We have just proved a theorem above:

2.3.2 Theorem. *If A, B are sets or atoms, then $\{A, B\}$ is a set.*

2.3.3 Exercise. Without referring to stages in your proof, prove that if A is a set or atom, then $\{A\}$ is a set. □

2.3.4 Remark. A very short digression into Boolean Logic —for now. It will be convenient —but not necessary; we are doing fine so far— to use *truth tables* to handle many simple situations that we will encounter where "logical connectives" such as "not", "and", "or", "implies" and "is equivalent" enter into our arguments.

We will put on record here how to compute things such as "S_1 and S_2", "S_1 implies S_2", etc., where S_1 and S_2 stand for two arbitrary statements of mathematics. In the process we will introduce the *mathematical symbols* for "and", "implies", etc.

The symbol translation table from English to symbol is:

NOT	$\neg$
AND	$\wedge$
OR	$\vee$
IMPLIES (IF...,THEN)	$\rightarrow$
IS EQUIVALENT	$\equiv$

The truth table below has a simple reading. For *all possible* truth values —true/false, for short **t**/**f**— of the "simpler" statements S_1 and S_2 we indicate the computed truth value of the compound (or "more complex)" statement that we obtain when we apply one or the other Boolean connective of the previous table.

S_1	S_2	$\neg S_1$	$S_1 \wedge S_2$	$S_1 \vee S_2$	$S_1 \rightarrow S_2$	$S_1 \equiv S_2$	$S_2 \rightarrow S_1$
f	**f**	**t**	**f**	**f**	**t**	**t**	**t**
f	**t**	**t**	**f**	**t**	**t**	**f**	**f**
t	**f**	**f**	**f**	**t**	**f**	**f**	**t**
t	**t**	**f**	**t**	**t**	**t**	**t**	**t**

Comment. All the computations of truth values satisfy our intuition, except perhapsthat for "$\rightarrow$": $\neg$ flips the truth value as it should, $\wedge$ is eminently consistent with common sense, $\vee$ is the "inclusive or" of the mathematician, and $\equiv$ is just equality on the set $\{\mathbf{f}, \mathbf{t}\}$, as it should be.

The "problem" with $\rightarrow$ is that there is no *causality* from left to right. The only "sane" entry is for $\mathbf{t} \rightarrow \mathbf{f}$. The outcome should be false for a "bad implication" and so it is. But look at it this way:

- Looking at $\rightarrow$ also in the "red column" see how the given table for $\rightarrow$ is eminently consistent with that for $\equiv$. Intuitively $\equiv$ is $\rightarrow$ from left to right AND $\rightarrow$ from right to left. It IS!

- This version of $\rightarrow$ goes way back to Aristotle. It is the version used by the vast majority of practising mathematicians and is nicknamed "material implication".

Practical considerations. Thus

1. if you want to demonstrate that $S_1 \vee S_2$ is true, for any component statements S_1, S_2, then show that *at least one* of the S_1 and S_2 is true.

2. If you want to demonstrate that $S_1 \wedge S_2$ is true, then show that *both* of the S_1 and S_2 are true.

   Note, incidentally, the if we *know* that $S_1 \wedge S_2$ is true, then the truth table guarantees that each of S_1 and S_2 *must* be true.

3. If now you want to show the implication $S_1 \rightarrow S_2$ is true, **then the only real work is to show that if we assume S_1 is true, then S_2 is true too**.

   If S_1 is known to be false, then no work is required to prove the implication because of the first two lines of the truth table!!

4. If you want to show $S_1 \equiv S_2$, then —because the last three columns show that this is equivalent to (same truth values as) $\Big(S_1 \rightarrow S_2\Big) \wedge \Big(S_2 \rightarrow S_1\Big)$— that is, you just prove **each** of the two implications $S_1 \rightarrow S_2$ and $S_2 \rightarrow S_1$

   An important variant of $\rightarrow$ and $\equiv$ Pay attention to this point since almost everybody gets it wrong! In the literature and in the interest of creating a usable shorthand many practitioners of mathematical writing use notation

   $$S_1 \rightarrow S_2 \rightarrow S_3 \tag{1}$$

   attempting to convey the meaning

   $$(S_1 \rightarrow S_2) \wedge (S_2 \rightarrow S_3) \tag{2}$$

   Alas, (2) is not the same as (1)! But what about $a < b < c$ ostensibly meaning $a < b \wedge b < c$? That is wrong too!

   Back to $\rightarrow$-chains like (1) vs. chains like (2): Take S_1 to be $\mathbf{t}$ (true), S_2 to be $\mathbf{f}$ and S_3 to be $\mathbf{t}$. Then (1) is true because in a chain using the same Boolean connective *we put brackets from right to left*: (1) is $S_1 \rightarrow (S_2 \rightarrow S_3)$ and evaluates to $\mathbf{t}$, while (2) evaluates clearly to false ($\mathbf{f}$) since $S_1 \rightarrow S_2 = \mathbf{f}$ and $S_2 \rightarrow S_3 = \mathbf{t}$.

So we need a special symbol to denote (2) "economically". We need a *conjunctional implies*! Most people use (correctly) $\Longrightarrow$ for that:

$$S_1 \Longrightarrow S_2 \Longrightarrow S_3 \tag{3}$$

that means, by **definition**, (2) above.

Similarly,

$$S_1 \equiv S_2 \equiv S_3 \tag{4}$$

is **NOT** conjunctional. It is **not** two equivalences —two statements— connected by an *implied* "$\wedge$", rather it says

$$S_1 \equiv (S_2 \equiv S_3)$$

Now if $S_1 = \mathbf{f}$, $S_2 = \mathbf{f}$ and $S_3 = \mathbf{t}$, then (4) evaluates as $\mathbf{t}$ but the conjunctional version

$$(S_1 \equiv S_2) \wedge (S_2 \equiv S_3) \tag{5}$$

evaluates as $\mathbf{f}$ since the second side of $\wedge$ is $\mathbf{f}$.

So how do we denote (5) correctly without repeating the consecutive S_2's and omitting the implied "$\wedge$"? This way:

$$S_1 \Longleftrightarrow S_2 \Longleftrightarrow S_3 \tag{4}$$

By definition, "$\Longleftrightarrow$" is conjunctional: It applies to two statements only —S_i and S_{i+1}— and implies an $\wedge$ before the adjoining next similar equivalence $S_{i+1} \Longleftrightarrow S_{i+2}$. □

2.3.5 Theorem. (The subclass theorem) *Let $\mathbb{A} \subseteq B$ (B a set). Then $\mathbb{A}$ is a set.*

Proof. Well, B being a set there is a stage Σ where it is built (Principle 1). By Principle 0, all members of B are available or built before stage Σ.

But by $\mathbb{A} \subseteq B$, all the members of $\mathbb{A}$ are among those of B.

Hey! By Principle 0 we can build $\mathbb{A}$ at stage Σ, so *it is a set.* □

Some corollaries are useful:

2.3.6 Corollary. (Modified comprehension I) *If for all x we have*

$$P(x) \rightarrow x \in A \tag{1}$$

for some set A, then $\mathbb{B} = \{x : P(x)\}$ is a set.

Proof. I will show that $\mathbb{B} \subseteq A$, that is,

$$x \in \mathbb{B} \rightarrow x \in A$$

Indeed (see 3 under **Practical considerations** in 2.3.4), let $x \in \mathbb{B}$. Then $P(x)$ is true, hence $x \in A$ by (1). Now invoke 2.3.5. □

2.3.7 Corollary. (Modified comprehension II) *If A is a set, then so is $\mathbb{B} = \{x : x \in A \wedge P(x)\}$ for any property $P(x)$.*

Proof. The defining property here is "$x \in A \wedge P(x)$". This implies $x \in A$ —by 2 in 2.3.4— that is, we have

$$(x \in A \wedge P(x)) \to x \in A$$

Now invoke 2.3.6. □

2.3.8 Remark. (*The* empty set) The class $\mathbb{E} = \{x : x \neq x\}$ has no members at all; it is empty. Why? Because

$$x \in \mathbb{E} \equiv x \neq x$$

but the condition $x \neq x$ is always false, therefore so is the statement

$$x \in \mathbb{E} \tag{1}$$

Is the class $\mathbb{E}$ a set?

Well, take $A = \{1\}$. This is a set as the atom 1 is given at stage 0, and thus we can construct the *set* A at stage 1.

Note that, by (1) and 3 in 2.3.4 we have that

$$x \in \mathbb{E} \to x \in \{1\}$$

is true (for all x). That is, $\mathbb{E} \subseteq \{1\}$.

By 2.3.5, $\mathbb{E}$ *is a set.*

But is it unique so we can justify the use of the definite article "the"? Yes. The specification of the empty set is a class with no members. So if D is another empty set, then we will have $x \in D$ always false. But then

$$x \in \mathbb{E} \equiv x \in D \text{ (both sides of } \equiv \text{ are false)}$$

and we have $\mathbb{E} = D$ by 2.1.1.

The *unique* empty set is denoted by the symbol $\emptyset$ in the literature. □

2.4. Operations on classes and sets

The reader probably has seen before (perhaps in calculus) the operations on sets denoted by $\cap, \cup, -$ and others. We will look into them in this section.

2.4.1 Definition. (Intersection of two classes) We define for any classes $\mathbb{A}$ and $\mathbb{B}$

$$\mathbb{A} \cap \mathbb{B} \stackrel{Def}{=} \Big\{x : x \in \mathbb{A} \wedge x \in \mathbb{B}\Big\}$$

We call the operator $\cap$ *intersection* and the result $\mathbb{A} \cap \mathbb{B}$ the intersection of $\mathbb{A}$ and $\mathbb{B}$.

If $\mathbb{A} \cap \mathbb{B} = \emptyset$ —which happens precisely when the two classes have no common elements— we call the classes *disjoint*.

It is meaningless to have $\cap$ operate on atoms.[†] □

We have the easy theorem below:

2.4.2 Theorem. *If B is a set, as its notation suggests, then $\mathbb{A} \cap B$ is a set.*

Proof. I will prove $\mathbb{A} \cap B \subseteq B$ which will rest the case by 2.3.5. So, I want

$$x \in \mathbb{A} \cap B \rightarrow x \in B$$

To this end, let then $x \in \mathbb{A} \cap B$ (cf. 3 in 2.3.4). This says that $x \in \mathbb{A} \wedge x \in B$ is true, so $x \in B$ is true (cf. 2 in 2.3.4). □

2.4.3 Corollary. *For sets A and B, $A \cap B$ is a set.*

2.4.4 Definition. (Union of two classes) We define for any classes $\mathbb{A}$ and $\mathbb{B}$

$$\mathbb{A} \cup \mathbb{B} \stackrel{Def}{=} \Big\{x : x \in \mathbb{A} \vee x \in \mathbb{B}\Big\}$$

We call the operator $\cup$ *union* and the result $\mathbb{A} \cup \mathbb{B}$ the union of $\mathbb{A}$ and $\mathbb{B}$.

It is meaningless to have $\cup$ operate on atoms. □

2.4.5 Theorem. *For any* sets *A and B, $A \cup B$ is a set.*

Proof. By assumption say A is built at stage Σ while B is built at stage Σ'. Without loss of generality (for short, "wlg") say Σ is no later than Σ', that is, $\Sigma \leq \Sigma'$.

By Principle 2 I can pick a state $\Sigma'' > \Sigma'$, thus

$$\Sigma'' > \Sigma' \tag{1}$$

and

$$\Sigma'' > \Sigma \tag{2}$$

Lets us examine any item $x \in A \cup B$:

I have two (not necessarily mutually exclusive) cases (by 2.4.4):

[†] The definition expects $\cap$ to *operate on classes.* As we know, atoms (by definition) *have no set/class structure* thus no class and no set is an atom.

- $x \in A$. Then x was available or built[†] at a stage $< \Sigma$,

$$\text{hence, by (2), } \underline{x \text{ is available } \textit{before } \Sigma''} \tag{3}$$

- $x \in B$. Then x was available or built at a stage $< \Sigma'$,

$$\text{hence, by (1), } \underline{x \text{ is available } \textit{before } \Sigma''} \tag{4}$$

In either case, (3) or (4), the arbitrary x from $A \cup B$ is built before Σ'', so we can collect all those x-values at stage Σ'' in order to form a *set*: $A \cup B$. □

2.4.6 Definition. (Difference of two classes) We define for any classes $\mathbb{A}$ and $\mathbb{B}$

$$\mathbb{A} - \mathbb{B} \stackrel{Def}{=} \Big\{ x : x \in \mathbb{A} \wedge x \notin \mathbb{B} \Big\}$$

We call the operator $-$ *difference* and the result $\mathbb{A} - \mathbb{B}$ the difference of $\mathbb{A}$ and $\mathbb{B}$, in that order.

It is meaningless to have "$-$" operate on atoms. □

2.4.7 Theorem. *For any* set A *and class* $\mathbb{B}$*,* $A - \mathbb{B}$ *is a set.*

Proof. The reader is asked to verify that $A - \mathbb{B} \subseteq A$. We are done by 2.3.5. □

Notation. The definitions of $\cap$ and "$-$" suggest a shorter notation for the rhs for $\mathbb{A} \cap \mathbb{B}$ and $\mathbb{A} - \mathbb{B}$. That is, respectively, it is common to write instead

$$\Big\{ x \in \mathbb{A} : x \in \mathbb{B} \Big\}$$

and

$$\Big\{ x \in \mathbb{A} : x \notin \mathbb{B} \Big\}$$

2.4.8 Exercise. Demonstrate —using Definition 2.4.1— that for any $\mathbb{A}$ and $\mathbb{B}$ we have $\mathbb{A} \cap \mathbb{B} = \mathbb{B} \cap \mathbb{A}$. □

2.4.9 Exercise. Demonstrate —using Definition 2.4.4— that for any $\mathbb{A}$ and $\mathbb{B}$ we have $\mathbb{A} \cup \mathbb{B} = \mathbb{B} \cup \mathbb{A}$. □

2.4.10 Exercise. By picking two particular very small sets A and B show that $A - B = B - A$ is not true for all sets A and B.

Is it true of all classes? □

Let us generalise unions and intersections next. First a definition:

[†] As x may be an atom, we allow the *possibility* that it was available *with no building involved*, hence we said "available or built". For A and B though we are told they are *sets*, so they *were* built at some stage, by Principle 1!

2.4.11 Definition. (Family of sets) A class $\mathbb{F}$ is called a *family of sets* iff it contains no atoms. The letter F is here used *generically* ("F" for family), and a family may be given any name, usually capital (blackboard bold if we have not said it is a set). □

2.4.12 Example. Thus, $\emptyset$ is a family of sets; the empty family.

So are $\{\{2\},\{2,\{3\}\}\}$ and $\mathbb{V}$, the latter given by

$$\mathbb{V} \stackrel{Def}{=} \Big\{x : x \text{ is a set}\Big\}$$

BTW, as $\mathbb{V}$ contains *all* sets (but *no* atoms!) it is a proper class! Why? Well, if it is a set, then it is one of the x-values that we are collecting, thus $\mathbb{V} \in \mathbb{V}$. But we saw that this statement is false for sets!

Here are some classes that are *not* families: $\{1\}$, $\{2,\{\{2\}\}\}$ and $\mathbb{U}$, the latter being the universe of *all objects* —sets and atoms— and equals Russell's "R" as we saw in Section 2.2. These all are disqualified as they contain atoms. □

2.4.13 Definition. (Intersection and union of families) Let $\mathbb{F}$ be a family of sets. Then

(i) the symbol $\bigcap \mathbb{F}$ denotes the class that contains *all the objects* that *are common to all* $A \in \mathbb{F}$.

In symbols the definition reads:

$$\bigcap \mathbb{F} \stackrel{Def}{=} \Big\{x : \text{for all } A, A \in \mathbb{F} \to x \in A\Big\} \tag{1}$$

(ii) the symbol $\bigcup \mathbb{F}$ denotes the class that contains *all the objects* that *are found among the various* $A \in \mathbb{F}$. That is, imagine that the members of *each* $A \in \mathbb{F}$ are "emptied" into a single —originally empty— container $\{\ldots\}$. The class we get this way is what we denote by $\bigcup \mathbb{F}$.

In symbols the definition reads (and I think it is clearer):

$$\bigcup \mathbb{F} \stackrel{Def}{=} \Big\{x : \text{for some } A, A \in \mathbb{F} \wedge x \in A\Big\} \tag{2}$$

□

2.4.14 Example. Let $\mathbb{F} = \{\{1\},\{1,\{2\}\}\}$. Then emptying all the contents of the members of $\mathbb{F}$ in some (originally) empty container we get

$$\{1, 1, \{2\}\} \tag{3}$$

This is $\bigcup \mathbb{F}$.

Would we get the same answer from the mathematical definition (2)? Of course:

1 *is* in some member of $\mathbb{F}$, indeed in both of the members $\{1\}$ and $\{1, \{2\}\}$, and in order to emphasise this I wrote two copies of 1 —it is empties/contributed twice. Then $\{2\}$ is the member that only $\{1, \{2\}\}$ of $\mathbb{F}$ contributes.

What is $\bigcap \mathbb{F}$? Well, only 1 is common between the two sets —$\{1\}$ and $\{1, \{2\}\}$— that are in $\mathbb{F}$. So, $\bigcap \mathbb{F} = \{1\}$. □

2.4.15 Exercise.

1. Prove that $\bigcup \Big\{A, B\Big\} = A \cup B$.

2. Prove that $\bigcap \Big\{A, B\Big\} = A \cap B$.

Hint. In each of part 1. and 2. show that lhs $\subseteq$ rhs and rhs $\subseteq$ lhs. For that analyse membership, i.e., "assume $x \in$ lhs and prove $x \in$ rhs", and conversely (cf. 2.1.1 and 2.1.2.) □

2.4.16 Theorem. *If the set F is a family of sets, then $\bigcup F$ is a set.*

Proof. Let F be built at stage Σ. Now,

$$x \in \bigcup F \equiv x \in \overset{\text{some}}{\overset{\downarrow}{A}} \in F$$

Thus x is available or built before A which is built before stage Σ since that is when F was built. x being arbitrary, all members of $\bigcup F$ are available/built before Σ, so we can build $\bigcup F$ as a *set* at stage Σ. □

2.4.17 Theorem. *If the class $\mathbb{F} \neq \emptyset$ is a family of sets, then $\bigcap \mathbb{F}$ is a set.*

Proof. By assumption there is some set in $\mathbb{F}$. Fix one such and call it D.

First note that

$$x \in \bigcap \mathbb{F} \rightarrow x \in D \tag{$*$}$$

Why? Because (1) of Definition 2.4.13 says that

$$x \in \bigcap \mathbb{F} \equiv \text{for all } A \in \mathbb{F} \text{ we have } x \in A$$

Well, D *is* one of those "A" sets in $\mathbb{F}$, so if $x \in \bigcap \mathbb{F}$ then $x \in D$. We established $(*)$ and thus we established

$$\bigcap \mathbb{F} \subseteq D$$

by 2.1.1. We are done by 2.3.5. □

2.4.18 Remark. What if $\mathbb{F} = \emptyset$? Does it affect Theorem 2.4.17? Yes, **badly**!
In Definition 2.4.13 we read

$$\bigcap \mathbb{F} \stackrel{Def}{=} \Big\{x : \text{for all } A, A \in \mathbb{F} \to x \in A\Big\} \qquad (**)$$

However, as the hypothesis (i.e., lhs) of the implication in $(**)$ is **false**, the implication itself is **true**. Thus the entrance condition "for all $A, A \in \mathbb{F} \to x \in A$" is true for *all* x and thus allows *ALL* objects x to get into $\bigcap \mathbb{F}$,

Thus $\bigcap \mathbb{F} = \mathbb{U}$, the universe of *all* objects which we saw that (cf. Section 2.2) it is a *proper class*. □

2.4.19 Exercise. What is $\bigcup F$ if $F = \emptyset$? Set or proper class? Can you "compute" which class exactly it is? □

2.4.20 Remark. (More notation)
Suppose the family of sets Q is a set of sets A_i, for $i = 1, 2, \ldots, n$ where $n \geq 3$.

$$Q = \{A_1, A_2, \ldots, A_n\}$$

Then we have a few alternative notations for $\bigcap Q$:

(a)
$$A_1 \cap A_2 \cap \ldots \cap A_n$$

or, more elegantly,

(b)
$$\bigcap_{i=1}^{n} A_i$$

or also

(c)
$$\textstyle\bigcap_{i=1}^{n} A_i$$

Similarly for $\bigcup Q$:

(i)
$$A_1 \cup A_2 \cup \ldots \cup A_n$$

or, more elegantly,

(ii)
$$\bigcup_{i=1}^{n} A_i$$

or also

(iii)

$$\bigcup\nolimits_{i=1}^{n} A_i$$

If the family has so many elements that *all* the natural numbers are needed to index the sets in the set family Q we will write

$$\bigcap_{i=0}^{\infty} A_i$$

or

$$\bigcap\nolimits_{i=0}^{\infty} A_i$$

or

$$\bigcap_{i\geq 0} A_i$$

or

$$\bigcap\nolimits_{i\geq 0} A_i$$

for $\bigcap Q$ and

$$\bigcup_{i=0}^{\infty} A_i$$

or

$$\bigcup\nolimits_{i=0}^{\infty} A_i$$

or

$$\bigcup_{i\geq 0} A_i$$

or

$$\bigcup\nolimits_{i\geq 0} A_i$$

for $\bigcup Q$ □

2.4.21 Example. Thus, for example, $A \cup B \cup C \cup D$ can be seeing —just changing the notation— as $A_1 \cup A_2 \cup A_3 \cup A_4$, therefore it means, $\bigcup\{A_1, A_2, A_3, A_4\}$, or $\bigcup\{A, B, C, D\}$.

Same comment for $\cap$. □

Pause. How come for the case for $n = 2$ we *proved*† $A \cup B = \bigcup\{A, B\}$ (2.4.15) but *here* we say ($n \geq 3$) that something like the content of the previous remark and example are *notation* (*definitions*)?

Well, we had *independent* definitions (and associated theorems re set status for each, 2.4.5 and 2.4.16) for $A \cup B$ and $\bigcup\{A, B\}$ so it makes sense to compare the two definitions after the fact and see if we can *prove* that they say the same thing. For $n \geq 3$ we opted to *NOT* give a definition for $A_1 \cup \ldots \cup A_n$ that is independent of $\bigcup\{A_1 \cup \ldots \cup A_n\}$, rather we gave the definition of the former in terms of the latter. No independent definitions, no theorem to compare the two!◀

†Well, *you* proved! Same thing :-)

2.5. The powerset

2.5.1 Definition. For any set A the symbol $\mathscr{P}(A)$ —pronounced the *powerset* of A— is defined to be the class

$$\mathscr{P}(A) \stackrel{Def}{=} \Big\{x : x \subseteq A\Big\}$$

Thus we collect *all* the subsets x of A to form $\mathscr{P}(A)$.

The literature most frequently uses the symbol 2^A in place for of $\mathscr{P}(A)$. □

(1) The term "power*set*" is slightly premature, but it is apt. Under the conditions of the definition —A a set— 2^A is a *set* as we prove immediately below.

(2) We said "*all* the subsets x of A" in the definition. This is correct. As we know from 2.3.5, if $\mathbb{X} \subseteq Y$ and Y is a set, then so is $\mathbb{X}$.

2.5.2 Theorem. *For any set A, its powerset $\mathscr{P}(A)$ is a set.*

Proof. Let A be built at stage Σ. Then each of its members y are given or built *before* Σ.

Thus, since *every* subset x of A is a set of y-values, **every such subset x can be built at stage Σ.**

But then, just take any $\Sigma' > \Sigma$. Since all x-values (such that $x \subseteq A$) are built *before* Σ', at stage Σ' we can collect them all and build the *set* 2^A. □

2.5.3 Example. Let $A = \{1, 2, 3\}$. Then

$$\mathscr{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{3, 2\}, \{1, 2, 3\}\Big\}$$

Thus the powerset of A has 8 elements.

We will later see that if A has n elements, for any $n \geq 0$, then 2^A has 2^n elements. This observation is at the root of the notation "2^A". □

2.5.4 Remark. For any set A it is trivial (verify!) that we have $\emptyset \subseteq A$ and $A \subseteq A$. Thus, for any A, $\{\emptyset, A\} \subseteq 2^A$. □

2.6. The ordered pair and finite sequences

To introduce the concepts of cartesian product —so that, in principle, plane analytic geometry can be developed within set theory— we need an object "(A, B)" that is *like* the set pair (2.3.1) in that it contains *two* objects, A and B ($A = B$ is a possibility), but in (A, B) order *and* length (here it is 2) matter!

> *We want* $(A, B) = (A', B')$ implies $A = A'$ *and* $B = B'$. *Moreover,* (A, A) *is not* $\{A\}$*! It is still an ordered* pair *but so happens that the first and second component, as we call the members of the ordered pair, are equal in this example.*

So, are we going to accept a new type of object in set theory? *Not at all*! We will build (A, B) so that it is a set!

2.6.1 Definition. (Ordered pair) *By definition*, (A, B) is the abbreviation (short name) given below:

$$(A, B) \stackrel{Def}{=} \Big\{A, \{A, B\}\Big\} \tag{1}$$

We call "(A, B)" an *ordered pair*, and A its first *component*, while B is its second component. □

2.6.2 Remark.

1. Note that $A \neq \{A, B\}$ and $A \neq \{A, A\}$, because in either case we would otherwise get $A \in A$, which is false for *sets or atoms* A. Thus (A, B) does contain exactly two members, or *has length 2*: A and $\{A, B\}$.

   Pause. We have *not* said in 2.6.1 that A and B are sets or atoms. So what right do we have in the paragraph above to so declare?◀

2. What about the desired property that

   $$(A, B) = (X, Y) \rightarrow A = X \wedge B = Y \tag{2}$$

   Well, **assume the lhs** of "→" in (2) and prove the rhs, "$A = X \wedge B = Y$". From our truth table we know that we do the latter by proving *each* of $A = X$ and $B = Y$ true (separately).

   The lhs that we assume translates to

   $$\Big\{A, \{A, B\}\Big\} = \Big\{X, \{X, Y\}\Big\} \tag{3}$$

   By the remark #1 above there are *two* distinct members in each of the two sets that we equate in (3).

   So since (3) is true (by assumption) we have (by definition of set equality) one of:

(a) $A = \{X, Y\}$ and $\{A, B\} = X$, that is, **1st listed element in lhs of "=" equals the 2nd listed in rhs; and 2nd listed element in lhs of "=" equals the 1st listed in rhs**.

(b) $A = X$ and $\{A, B\} = \{X, Y\}$.

Now case (a) above *cannot hold*, for it leads to $A = \{\{A, B\}, Y\}$. This in turn leads to

$$\{A, B\} \in A$$

and thus the set $\{A, B\}$ is built before of its member A, which contradicts Principle 0.

Let's then work with case (b).

We have

$$\{A, B\} = \{A, Y\} \tag{4}$$

Well, all the members on the lhs must also be on the rhs. I note that A is.

- What if B is also equal to A? Then we have $\{B\} = \{A, Y\}$ and thus $Y \in \{B\}$ (why?). Hence $Y = B$. We showed so far $A = X$ (listed in case (b)) and $B = Y$ (proved here); great!

- Here B is *not* equal to A. But B must be in the rhs of (4), so the only way is $B = Y$. *All Done*! □

Worth noting as a theorem what we proved above:

2.6.3 Theorem. *If* $(A, B) = (X, Y)$*, then* $A = X$ *and* $B = Y$*.*

But is (A, B) a set? (atom it is not, of course!) Yes!

2.6.4 Theorem. (A, B) *is a set.*

Proof. Now $(A, B) = \Big\{A, \{A, B\}\Big\}$. By 2.3.1, $\{A, B\}$ is set. Applying 2.3.1 once more, $\Big\{A, \{A, B\}\Big\}$ is a set. □

2.6.5 Example. So, $(1, 2) = \{1, \{1, 2\}\}$, $(1, 1) = \{1, \{1\}\}$, and $(\{a\}, \{b\}) = \{\{a\}, \{\{a\}, \{b\}\}\}$. □

2.6.6 Remark. We can extend the ordered pair to ordered *triple*, ordered *quadruple*, and beyond!

We take this approach in these notes:

$$(A, B, C) \stackrel{Def}{=} \Big((A, B), C\Big) \tag{1}$$

$$(A, B, C, D) \stackrel{Def}{=} \Big((A, B, C), D\Big) \tag{2}$$

$$(A, B, C, D) \overset{Def}{=} \Big((A, B, C), D\Big) \tag{3}$$

etc. So suppose we defined what an n-tuple is, for *some fixed unspecified* n, and denote it by $(A_1, A_2, \ldots, A_n)$ for convenience. Then

$$(A_1, A_2, \ldots, A_n, A_{n+1}) \overset{Def}{=} \Big((A_1, A_2, \ldots, A_n), A_n\Big) \tag{$*$}$$

This is an "*inductive*" or "*recursive*" definition, defining a concept ($n+1$-tuple) in terms of *a smaller instance of itself*, namely, in terms of the concept for an n-tuple, and in terms of the case $n = 2$ that we dealt with by *direct* definition (*not* in terms of the concept itself!) in 2.6.1.

Suffice it to say this "case of $n + 1$ in terms of case of n" provides just *shorthand notation* to take the mystery out of the red "etc." above. We **condense**/*codify* infinitely many definitions (1), (2), (3), ... into just **two**:

- 2.6.1

  and

- $(*)$

The reader has probably seen such recursive definitions before (likely in calculus and/or high school).

The most frequent example that occurs is to define, for any natural number n and any real number $a > 0$, what a^n means. One goes like this:

$$\begin{aligned} a^0 &= 1 \\ a^{n+1} &= a \cdot a^n \end{aligned}$$

The above condenses infinitely many definitions such as

$$\begin{aligned} a^0 &= 1 \\ a^1 &= a \cdot a^0 = a \\ a^2 &= a \cdot a^1 = a \cdot a \\ a^3 &= a \cdot a^2 = a \cdot a \cdot a \\ a^4 &= a \cdot a^3 = a \cdot a \cdot a \cdot a \\ &\vdots \end{aligned}$$

into just two!

We will study *inductive definitions* and *induction* soon!

Before we exit this remark note that $(A, B, C) = (A', B', C')$ implies $A = A', B = B', C = C'$ because it implies

$$C = C' \text{ and } (A, B) = (A', B')$$

That is, (A, B, C) is an **ordered** triple (3-tuple).

We can also prove that $(A_1, A_2, \ldots, A_n, A_{n+1})$ is an **ordered** $n+1$-tuple, i.e.,

$$(A_1, A_2, \ldots, A_{n+1}) = (A'_1, A'_2, \ldots, A'_{n+1}) \to A_1 = A'_1 \land \ldots \land A_{n+1} = A'_{n+1}$$

if we have followed the "etc." all the way to the case of $(A_1, A_2, \ldots, A_n)$. We will do the "etc."-argument *elegantly* once we learn induction! □

2.6.7 Definition. (Finite sequences) An n-tuple for $n \geq 1$ is called a finite sequence of length n, where we extend the concept to a *one element sequence* —**by definition**— to be

$$(A) \stackrel{Def}{=} A$$

□

Note that now we can redefine all sequences of lengths $n \geq 1$ using again $(*)$ above, but this time with starting condition that of 2.6.7. Indeed, for $n = 2$ we rediscover (A_1, A_2):

$$\text{the "new" 2-tuple pair: } (A_1, A_2) \stackrel{\text{by } (*)}{=} \Big((A_1), A_2\Big) \stackrel{\text{by 2.6.7 the "old"}}{=} \Big(A_1, A_2\Big)$$

The big red brackets are applications of the ordered pair defined in 2.6.1, just as it was in the general definition $(*)$.

2.7. The Cartesian product

We are ready to define classes of pairs.

2.7.1 Definition. (Cartesian product of classes) Let $\mathbb{A}$ and $\mathbb{B}$ be classes. Then we define

$$\mathbb{A} \times \mathbb{B} \stackrel{Def}{=} \Big\{(x, y) : x \in \mathbb{A} \land y \in \mathbb{B}\Big\}$$

The definition requires both sides of $\times$ to be classes. It makes no sense if one or both are atoms. □

2.7.2 Theorem. *If A and B are sets, then so is $A \times B$.*

Proof. By 2.7.1 and 2.6.1

$$A \times B = \Big\{\big\{x, \{x, y\}\big\} : x \in A \land y \in B\Big\} \qquad (1)$$

So, for each $\big\{x, \{x, y\}\big\} \in A \times B$ we have $x \in A$ and $\{x, y\} \subseteq A \cup B$, or $x \in A$ and $\{x, y\} \in 2^{A \cup B}$. Thus $\big\{x, \{x, y\}\big\} \subseteq A \cup 2^{A \cup B}$ and hence (changing notation) $(x, y) \in 2^{A \cup 2^{A \cup B}}$.

We have established that

$$A \times B \subseteq 2^{A \cup 2^{A \cup B}}$$

thus $A \times B$ is a set by 2.3.5, 2.4.5 and 2.5.2. □

2.7.3 Definition. Mindful of the Remark 2.6.6 where $\Big((A,B),C\Big)$, $\Big((A,B,C),D\Big)$, etc. were defined, we define here $A_1 \times \ldots \times A_n$ for any $n \geq 3$ as follows:

$$
\begin{aligned}
&A \times B \times C &&\overset{Def}{=} (A \times B) \times C \\
&A \times B \times C \times D &&\overset{Def}{=} (A \times B \times C) \times D \\
&\vdots \\
&A_1 \times A_2 \times \ldots \times A_n \times A_{n+1} &&\overset{Def}{=} (A_1 \times A_2 \times \ldots \times A_n) \times A_{n+1} \\
&\vdots
\end{aligned}
$$

$$\text{We may write } \bigtimes_{i=1}^{n} A_i \text{ for } A_1 \times A_2 \times \ldots \times A_n$$

If $A_1 = \ldots = A_n = B$ we may write B^n for $A_1 \times A_2 \times \ldots \times A_n$. □

2.7.4 Remark. Thus, what we learnt in 2.7.3 is, in other words,

$$\bigtimes_{i=1}^{n} A_i \overset{Def}{=} \Big\{(x_1, \ldots, x_n) : x_i \in A_i, \text{ for } i = 1, 2, \ldots, n\Big\}$$

and

$$B^n \overset{Def}{=} \Big\{(x_1, \ldots, x_n) : x_i \in B\Big\}$$

□

2.7.5 Theorem. *If A_i, for $i = 1, 2, \ldots, n$ is a set, then so is $\bigtimes_{i=1}^{n} A_i$.*

Proof. $A \times B$ is a set by 2.7.2. By 2.7.3, **and in this order**, we verify that so is $A \times B \times C$ and $A \times B \times C \times D$ and … and $A_1 \times A_2 \times \ldots \times A_n$ and … □

If we had inductive definitions available already, then Definition 2.7.3 would simply read

$$
\begin{aligned}
&A_1 \times A_2 &&\overset{Def}{=} \Big\{(x_1, x_2) : x_1 \in A_1 \wedge x_2 \in A_2\Big\} \\
&\quad \text{and, for } n \geq 2, \\
&A_1 \times A_2 \times \ldots \times A_n \times A_{n+1} &&\overset{Def}{=} (A_1 \times A_2 \times \ldots \times A_n) \times A_{n+1}
\end{aligned}
$$

Correspondingly, the proof of 2.7.5 would be far more elegant, via induction.

Bibliography

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