begin{cases} f(n) & \text{if } f(n) \downarrow \\ a & \text{if } f(n) \uparrow \end{cases}$$

□

Some set theorists also define sets that can be enumerated using *all* the elements of $\mathbb{N}$ as indices *without repetitions.*

3.3.12 Definition. (Enumerable or denumerable sets) A set A is *enumerable* iff $A \sim \mathbb{N}$. □

3.3.13 Example. Every enumerable set is countable, but the converse fails. For example, $\{1\}$ is countable but not enumerable due to 3.3.8. $\{2n : n \in \mathbb{N}\}$ is enumerable, with $f(n) = 2n$ effecting the 1-1 correspondence $f : \mathbb{N} \to \{2n : n \in \mathbb{N}\}$. □

[†]Since we are constructing a *total* onto function to A we need to assume the case $A \neq \emptyset$ as we cannot put any outputs into $\emptyset$.

3.3.14 Theorem. *If A is an infinite subset of $\mathbb{N}$, then $A \sim \mathbb{N}$.*

Proof. We will build a 1-1 and total enumeration of A, presented in a finite manner as a (pseudo) program below, which enumerates all the members of A in strict ascending order and arranges them in an array

$$a(0), a(1), a(2), \ldots \tag{1}$$

n	$\leftarrow 0$
while	$A \neq \emptyset$
$a(n)$	$\leftarrow \min A$ **Comment**. Inside the loop $\emptyset \neq A \subseteq \mathbb{N}$, hence min exists.
A	$\leftarrow A - \{a(n)\}$
n	$\leftarrow n + 1$
end while	

Note that the sequence $\{a(0), a(1), \ldots, a(m)\}$ is **strictly increasing** for any m, since $a(0)$ is smallest in A, $a(1)$ is smallest in $A - \{a(0)\}$ and hence the next higher than $a(0)$ in A, etc.

Will this loop ever exit? **Say, yes**, when it starts (but does not complete) the k-th pass through the loop. Thus A became empty when we did $A \leftarrow A - \{a(k-1)\}$ in the previous pass, that is $A = \{a(0), a(1), \ldots, a(k-1)\}$ and thus, since

$$a(0) < a91) < \ldots < a(k-1)$$

we have that the function $f : \{0, \ldots, k-1\} \to A$ given by

$$f = \{(0, a(0)), (1, a(1)), \ldots (k-1, a(k-1))\}$$

is total, 1-1 and onto, thus, $A \sim \{0, \ldots, k-1\}$ **contradicting that A is infinite**!

Thus, we never exit the loop!

Thus, by the remark in the paragraph above, (1) enumerates A in strict ascending order, that is,

$$\text{if we define } f : \mathbb{N} \to A \text{ by } f(n) = a(n), \text{ for all } n$$

then f is 1-1 (by strict increasing property: distinct inputs cause distinct outputs), and is trivially total, and onto. Why the latter? Every $a \in A$ is reached in ascending order, and assigned an "n" from $\mathbb{N}$. □

3.3.15 Theorem. *Every infinite countable set is enumerable.*

Proof. Let $f : \mathbb{N} \to A$ be onto and total (cf. 3.3.11), where A is infinite. Let $g : A \to \mathbb{N}$ such that $(fg) = \mathbf{1}_A$ (3.2.23). Thus, if we let $B = \text{ran}(g)$, we have that g is onto B, and thus by 3.2.18 is also 1-1 and total. Thus it is a 1-1 correspondence $g : A \to B$, that is,

$$A \sim B \tag{1}$$

B *must* be infinite, otherwise (3.3.1), for some n, $B \sim \{0, \ldots, n\}$ and by (1) via Exercise 3.2.13 we have $A \sim \{0, \ldots, n\}$, contradicting that A is infinite. Thus, by 3.3.14, $B \sim \mathbb{N}$, hence (again, Exercise 3.2.13 and (1)) $A \sim \mathbb{N}$. That is, A is enumerable. □

So, if we can enumerate an infinite set at all, then we can enumerate it without repetitions.

We can linearise an infinite square matrix of elements in each location (i, j) by devising a traversal that will go through each (i, j) entry *once*, and will *not miss any entry*!

In the literature one often sees the method diagrammatically, see below, where arrows *clearly* indicate the sequence of traversing, with the understanding that we use the arrows by pick the first unused chain of arrows from left to right.

$$
\begin{array}{cccccccc}
(0,0) & & (0,1) & & (0,2) & & (0,3) & \ldots\\
 & \nearrow & & \nearrow & & \nearrow & & \\
(1,0) & & (1,1) & & (1,2) & & & \\
 & \nearrow & & \nearrow & & & & \\
(2,0) & & (2,1) & & & & & \\
 & \nearrow & & & & & & \\
(3,0) & & & & & & & \\
\vdots & & & & & & &
\end{array}
$$

So the linearisation induces a 1-1 correspondence between $\mathbb{N}$ *and the linearised sequence of matrix entries, that is, it shows that* $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. For short,

3.3.16 Theorem. *The set* $\mathbb{N} \times \mathbb{N}$ *is countable. In fact, it is enumerable.*

Is there a "mathematical" way to do this? Well, the above IS mathematical, don't get me wrong, but is given in *outline*. It is kind of an argument in geometry, where we rely on drawings (figures).

Here are the algebraic details:

Proof. (of 3.3.16 with an algebraic argument). Let us call $i+j+1$ the "*weight*" of a pair (i, j). The weight is the number of elements in the group:

$$(i+j, 0), (i+j-1, 1), (i+j-2, 2), \ldots, (i, j), \ldots, (0, i+j)$$

Thus the diagrammatic enumeration proceeds by enumerating *groups* by increasing weight

$$1, 2, 3, 4, 5, \ldots$$

and in each group of weight k we enumerate in *ascending order of the second component.*

Thus the (i, j) th entry occupies position j *in its group* —the first position in the group being the 0 th, e.g., in the group of $(3, 0)$ the first position is the 0 th— and this position *globally* is the number of elements in all groups *before* group $i+j+1$, *plus* j. Thus the first available position for the first entry of group (i,j) members is just after this many occupied positions:

$$1+2+3+\ldots(i+j)=\frac{(i+j)(i+j+1)}{2}$$

That is,

$$\text{global position of } (i,j) \text{ is this: } \frac{(i+j)(i+j+1)}{2}+j$$

The function f which for all i,j is given by

$$f(i,j)=\frac{(i+j)(i+j+1)}{2}+j$$

is the algebraic form of the above enumeration. □

There is an easier way to show that $\mathbb{N}\times\mathbb{N}\sim\mathbb{N}$ without diagrams:

By the unique factorisation of numbers into products of primes (Euclid) the function $g:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ given for all m,n by $g(m,n)=2^m3^n$ is 1-1, since Euclid proved that $2^m3^n=2^{m'}3^{n'}$ implies $m=m'$ and $n=n'$. It is not onto as it never outputs, say, 5, but $\operatorname{ran}(g)$ is an *infinite* subset of $\mathbb{N}$ (Exercise!).

Thus, trivially, $\mathbb{N}\times\mathbb{N}\sim\operatorname{ran}(g)\sim\mathbb{N}$, the latter "$\sim$" by 3.3.14.

3.3.17 Exercise. If A and B are enumerable, so is $A\times B$.

Hint. So, $\mathbb{N}\sim A$ and $\mathbb{N}\sim B$. Can you show now that $\mathbb{N}\times\mathbb{N}\sim A\times B$? □

With little additional effort one can generalise to the case of $\bigtimes_{i=1}^{n} A_i$.

3.3.18 Remark.

1. Let us collect a few more remarks on countable sets here. Suppose now that we start with a countable set A. Is every subset of A countable? Yes, because the composition of onto functions is onto.

2. **3.3.19 Exercise.** What does composition of onto functions have to do with this? Well, if $B\subseteq A$ then there is a *natural* onto function $g:A\to B$. Which one? Think "natural"! Get a *natural* total and 1-1 function $f:B\to A$ and then use f to get g. □

3. As a special case, if A is countable, then so is $A\cap B$ for any B, since $A\cap B\subseteq A$.

4. How about $A\cup B$? If both A and B are countable, then so is $A\cup B$. Indeed, and without inventing a new technique, let

$$a_0,a_1,\ldots$$

be an enumeration of A and

$$b_0, b_1, \ldots$$

for B. Now form an infinite matrix with the A-enumeration as the 1st row, while each remaining row is the same as the B-enumeration. Now linearise this matrix!

Of course, we may alternatively adapt the unfolding technique to an infinite matrix of just two rows. **How?**

5. **3.3.20 Exercise.** Let A be enumerable and an enumeration of A

$$a_0, a_1, a_2, \ldots \tag{1}$$

is given.

So, this is an enumeration without repetitions.

Use techniques we employed in this section to propose a new enumeration in which *every* a_i is listed *infinitely many times* (this is useful in some applications of logic). □

3.4. Diagonalisation and uncountable sets

3.4.1 Example. Suppose we have a 3×3 matrix

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

and we are asked: Find a sequence of three numbers, *using only* 0 *or* 1, that does not *fit* as a row of the above matrix —i.e., is *different from all rows.*

Sure, you reply: Take 1 1 1. Or, take 0 0 0.

That is correct. But what if the matrix were big, say, $10^{350000} \times 10^{350000}$, or even *infinite*?

Is there a *finitely describable technique* that can produce an "unfit" row for any square matrix, even an infinite one? Yes, it is Cantor's *diagonal method* or technique.

He noticed that any row that fits in the matrix as the, say, i-th row, intersects the main diagonal at the same spot that the i-th column does.

That is, at entry (i, i).

Thus if we take the main diagonal —a sequence that has the same length as any row— and *change every one of its entries*, then it will not fit *anywhere* as a row! *Because no row can have an entry that is different than the entry at the location where it intersects the main diagonal!*

This idea would give the answer 0 1 0 to our original question. While 1000 11 3 also follows the principle "change all the entries of the diagonal" and works, we are constrained here to "use only 0 or 1" as entries. More seriously, in a case of a very large or infinite matrix it is best to have a simple technique that works even if we do not know much about the elements of the matrix. Read on! □

3.4.2 Example. We have an infinite matrix of 0-1 entries. Can we produce an infinite sequence of 0-1 entries that does not match *any* row in the matrix? Yes, take the main diagonal and *flip every entry* (0 to 1; 1 to 0).

If we think that, yes, it fits as row i, then we get a contradiction:

Say the original row has an a as entry (i, i). But, by our construction, the *new* row has an $1 - a$ in as entry (i, i), so it will not fit as row i after all. So it fits nowhere, i being arbitrary. □

3.4.3 Example. (Cantor) Let S denote the set of all infinite sequences of 0s and 1s.

Pause. What is an *infinite sequence*? Our intuitive understanding of the term is captured mathematically by the concept of a total function f with left field (and hence domain) $\mathbb{N}$. The n-th member of the sequence is $f(n)$.◀

Can we arrange *all* of S in an infinite matrix —one element per row? No, since the preceding example shows that we would miss at least one infinite sequence (i.e., we would fail to list it as a row), for a sequence of infinitely many 0s and/or 1s can be found, that does not match any row!

But arranging all members of S as an infinite matrix —one element per row— is tantamount to saying that we can enumerate all the members of S using members of $\mathbb{N}$ as indices.

So we cannot do that. S is not countable! □

3.4.4 Definition. (Uncountable Sets) A set that is not countable is called *uncountable*. □

So, an uncountable set is neither finite, nor enumerable. The first observation makes it infinite, the second makes it "more infinite" than the set of natural numbers since it is not in 1-1 correspondence with $\mathbb{N}$ (else it would be enumerable, hence countable) nor with a subset of $\mathbb{N}$: If the latter, our uncountable set would be finite or enumerable (which is absurd) according as it is in 1-1 correspondence with a finite subset or an infinite subset (cf. 3.3.14 and Exercise 3.2.13).

Example 3.4.3 shows that uncountable sets exist. Here is a more interesting one.

3.4.5 Example. (Cantor) The set of real numbers in the interval

$$(0,1) \overset{\text{Def}}{=} \{x \in \mathbb{R} : 0 < x < 1\}$$

is uncountable. This is done via an elaboration of the argument in 3.4.3.

Think of a member of $(0,1)$, *in form*, as an infinite sequence of numbers from the set $\{0,1,2,3,4,5,6,7,8,9\}$ prefixed with a dot; that is, think of the number's decimal notation.

Some numbers have representations that end in 0s after a certain point. We call these representations *finite*. Every such number has also an "infinite representation" since the non zero digit d immediately to the left of the infinite tail of 0s can be converted to $d-1$, and the infinite tail into 9s, without changing the value of the number.

Allow only infinite representations.

Assume now by way of contradiction that a listing of all members of $(0,1)$ exists, listing them via their infinite representations

$$\begin{array}{c} .a_{00}a_{01}a_{02}a_{03}a_{04}\ldots \\ .a_{10}a_{11}a_{12}a_{13}a_{14}\ldots \\ .a_{20}a_{21}a_{22}a_{23}a_{24}\ldots \\ .a_{30}a_{31}a_{32}a_{33}a_{34}\ldots \\ \vdots \end{array}$$

The argument from 3.4.3 can be easily modified to get a "row that does not fit", that is, a representation

$$.d_0d_1d_2\cdots$$

not in the listing.

Well, just let

$$d_i = \begin{cases} 2 & \text{if } a_{ii} = 0 \lor a_{ii} = 1 \\ 1 & \text{otherwise} \end{cases}$$

Clearly $.d_0d_1d_2\cdots$ does not fit in any row i as it differs from the expected digit at the i-th decimal place: should be a_{ii}, but $d_i \neq a_{ii}$. It is, on the other hand, an infinite decimal expansion, being devoid of zeros, and thus *should* be listed. This contradiction settles the issue. □

3.4.6 Example. (3.4.3 Revisited) Consider the set of all total functions from $\mathbb{N}$ to $\{0,1\}$. Is this countable?

Well, if there is an enumeration of these one-variable functions

$$f_0, f_1, f_2, f_3, \ldots \tag{1}$$

consider the function $g : \mathbb{N} \to \{0,1\}$ given by $g(x) = 1 - f_x(x)$. Clearly, this *must* appear in the listing (1) since it has the correct left and right fields, and is total.

Too bad! If $g = f_i$ then $g(i) = f_i(i)$. By definition, it is also $1 - f_i(i)$. A contradiction.

This is just version of 3.4.3; as already noted there, an infinite sequence of 0s and 1s is just a total function from $\mathbb{N}$ to $\{0,1\}$. □

The same argument as above shows that the set of all functions from $\mathbb{N}$ to itself is uncountable. Taking $g(x) = f_x(x) + 1$ also works here to "systematically change the diagonal" $f_0(0), f_1(1), \ldots$ since we are not constrained to keep the function values in $\{0, 1\}$.

3.4.7 Remark. Worth Emphasizing. Here is how we constructed g: We have a list of *in principle available f-indices* for g. We want to make sure that *none of them applies.*

A convenient method to do that is to inspect each available index, i, and using the diagonal method do this: Ensure that g differs from f_i at input i, by setting $g(i) = 1 - f_i(i)$.

This ensures that $g \neq f_i$; period. We say that *we cancelled the index i* as a *possible* "f-index" of g.

Since the process is applied *for each i*, *we have cancelled* all *possible indices for g*: For no i can we have $g = f_i$. □

3.4.8 Example. (Cantor) What about the set of all subsets of $\mathbb{N}$ —$\mathcal{P}(\mathbb{N})$ or $2^{\mathbb{N}}$?

Cantor showed that this is uncountable as well: If not, we have an enumeration of its members as

$$S_0, S_1, S_2, \ldots \tag{1}$$

Define the set

$$D \stackrel{\text{Def}}{=} \{x \in \mathbb{N} : x \notin S_x\} \tag{2}$$

So, $D \subseteq \mathbb{N}$, thus it must appear in the list (1) as an S_i. But then $i \in D$ iff $i \in S_i$ by virtue of $D = S_i$. However, also $i \in D$ iff $i \notin S_i$ by (2). This contradiction establishes that a legitimate subset of $\mathbb{N}$, namely D, is *not* an S_i. That is, $2^{\mathbb{N}}$ *cannot* be so enumerated; it is uncountable. □

3.4.9 Example. (Characteristic functions) Let $S \subseteq \mathbb{N}$. We can represent S as an infinite 0/1 array:

$$\begin{array}{rccccc}
\text{array position} & \ldots & i & \ldots & j & \ldots \\
\text{array content} & \ldots & 0 & \ldots & 1 & \ldots \\
 & \ldots & \uparrow & \ldots & \uparrow & \ldots \\
\text{means} & \multicolumn{5}{c}{\ldots i \notin S \ldots j \in S \ldots}
\end{array}$$

This array faithfully represents S —tells all we need to know about what S contains— since it contains a "1" in location x iff $x \in S$; contains "0" otherwise.

The array viewed as a total function from $\mathbb{N}$ to $\{0, 1\}$ is called the *characteristic function of S*, denoted by c_S:

$$c_S(x) = \begin{cases} 1 & \textbf{if } x \in S \\ 0 & \textbf{if } x \in \mathbb{N} - S \end{cases}$$

Note that there is a 1-1 correspondence, let's call it F, between subsets of $\mathbb{N}$ and the total 0-1-valued functions from $\mathbb{N}$ simply given by $F(S) = c_S$. (Exercise!)

Thus

$$\{f : f : \mathbb{N} \to \{0,1\} \text{ and } f \text{ is total}\} \sim 2^{\mathbb{N}}$$

In particular, the concept of characteristic functions shows that Example 3.4.8 fits the diagonalization methodology. Indeed, the argument in 3.4.8 sets $c_D(x) = 1 - c_{S_x}(x)$, for all x, because

$$c_D(x) = 1 \text{ iff } x \in D \text{ iff } x \notin S_x \text{ iff } c_{S_x}(x) = 0 \text{ iff } 1 - c_{S_x}(x) = 1$$

But then, the argument in 3.4.8 essentially applies the diagonal method to the list of 0/1 functions c_{S_x}, for $x = 0, 1, 2, \ldots$, to show that some 0/1 function, namely, c_D cannot be in the list. □