3.1.3. Partial orders

This subsection introduces one of the most important kind of binary relations in set theory and mathematics in general: The *partial order* relations.

We will find the following definitions and notation useful in this subsection:

3.1.50 Definition. (Converse or inverse relation of $\mathbb{P}$) For any relation $\mathbb{P}$, the symbol $\mathbb{P}^{-1}$ stands for the *converse* or *inverse* relation of $\mathbb{P}$ and is defined as

$$\mathbb{P}^{-1} = \{(x, y) : y\mathbb{P}x\} \tag{1}$$

$\underline{x\mathbb{P}^{-1}y \text{ iff } y\mathbb{P}x}$ is an equivalence that says exactly what (1) does. □

3.1.51 Definition. ("$(a)\mathbb{P}$" notation) For any relation $\mathbb{P}$ we write "$(a)\mathbb{P}$" to indicate the *class* —might fail to be a set— of *all outputs* of $\mathbb{P}$ *on (caused by) input* a. That is,

$$(a)\mathbb{P} \stackrel{Def}{=} \{y : a\,\mathbb{P}\,y\}$$

If $(a)\mathbb{P} = \emptyset$, then $\mathbb{P}$ is *undefined* at a —that is, $\underline{a \notin \mathrm{dom}(\mathbb{P})}$. The underlined statement is often denoted simply by "$\underline{(a)\mathbb{P}\uparrow}$" and is naturally read as "$\mathbb{P}$ is *undefined* at a".

If $(a)\mathbb{P} \neq \emptyset$, then $\mathbb{P}$ is *defined* at a —that is, $\underline{a \in \mathrm{dom}(\mathbb{P})}$. The underlined statement is often denoted simply by "$\underline{(a)\mathbb{P}\downarrow}$" and is naturally read as "$\mathbb{P}$ is *defined* at a". □

3.1.52 Exercise. Give an example of a specific relation $\mathbb{P}$ and one specific object (set or atom) a such that $(a)\mathbb{P}$ is a proper class. □

3.1.53 Remark. We note that for any $\mathbb{P}$ and a,

$$(a)\mathbb{P}^{-1} = \{y : a\mathbb{P}^{-1}y\} = \{y : y\mathbb{P}a\}$$

Thus,

$$(a)\mathbb{P}^{-1}\uparrow \text{ iff } a \notin \mathrm{ran}(\mathbb{P})$$

and

$$(a)\mathbb{P}^{-1}\downarrow \text{ iff } a \in \mathrm{ran}(\mathbb{P})$$

□

3.1.54 Definition. (Partial order) A relation $\mathbb{P}$ is called a *partial order* or just an *order*, iff it is

(1) *irreflexive* (i.e., $x\mathbb{P}y \to x \neq y$ for all x, y), and

(2) *transitive*.

It is emphasised that in the interest of generality —for much of this subsection (until we say otherwise)— $\mathbb{P}$ need not be a set.

Some people call this a *strict order* as it imitates the "$<$" on, say, the natural numbers. □

3.1.55 Remark. (1) We will normally use the symbol "<" in *the abstract setting* to denote *any* unspecified order $\mathbb{P}$, and it will be pronounced "less than".

It is **hoped** that the context will not allow confusion with any concrete use of the symbol $<$ on numbers (say, on the reals, natural numbers, etc.).

(2) If the order $<$ is a subclass of $\mathbb{A} \times \mathbb{A}$ —i.e., it is $<: \mathbb{A} \to \mathbb{A}$— then we say that $<$ *is an order on* $\mathbb{A}$.

(3) Clearly, for any order $<$ and any class $\mathbb{B}$, $< \cap (\mathbb{B} \times \mathbb{B})$ is an order on $\mathbb{B}$. □

3.1.56 Exercise. How clearly? (re (3) above.) Give a simple, short proof. □

3.1.57 Example. The concrete "less than", $<$, on $\mathbb{N}$ is an order, but $\leq$ is not (it is *not* irreflexive). The "greater than" relation, $>$, on $\mathbb{N}$ is also an order, but $\geq$ is not. Of course, $> = <^{-1}$.

In general, it is trivial to verify that $\mathbb{P}$ is an order iff $\mathbb{P}^{-1}$ is an order. *Exercise*! □

3.1.58 Example. $\emptyset$ is an order. Since for any $\mathbb{A}$, $\emptyset \subseteq \mathbb{A} \times \mathbb{A}$, $\emptyset$ is also an order *on* $\mathbb{A}$ for the arbitrary $\mathbb{A}$. □

3.1.59 Example. The relation $\in$ is irreflexive by the well known $A \notin A$, for all A. It is not transitive though. For example, if a is a set (or atom), then $a \in \{a\} \in \{\{a\}\}$ but $a \notin \{\{a\}\}$. *So it is not an order.*

Let $M = \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\Big\}$. The relation $\varepsilon = \in \cap (M \times M)$ *is* transitive and irreflexive, hence it is an order (on M). *Verify*! □

3.1.60 Example. $\subset$ is an order, $\subseteq$—failing irreflexivity— is not. □

3.1.61 Example. Consider the order $\subset$ again. In this case we have **none** of $\{\emptyset\} \subset \{\{\emptyset\}\}$, $\{\{\emptyset\}\} \subset \{\emptyset\}$ or $\{\{\emptyset\}\} = \{\emptyset\}$. That is, $\{\emptyset\}$ and $\{\{\emptyset\}\}$ are *non comparable* items. This justifies the qualification *partial* for orders in general (Definition 3.1.66).

On the other hand, the "natural" $<$ on $\mathbb{N}$ is such that one of $x = y$, $x < y$, $y < x$ always holds for any x, y. That is, all (unordered) pairs x, y of $\mathbb{N}$ *are* comparable under $<$. This is a concrete example of a *total* order (see the "official definition" below: 3.1.67).

While *all* orders are "partial", some are total ($<$ above) and others are *nontotal* ($\subset$ above). □

3.1.62 Definition. Let $<$ be a partial order on $\mathbb{A}$. We set

$$\leq \stackrel{Def}{=} \mathbf{\Delta}_{\mathbb{A}} \cup <$$

We pronounce $\leq$ "less than or equal". $\mathbf{\Delta}_{\mathbb{A}} \cup >$ is denoted by $\geq$ and is pronounced "greater than or equal".

Let us call $\leq$ a *reflexive order.* □

(1) In plain English, given $<$ on $\mathbb{A}$, we define $x \leq y$ to mean

$$x < y \vee \overbrace{x = y}^{\text{equality is } \Delta_{\mathbb{A}}}$$

for all x, y in $\mathbb{A}$.
(2) The definition of $\leq$ depends on $\mathbb{A}$ due to the presence of $\boldsymbol{\Delta}_{\mathbb{A}}$. **There is no such dependency on a "reference" class in the case of** $<$.
(3) We remind ourselves once more here that the symbols $<$ and $\leq$ —and their pronunciations— do *NOT* imply that we are talking about the specific ones on *numbers*. It is just a harmless (I hope) notational devise, but **unless said explicitly otherwise, "$<$" and "$\leq$" are <u>any</u> orders**.

3.1.63 Lemma. *For any* $<: \mathbb{A} \to \mathbb{A}$*, the associated relation* $\leq$ *on* $\mathbb{A}$ *is* reflexive*,* antisymmetric *and* transitive*.*

Proof. (1) Reflexivity is trivial.

(2) For antisymmetry, let $x \leq y$ and $y \leq x$. If $x = y$ then we are done, so assume the remaining case $x \neq y$ (i.e., $(x, y) \notin \boldsymbol{\Delta}_{\mathbb{A}}$). Then the hypothesis becomes $x < y$ and $y < x$, therefore $x < x$ by transitivity, contradicting the irreflexivity of $<$.

(3) As for transitivity let $x \leq y$ and $y \leq z$.

(a) If $x = z$, then $x \leq z$ (see the ☡-remark after 3.1.62) and we are done.

(b) The remaining case is $x \neq z$. Now, if it is $x = y$ or $y = z$ (but not both (why?)), then we are done again. So it remains to consider $x < y$ and $y < z$. By transitivity of $<$ we get $x < z$, hence $x \leq z$, since $< \subseteq \leq$. □

3.1.64 Lemma. *Let* $\mathbb{P}$ *on* $\mathbb{A}$ *be reflexive, antisymmetric and transitive.*
Then $\mathbb{P} - \boldsymbol{\Delta}_{\mathbb{A}}$ *is an order on* $\mathbb{A}$*.*

Proof. Since

$$\mathbb{P} - \boldsymbol{\Delta}_{\mathbb{A}} \subseteq \mathbb{P} \tag{1}$$

it is clear that $\mathbb{P} - \boldsymbol{\Delta}_{\mathbb{A}}$ is *on* $\mathbb{A}$. It is also clear that it is irreflexive. We only need verify that it is transitive.

So let

$$(x, y) \text{ and } (y, z) \text{ be in } \mathbb{P} - \boldsymbol{\Delta}_{\mathbb{A}} \tag{2}$$

By (1) (or (2))

$$(x, y) \text{ and } (y, z) \text{ are in } \mathbb{P} \tag{3}$$

hence

$$(x, z) \in \mathbb{P}$$

by transitivity of $\mathbb{P}$.

Can $(x, z) \in \boldsymbol{\Delta}_{\mathbb{A}}$, i.e., can $x = z$? No, for antisymmetry of $\mathbb{P}$ and (3) would imply $x = y$, i.e., $(x, y) \in \boldsymbol{\Delta}_{\mathbb{A}}$ contrary to (2).

So, $(x, z) \in \mathbb{P} - \boldsymbol{\Delta}_{\mathbb{A}}$. □

3.1.65 Remark. Often in the literature, but decreasingly so, it is the "reflexive order" $\leq : \mathbb{A} \to \mathbb{A}$ that is defined as a "partial order" by the requirements that it is *reflexive*, *antisymmetric* and *transitive*. Then $<$ is obtained as in Lemma 3.1.64, namely, as "$\leq - \mathbf{\Delta}_{\mathbb{A}}$". Lemmas 3.1.63 and 3.1.64 show that the two approaches are interchangeable, but the "modern" approach of Definition 3.1.54 avoids the nuisance of having to tie the notion of order to some particular "field" $\mathbb{A}$ (3.1.6).

For us "$\leq$" is the *derived* notion defined in 3.1.62. □

3.1.66 Definition. (PO Class) If $<$ is an order on a class $\mathbb{A}$, we call the *informal* pair $(\mathbb{A}, <)$[†] a *partially ordered class*, or *PO class*.

If $<$ is an order on a *set* A, we call the pair $(A, <)$ a *partially ordered set* or *PO set*. Often, if the order $<$ is understood as being on $\mathbb{A}$ or A, one says that "$\mathbb{A}$ is a PO class" or "A is a PO set" respectively. □

3.1.67 Definition. (Linear order) A relation $<$ on $\mathbb{A}$ is a *total* or *linear* order *on* $\mathbb{A}$ iff it is

(1) An order, and

(2) For any x, y in $\mathbb{A}$ one of $x = y$, $x < y$, $y < x$ holds —this is the so-called "*trichotomy*" property.

If $\mathbb{A}$ is a class, then the informal pair $(\mathbb{A}, <)$ is a *linearly ordered class* —for short, a *LO class*.

If $\mathbb{A}$ is a set, then the pair $(\mathbb{A}, <)$ is a *linearly ordered set* —for short, a *LO set*.

One often calls just $\mathbb{A}$ a LO class or LO set (as the case warrants) when $<$ is understood from the context. □

3.1.68 Example. The standard $<: \mathbb{N} \to \mathbb{N}$ is a total order, hence $(\mathbb{N}, <)$ is a LO set.

3.1.69 Definition. (Minimal and minimum elements) Let $<$ be an order and $\mathbb{A}$ some class.

We are not postulating that $<$ is on $\mathbb{A}$.

An element $a \in \mathbb{A}$ is *a $<$-minimal element in* $\mathbb{A}$, or *a $<$-minimal element of* $\mathbb{A}$, iff $\neg(\exists x \in \mathbb{A}) x < a$ —in words, there is nothing below a in $\mathbb{A}$.

$m \in \mathbb{A}$ is *a $<$-minimum element in* $\mathbb{A}$ iff $(\forall x \in \mathbb{A}) m \leq x$.

We also use the terminology minimal or minimum *with respect to* $<$, instead of $<$-minimal or $<$-minimum.

[†]Formally, $(\mathbb{A}, <)$ is *not* an ordered pair since $\mathbb{A}$ may be a proper class and we do not allow class *members* —e.g., in $\{\mathbb{A}, \{\mathbb{A}, <\}\}$— to be proper classes. We may think then of "$(\mathbb{A}, <)$" as *informal* notation that simply "ties" $\mathbb{A}$ and $<$ together. Alternatively, if we are really determined to have class pairs (we are not!), we can *define* pairing with proper classes as components, for example as $(\mathbb{A}, \mathbb{B}) =^{Def} (\mathbb{A} \times \{0\}) \cup (\mathbb{B} \times \{1\})$. For our part we will have no use for such formality, and will consider $(\mathbb{A}, <)$ in only the *informal* sense.

If $a \in \mathbb{A}$ is $>$-minimal in $\mathbb{A}$, that is $\neg(\exists x \in \mathbb{A})x > a$, we call a a *$<$-maximal* element in $\mathbb{A}$. Similarly, a $>$-minimum element is called a *$<$-maximum*.

If the order $<$ is understood, then the qualification “$<$-” is omitted. □

3.1.70 Remark. In particular, if a ($\in \mathbb{A}$) is *not* in the *field* $\mathrm{dom}(<) \cup \mathrm{ran}(<)$ (cf. 3.1.6) of $<$, then a is *both* $<$-minimal and $<$-maximal *in* $\mathbb{A}$. For example, $(\exists x \in \mathbb{A})x < a$ is false in this case since if, for some x, we have $x \in \mathbb{A}$ and also $x < a$, then $a \in \mathrm{ran}(<)$; impossible.

Because of the duality between the notions of minimal/maximal and minimum/maximum, we will mostly deal with the $<$-notions whose results can be trivially translated for the $>$-notions.

Note how the notation learnt from 3.1.51 and 3.1.50 and 3.1.53 can *simplify*

$$\neg(\exists x \in \mathbb{A})x < a \tag{1}$$

(1) says that *no x is in **both** $\mathbb{A}$ and $(a)>$*.[†]

That is, a is $<$-minimal in $\mathbb{A}$ iff

$$\mathbb{A} \cap (a) > = \emptyset \tag{2}$$

□

3.1.71 Example. 0 is *minimal*, also *minimum*, in $\mathbb{N}$ with respect to the natural ordering.

In $\mathbf{P}(\mathbb{N})$, $\emptyset$ is both $\subset$-minimal and $\subset$-minimum. On the other hand, all of $\{0\}, \{1\}, \{2\}$ are $\subset$-minimal in $\mathbf{P}(\mathbb{N}) - \{\emptyset\}$ but *none* are *$\subset$-minimum* in that set.

Observe from this last example that minimal elements in a class are *not* unique. □

3.1.72 Remark. (Hasse diagrams) There is a neat pictorial way to depict orders on finite sets known as “*Hasse diagrams*”. To do so one creates a so-called “*graph*” of the finite PO set $(A, <)$ where $A = \{a_1, a_2, \ldots, a_n\}$.

How? The graph consists of n *nodes* —which are drawn as points— each labeled by one a_i. The graph also contains 0 or more *arrows* that connect nodes. These arrows are called *edges*.

When we depict an arbitrary R on a finite set like A we draw *one* arrow (edge) from a_i to a_j iff the two *relate*: $a_i R a_j$.

In Hasse diagrams for PO sets $(A, <)$ we are more selective: We say that b *covers* a iff $a < b$, but there is no c such that $a < c < b$. In a Hasse diagram we will

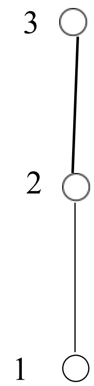
1. draw an edge from a_i to a_j iff a_j covers a_i.

2. by convention we will draw b higher than a on the page if b covers a.

[†] $(a) > = \{x : a > x\} = \{x : x < a\}$ (3.1.53).

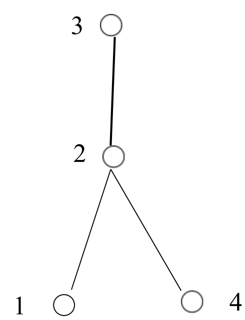
3. given the convention above, using "arrow-heads" is superfluous: our edges are plain line segments.

So, let us have $A = \{1, 2, 3\}$ and $<= \{(1, 2), (1, 3), (2, 3)\}$.



The above has a minimum (1) and a maximum (3) and is clearly a linear order.

A slightly more complex one is this $(A, <)$, where $A = \{1, 2, 3, 4\}$ and $<= \{(1, 2), (4, 2), (2, 3), (1, 3), (4, 3)\}$.



This one has a maximum (3), two minimal elements (1 and 4) but no minimum, and is not a linear order: 1 and 4 are not comparable. □

3.1.73 Lemma. *Given an order $<$ and a class $\mathbb{A}$.*
(1) *If m is a* minimum *in $\mathbb{A}$, then it is also* minimal.
(2) *If m is a* minimum *in $\mathbb{A}$, then it is unique.*

Proof. (1) Let m be mini*mum* in $\mathbb{A}$. Then

$$m \leq x, \text{ that is, } m = x \vee m < x \qquad (i)$$

for all $x \in \mathbb{A}$. Now, prove that there is no $x \in \mathbb{A}$ such that $x < m$.

OK, let us go by contradiction:

Let

$$\mathbb{A} \ni a < m \qquad (ii)$$

By (i) I also have

$$m = a \vee m < a \qquad (iii)$$

Now, by irreflexivity, (ii) rules out $a = m$. So, (iii) nets $m < a$. (ii) and (iii) and transitivity yield $a < a$; contradiction ($<$ is irreflexive). Done.

(2) Let m and n both be minima in $\mathbb{A}$. Then $m \le n$ (with m posing as minimum) and $n \le m$ (now n is so posing), hence $m = n$ by antisymmetry (Lemma 3.1.63). □

3.1.74 Example. Let m be $<$-minimal in $\mathbb{A}$.

Let us attempt to "show" that it is also $<$-minimum (this is, of course, doomed to fail due to 3.1.71 and 3.1.73(2) —but the "faulty proof" below is interesting):

By 3.1.69 we have that there is no x in $\mathbb{A}$ such that $x < m$.

Another way to say this is:

For all $x \in \mathbb{A}$, I have the negation of "$x < m$", that is, I have $\neg x < m$. (1)

But from "our previous math" (high school? university? Netflix?) $\neg x < m$ is equivalent to $m \le x$.

Thus (1) says $(\forall x \in \mathbb{A})m \le x$, in other words, m is the minimum in $\mathbb{A}$.

Do you believe this? (Don't!) If the order is not total, then I can *fail to have* all three of $x < m, x = m, m < x$ and thus $\neg m < x$ and $x < m \vee x = m$ are *NOT* equivalent. See the counterexample to such expectation in 3.1.61 and also 3.1.71. □

3.1.75 Lemma. *If $<$ is a* linear *order on $\mathbb{A}$, then every minimal element is also minimum.*

Proof. The "false proof" of the previous example is valid under the present circumstances. □

The following type of relation has fundamental importance for set theory, and mathematics in general.

3.1.76 Definition.

1. An order $<$ satisfies the *minimal condition*, for short *it has MC*, iff *every* nonempty $\mathbb{A}$ has $<$-minimal elements.
2. If a *total* order $<: \mathbb{B} \to \mathbb{B}$ has MC, then it is called a *well-ordering*† *on* (or *of*) the class $\mathbb{B}$.

†The term "well-ordering" is ungrammatical, but it is *the* terminology established in the literature!

3. If $(\mathbb{B}, <)$ is a LO class (or set) with MC, then it is a *well-ordered class* (or set), or *WO class* (or WO set).

□

3.1.77 Remark.

What Definition 3.1.76 says in case 1. is —see (2) in 3.1.70— "*if, for some fixed order* $<$ *the following statement*

$$\emptyset \neq \mathbb{A} \to (\exists a \in \mathbb{A})\mathbb{A} \cap (a) >= \emptyset \tag{1}$$

is provable in set theory, for any $\mathbb{A}$*, then we say that* $<$ *has MC*".

The following observation is very important *for future reference*:

If $\mathbb{A}$ is given via a defining property $F(x)$, as

$$\mathbb{A} \stackrel{Def}{=} \{x : F(x)\}$$

then (1) translates —in terms of $F(x)$— into

$$(\exists a)F(a) \to (\exists a)\Big(F(a) \wedge \neg(\exists y)\big(y < a \wedge F(y)\big)\Big) \tag{2}$$

Conversely, for each formula $F(x)$ we get a class $\mathbb{A} = \{x : F(x)\}$ and thus —if $\mathbb{A}$ has MC with respect to $<$— we may express this fact as in (2) above.