3.2. Functions

At last! We consider here a special case of relations that we know them as "functions". Many of you know already that a function is a relation with some special properties.

Let's make this official:

3.2.1 Definition. A *function* R is a *single-valued* relation. That is, whenever we have both xRy and xRz, we will also have $y = z$.

It is traditional to use, generically, lower case letters from among f, g, h, k to denote functions but this is by no means a requirement. □

Another way of putting it, using the notation from 3.1.51, is: A relation R is a function iff $(a)R$ is either *empty* or contains *exactly one* element.

3.2.2 Example. The empty set is a relation of course, the empty set of pairs. It is also a function since

$$(x, y) \in \emptyset \wedge (x, z) \in \emptyset \rightarrow y = z$$

vacuously, by virtue of the left hand side of $\rightarrow$ being false. □

We now turn to notation and concepts specific to functions.

3.2.3 Definition. (Function-specific notations) Let f be a function. First off, the *concepts* of domain, range, and —in case of a function $f : A \rightarrow B$— total and onto *are inherited from that of relations without change*. Even the notations "aRb" and "$(a, b) \in R$" transfer over to functions. And now we have an annoying *difference* in notation:

It is $f(a)$ that *normally* denotes the set $\{y : afy\}$ *in the literature*, NOT $(a)f$ (compare with 3.1.51). "Normally" allows some to differ: Notably, [Kur63] writes "af" for functions and relations, omitting even the brackets around a.

The reason for the preferred notation "$f(a)$" for functions will become more obvious once we consider composition of *functions*.

Can I use "$(a)f$" for a relation f regardless of whether it is also a function? YES! But once I proved (or I was told) that it is a function I ought to prefer to write $f(a)$.

If b is such that afb or $(a, b) \in f$ and f is a function, then seeing that b is unique we have $f(a) = \{b\}$.

However we will write

$$f(a) = b$$

That is,

$$\underbrace{f(a) = b}_{\text{functional notation}} \quad \text{iff} \quad \underbrace{(a)f = \{b\}}_{\text{relational notation}}$$

The notation "$(a)R\downarrow$" meaning $a \in \mathrm{dom}(R)$ is inherited by functions but for the flipping of the "(a)" part. Thus

Inherited from 3.1.51, $f(a)\downarrow$ iff $a \in \mathrm{dom}(f)$, pronounced "f is defined at a".

and, similarly to the notation $(a)R\uparrow$, we have

Inherited from 3.1.51, $f(a)\uparrow$ iff $a \notin \mathrm{dom}(f)$, pronounced "f is *UN*defined at a".

The set of *all* outputs of a function, *when the inputs come from a particular set* X, is called the *image of* X *under* f and is denoted by $f[X]$. Thus,

$$f[X] \stackrel{Def}{=} \{f(x) : x \in X\} \tag{1}$$

Note that careless notation (e.g., in our text) like $f(X)$ will *not* do. This means the input *IS* X. If I want the inputs to be *from inside* X I must change the round brackets notation; I did.

Pause. So far we have been giving definitions regarding functions of *one* variable. Or have we?◀

Not really: We have already said that the multiple-input case is subsumed by our notation. If $f : A \to B$ and A is a set of n-tuples, then f is a function of "n-variables", essentially. The binary relation that is the alias of f contains pairs like $\big((\vec{x}_n), x_{n+1}\big)$. However, we usually abuse the notation $f\Big((\vec{x}_n)\Big)$ and write instead $f(\vec{x}_n)$, omitting the brackets of the n-tuple $(\vec{x}_n)$.

The *inverse image* of a set Y under a function is useful as well, that is, the set of *all* inputs that generate f-outputs exclusively in Y. It is denoted by $f^{-1}[Y]$ and is defined as

$$f^{-1}[Y] \stackrel{Def}{=} \{x : f(x) \in Y\} \tag{2}$$

□

3.2.4 Remark. Regarding, say, the definition of $f[X]$:

What if $f(a)\uparrow$? How do you "collect" an <u>undefined</u> value *into a set?*

Well, you don't. Both (1) and (2) have a rendering that is independent of the notation "$f(a)$".

Never forget that a function is no mystery; it is a relation and we have access to relational notation. Thus,

$$f[X] = \{y : (\exists x \in X)xfy\} \tag{1$'$}$$

$$f^{-1}[Y] = \{x : (\exists y \in Y)xfy\} \tag{2$'$}$$

□

3.2.5 Example. Thus, $f[\{a\}] = \{f(x) : x \in \{a\}\} = \{f(x) : x = a\} = \{f(a)\}$.

Let now $g = \{\langle 1,2\rangle, \langle\{1,2\},2\rangle, \langle 2,7\rangle\}$, clearly a function. Thus, $g(\{1,2\}) = 2$, but $g[\{1,2\}] = \{2,7\}$. Also, $g(5)\uparrow$ and thus $g[\{5\}] = \emptyset$.

On the other hand, $g^{-1}[\{2,7\}] = \{1, \{1,2\}, 2\}$ and $g^{-1}[\{2\}] = \{1, \{1,2\}\}$, while $g^{-1}[\{8\}] = \emptyset$ since no input causes output 8. □

When $f(a)\downarrow$, then $f(a) = f(a)$ as is naturally expected. What about when $f(a)\uparrow$? This begs a more general question that we settle as follows:

3.2.6 Remark. This is the first (and probably last) time that we will view an $(m+n+1)$-ary relation $R(z_1,\ldots,z_m,x,y_1,\ldots,y_n)$ as a *function* with input values entered into *all* the variables $z_1,\ldots,z_m,x,y_1,\ldots,y_n$ and output values belonging to the set $\{\mathbf{t},\mathbf{f}\}$.

Such a relation, as we explained when we introduced relations, is always total, no matter what the input. That is, *any* input $a_1,\ldots,a_m,b,c_1,\ldots,c_n$ either *appears* in the *table of the relation*, or it does *not*. In other words, $R(a_1,\ldots,a_m,b,c_1,\ldots,c_n)$ is precisely *one* of true or false; there is no "maybe" or "I do not know".

Given such an $(m+n+1)$-ary relation, a function f, and an input u for f,

when is $R(z_1,\ldots,z_m,f(u),y_1,\ldots,y_n)$ true, for any given $z_1,\ldots,z_m,u,y_1,\ldots,y_n$?

Well, what we are saying in the notation (in blue) above is that if $f(u) = w$, for some w, then $R(z_1,\ldots,z_m,w,y_1,\ldots,y_n)$ is true.

Thus,

$$R(z_1,\ldots,z_m,f(u),y_1,\ldots,y_n) \text{ **iff** } (\exists w)\Big(w = f(u) \wedge R(z_1,\ldots,z_m,w,y_1,\ldots,y_n)\Big) \qquad (3)$$

Note that the part "for some w, $w = f(u)$" in (3) entails that $f(u)\downarrow$, so that if *no such w exists* [the case where $f(u)\uparrow$], then the rhs of (3) is *false*; **not** undefined!

This convention is prevalent in the modern literature (cf. [Hin78, p.9]). Contrast with the convention in [Kle43], where, for example, an expression like $f(a) = g(b)$ [and even $f(a) = b$] is allowed to be undefined! □

3.2.7 Example. Thus, applying the above twice, where our "R" is $x = y$, we get that $f(a) = g(b)$ means $(\exists u)(\exists w)(u = f(a) \wedge w = g(b) \wedge u = w)$ which simplifies to $(\exists u)(u = f(a) \wedge u = g(b))$. In particular, $f(a) = g(b)$ entails that $f(a)\downarrow$ and $g(b)\downarrow$ as we noted above.

Furthermore, using $x \neq y$ as R we get that $f(a) \neq g(b)$ means $(\exists u)(\exists w)(u = f(a) \wedge w = g(b) \wedge u \neq w)$. Again, if $f(a) \neq g(b)$ is true, its meaning implies $f(a)\downarrow$ and $g(b)\downarrow$. □

3.2.8 Example. Let $g = \{\langle 1,2\rangle, \langle\{1,2\},2\rangle, \langle 2,7\rangle\}$. Then, $g(1) = g(\{1,2\})$ and $g(1) \neq g(2)$. □

3.2.9 Definition. A function f is 1-1 if for all x and y, $f(x) = f(y)$ implies $x = y$. □

Note that $f(x) = f(y)$ implies that $f(x)\downarrow$ and $f(y)\downarrow$ (3.2.6).

3.2.10 Example. $\{\langle 1,1\rangle\}$ and $\{\langle 1,1\rangle, \langle 2,7\rangle\}$ are 1-1. $\{\langle 1,0\rangle, \langle 2,0\rangle\}$ is not. $\emptyset$ is 1-1 vacuously. □

3.2.11 Exercise. Prove that if f is a 1-1 function, then the relation converse f^{-1} is a function (that is, single-valued). □

3.2.12 Definition. (1-1 Correspondence) A function $f : A \to B$ is called a *1-1 correspondence* iff it is all three: 1-1, total and onto.

Often we say that A and B are *in 1-1 correspondence* writing $A \sim B$, often omitting mention of the function that *is* the 1-1 correspondence. □

The terminology is derived from the fact that every element of A is paired with precisely one element of B and vice versa.

3.2.13 Exercise. Show that $\sim$ is a symmetric and transitive relation on sets. □

3.2.14 Remark. Composition of functions is inherited from the composition of relations. Thus, $f \circ g$ for two functions still means

$$x\, f \circ g\, y \text{ iff, for some } z,\ x\, f\, z\, g\, y \tag{1}$$

In particular,

$f \circ g$ is also a function. Indeed, if we have

$$x\, f \circ g\, y \text{ and } x\, f \circ g\, y'$$

then

$$\text{for some } z, x\, f\, z\, g\, y \tag{1}$$

and

$$\text{for some } w, x\, f\, w\, g\, y' \tag{2}$$

As f is a function, (1) and (2) give $z = w$. In turn, this (g is a function too!) gives $y = y'$. □

The notation (as in 3.1.51) "$(a)f$" for relations is awkward when applied to functions —awkward but correct— where we prefer to use "$f(a)$" instead. The awkwardness manifests itself when we compose functions: In something like

$$x \to \boxed{f} \to z \to \boxed{g} \to y$$

that represents (1) above, note that f **acts first**. Its result $z = f(x)$ is then inputed to g —that is, we do $g(z) = g\Big(f(x)\Big)$ to obtain output y. Thus the first acting function f is "called" first with argument x and then g is called with argument $f(x)$. "Everyday math" notation places the two calls as in the red type above: The first call to the right of the 2nd call —order reversal vis a vis relational notation!

So, set theory heeds these observations and defines:

3.2.15 Definition. (Composition of functions; Notation) We just learnt (3.2.14) that the composition of two functions produces a function. The present definition is *about notation only*.

Let $f : A \to B$ and $g : B \to C$ be two functions. The relation $f \circ g : A \to C$, their *relational composition* is given in 3.1.15.

For composition of *functions*, we have the alternative —so-called *functional notation for composition*: "gf" for "$f \circ g$"; *note the order reversal* and the absence of "$\circ$", the composition symbol. In particular we write $(gf)(a)$ for $(a)(f \circ g)$ —cf. 3.2.3. Thus

$$a(gf)y \overset{Def}{\iff} a\ f \circ g\ y \iff (\exists z)(afz \wedge z\,g\,y)$$

also

$$a(gf)y \overset{Def}{\iff} a\ f \circ g\ y \overset{Def\ 3.1.51}{\iff} (a)(f \circ g) = \{y\}$$

In particular, we have that $(a)(f \circ g)$ of 3.1.51 is the same as $(gf)(a) = g\Big(f(a)\Big)$ as seen through the "computation"

$$\begin{aligned}(a)(f \circ g) =^{3.2.14} \{y\} &\iff \text{for some } z,\ a\,f\,z \wedge z\,g\,y \\ &\iff^{3.2.3} \text{for some } z,\ f(a) = z \wedge g(z) = y \\ &\iff^{\text{subst. } z \text{ by } f(a)} g\Big(f(x)\Big) = y \end{aligned} \qquad (1)$$

Conclusion:

$$(gf)(a) \overset{\text{blue type above}}{=} (a)(f \circ g) \overset{(1)}{=} g\Big(f(x)\Big)$$

Thus the "reversal" $gf = f \circ g$ now makes sense! So does $(gf)(a) = g\big(f(a)\big)$. □

3.2.16 Theorem. *Functional composition is associative, that is,* $(gf)h = g(fh)$.

Proof. Exercise!

Hint. Note that by, 3.2.15, $(gf)h = h \circ (f \circ g)$. Take it from here. □

3.2.17 Example. The *identity relation* on a set A is a function since $(a)\mathbf{1}_A$ is the singleton $\{x\}$. □

The following interesting result connects the notions of ontoness and 1-1ness with the "algebra" of composition.

3.2.18 Theorem. *Let $f : A \to B$ and $g : B \to A$ be functions. If*

$$(gf) = \mathbf{1}_A \tag{1}$$

then g is <u>onto</u> while f is <u>total</u> and *<u>1-1</u>.*

We say that g is a *left inverse* of f and f is a *right inverse* of g. "A" because these are not in general unique! Stay tuned on this!

Proof. **About g:** Our goal, ontoness, means that, for each $x \in A$, I can "solve the equation $g(y) = x$ for y". Indeed I can: By definition of $\mathbf{1}_A$,

$$g\Big(f(x)\Big) \stackrel{3.2.15}{=} (gf)(x) \stackrel{(1)}{=} \mathbf{1}_A(x) = x$$

So to solve, take $y = f(x)$.

About f: As seen above, $x = g(f(x))$, <u>for each $x \in A$</u>. Since this is the same as "$x \, f \circ g \, x$ is true", there must be a z such that $x \, f \, z$ and $z \, g \, x$. The first of these says $f(x) = z$ and therefore $f(x) \downarrow$. This settles totalness.

For the 1-1ness, let $f(a) = f(b)$. Applying g to both sides we get $g(f(a)) = g(f(b))$. But this says $a = b$, by $(gf) = \mathbf{1}_A$, and we are done. □

3.2.19 Example. *The above is as much as can be proved.* For example, say $A = \{1, 2\}$ and $B = \{3, 4, 5, 6\}$. Let $f : A \to B$ be $\{\langle 1, 4\rangle, \langle 2, 3\rangle\}$ and $g : B \to A$ be $\{\langle 4, 1\rangle, \langle 3, 2\rangle, \langle 6, 1\rangle\}$, or in friendlier notation

$f(1) = 4$
$f(2) = 3$
and
$g(3) = 2$
$g(4) = 1$
$g(5)\uparrow$
$g(6) = 1$

Clearly, $(gf) = \mathbf{1}_A$ holds, but note:

(1) f is not onto.
(2) g is neither 1-1 nor total. □

3.2.20 Example. With $A = \{1, 2\}$, $B = \{3, 4, 5, 6\}$ and $f : A \to B$ and $g : B \to A$ as in the previous example, consider also the functions $\tilde{f}$ and $\tilde{g}$ given by

$\tilde{f}(1) = 6$
$\tilde{f}(2) = 3$
and
$\tilde{g}(3) = 2$
$\tilde{g}(4) = 1$
$\tilde{g}(5)\uparrow$
$\tilde{g}(6) = 2$

Clearly, $(\tilde{g}f) = \mathbf{1}_A$ and $(g\tilde{f}) = \mathbf{1}_A$ hold, but note:

(1) $f \neq \tilde{f}$.

(2) $g \neq \tilde{g}$.

Thus, neither left nor right inverses need to be unique. The article "a" in the definition of said inverses was well-chosen. □

The following two partial converses of 3.2.18 are useful.

3.2.21 Theorem. *Let $f : A \to B$ be total and 1-1. Then there is an* onto *$g : B \to A$ such that $(gf) = \mathbf{1}_A$.*

Proof. Consider the converse relation (3.1.50) of f —that is, the *relation* f^{-1}— and call it g:

$$x\, g\, y \stackrel{\text{Def}}{\text{iff}}\ y\, f\, x \tag{1}$$

By Exercise 3.2.11, $g : B \to A$ is a (possibly nontotal) function so we can write (1) as $g(x) = y$ iff $f(y) = x$, from which, substituting $f(y)$ for x in $g(x)$ we get $g(f(x)) = x$, for all $x \in A$, that is $gf = \mathbf{1}_A$, hence g is onto by 3.2.18. We got both statements that we needed to prove. □

3.2.22 Remark. By (1) above, $\mathrm{dom}(g) = \{x : (\exists y)g(x) = y\} = \{x : (\exists y)f(y) = x\} = \mathrm{ran}(f)$. □

3.2.23 Theorem. *Let $f : A \to B$ be onto. Then there is a* total and 1-1 *$g : B \to A$ such that $(fg) = \mathbf{1}_B$.*

Proof. By assumption, $\emptyset \neq f^{-1}[\{b\}] \subseteq A$, for all $b \in B$. To define $g(b)$ choose *one* $c \in f^{-1}[\{b\}]$ and set $g(b) = c$. Since $f(c) = b$, we get $f(g(b)) = b$ for all $b \in B$, and hence g is 1-1 and total by 3.2.18. □