Chapter 1

Some Elementary Informal Set Theory

Set theory is due to Georg Cantor. "Elementary" in the title above does not apply to the body of his work, since he went into considerable technical depth in this, his new theory. It applies however to *our* coverage as we are going to restrict ourselves to elementary topics only.

Cantor made many technical mistakes in the process of developing set theory, some of considerable consequence. The next section is about the easiest and most fundamental of his mistakes.

How come he made mistakes? The reason is that his theory was not based on axioms and rigid rules of reasoning —a state of affairs for a theory that we loosely characterise as "informal".

At the opposite end of informal we have the *formal* theories that are based on axioms *and* logic and are thus "safer" to develop (they do not lead to *obvious* contradictions).

One *cannot* fault Cantor for not using logic in arguing his theorems —that process was not invented when he built his theory— but then, *a fortiori*, mathematical logic was not invented in Euclid's time either, *and yet* he did use axioms that stated how his building blocks, *points*, *lines* and *planes* interacted and behaved!

Guess what: Euclidean Geometry leads to no contradictions.

The problem with Cantor's set theory is that anything goes as to what sets are and how they come about. He neglected to ask the most fundamental question: "How are sets formed?"[†] He just sidestepped this and simply said that a *set* is any collection. In fact he took the term "set" as just a synonym for "collection", "class", "aggregate", etc.

[†]It's amazing how much trouble could be avoided if he had done so!

Failure to ask and answer this question leads to “trouble”, which is the subject matter of the next section.

One can still do “safe” set theory —devoid of “trouble”, that is— within an *informal* (non axiomatic) setting, but we have to ask and answer how sets are built *first* and derive from our answer some *principles* that will guide (and protect!) the theory’s development! We will do so.

1.1. Russell’s “Paradox”

Cantor’s *naïve* (this adjective is not derogatory but is synonymous in the literature with *informal* and *non axiomatic*) set theory was plagued by *paradoxes*, the most famous of which (and the *least* “technical”) being pointed out by Bertrand Russell and thus nicknamed “Russell’s paradox”.[†]

His theory is the theory of collections (i.e., sets) of objects, as we mentioned above, terms that were neither defined nor how they were built.[‡]

This theory studies operations on sets, properties of sets, and aims to use set theory as the foundation *of all mathematics*. Naturally, mathematicians “do” set theory of *mathematical object collections* —not collections of birds and other beasts. We have learnt some elementary aspects of set theory at high school. We will learn more in this course.

1. **Variables**. Like any theory, informal or not, informal set theory —a safe variety of which we will develop here— uses *variables* just as algebra does. There is only *one type* of variable that varies over set and over atomic objects too, the latter being objects that have no set structure. For example integers. We use the names $A, B, C, \ldots$ and $a, b, c, \ldots$ for such variables, sometimes with primes (e.g., A'') or subscripts (e.g., x_{23}), or both (e.g., x'''_{22}, Y'_{42}).

2. **Notation**. *Sets given by listing.* For example, $\{1, 2\}$ is a set that contains precisely the objects 1 and 2, while $\{1, \{5, 6\}\}$ is a set that contains precisely the objects 1 and $\{5, 6\}$. The braces $\{$ and $\}$ are used to show the collection/set by outright listing.

3. **Notation**. *Sets given by “defining property”.* But what if we cannot (or will not) explicitly list all the members of a set? Then we may define

[†]From the Greek word “paradoxo” (παράδοξο) meaning against one’s belief or knowledge; a contradiction.

[‡]This is not a problem *in itself.* Euclid too did not say *what* points and lines *were*; but his axioms did characterise their nature and interrelationships: For example, he started from these (among a few others) *a priori truths* (axioms): *a unique line passes through two distinct points*; also, *on any plane, a unique line l can be drawn parallel to another line k on the plane if we want l to pass through a given point A that is not on k.*

The point is:

You cannot leave out *both* what the nature of your objects is and *how* they behave/interrelate and get away with it! Euclid omitted the former but provided the latter, so all worked out.

what objects x get in the set/collection by having them to *pass an entrance requirement*, $P(x)$:

> **An object x gets in the set *iff* (*if and only if*) $P(x)$ is true of said object.**

Let us parse "iff":

(a) The *IF*: So, IF $P(x)$ is true, then x gets in the set (it passed the "admission requirement").

(b) The *ONLY IF*: So, IF x gets in the set, then the ***only*** *way for this to happen* is for it to pass the "admission requirement"; that is, $P(x)$ is true.

In other words, "iff" (as we probably learnt in high school or some previous university course such as calculus) is the same thing as "is equivalent":

"x is in the set" is equivalent to "$P(x)$ is true".

We denote the collection/set[†] defined by the entrance condition $P(x)$ by

$$\{x : P(x)\} \tag{1}$$

but also as

$$\{x \mid P(x)\} \tag{1'}$$

reading it "the set of all x *such that* (this "such that" is the ":" or "|") $P(x)$ is true [or holds]"

4. "$x \in A$" is the assertion that "object x is in the set A". Of course, this assertion may be true or false or "it depends", just like the assertions of algebra $2 = 2$, $3 = 2$ and $x = y$ are so (respectively).

5. $x \notin A$ is the negation of the assertion $x \in A$.

6. **Properties**

   - Sets are *named* by letters of the Latin alphabet (cf. **Variables**, above). Naming is pervasive in mathematics as in, e.g., "let $x = 5$" in algebra.
     So we can write "let $A = \{1, 2\}$" and let "$c = \{1, \{5, 6\}\}$" to give the names A and c to the two example sets above, ostensibly because we are going to discuss these sets, and refer to them often, and it is cumbersome to keep writing things like $\{1, \{5, 6\}\}$. Names are *not permanent*;[‡] they are *local* to a discussion (argument).

[†] We have not yet reached Russell's result, so keeping an open mind and humouring Cantor we still allow ourselves to call said collection a "set".

[‡] OK, there *are* exceptions: $\emptyset$ is the permanent name for the *empty set* —the set with no elements at all— and for that set only; $\mathbb{N}$ is the permanent name of the set of all *natural numbers*.

- **Equality of sets** (repetition and permutation do not matter!)

  Two sets A and B are equal iff they have the same members. Thus order and multiplicity do not matter! E.g., $\{1\} = \{1, 1, 1\}$, $\{1, 2, 1\} = \{2, 1, 1, 1, 1, 2\}$.

- The fundamental equivalence pertaining to definition of sets by "defining property": So, if we name the set in (1) above, S, that is, if we say "let $S = \{x : P(x)\}$", then "$x \in S$ iff $P(x)$ is true"

  By the way, we almost *never say* "is true" unless we want to shout out this fact. We would say instead: "$x \in S$ iff $P(x)$".

  Equipped with the knowledge of the previous bullet, we see that the symbol $\{x : P(x)\}$ defines a *unique* set/collection: Well, say A and B are so defined, that is, $A = \{x : P(x)\}$ and $B = \{x : P(x)\}$. Thus

$$x \in A \overset{A=\{x:P(x)\}}{\text{iff}} P(x) \overset{B=\{x:P(x)\}}{\text{iff}} x \in B$$

  thus

$$x \in A \text{ iff } x \in B$$

  and thus $A = B$.

Let us pursue, as Russell did, the point made in the last bullet above. Take $P(x)$ to be specifically the assertion $x \notin x$. He then gave a name to

$$\{x : x \notin x\}$$

say, R. But then, by the last bullet above,

$$x \in R \text{ iff } x \notin x \tag{2}$$

If we now *believe*,[‡] as *Cantor*, the father of set theory did not question and went ahead with it, that every $P(x)$ defines a *set*, *then R is a set.*

What is wrong with that?

Well, if R is a set then this object has the proper *type* to be plugged into the *variable of type "math object"*, namely, x, throughout the equivalence (2) above. But this yields the contradiction

$$R \in R \text{ iff } R \notin R \tag{3}$$

This contradiction is called the Russell's Paradox.

[‡]Informal mathematics often relies on "I know so" or "I believe" or "it is 'obviously' true". Some people call "proofs" like this —i.e., "proofs" without justification(s)— "proofs by intimidation". Nowadays, with the ubiquitousness of the qualifier "fake", one could also call them "fake proofs".

This and similar paradoxes motivated mathematicians to develop formal symbolic logic and look to axiomatic set theory[†] as a means to avoid paradoxes like the above.

Other mathematicians who did not care to use mathematical logic and axiomatic theories found a way to do set theory *informally*, yet *safely*.

See, they asked *and* answered "how are sets formed?"[‡]

Read on!

[†]There are many flavours or axiomatisations of set theory, the most frequently used being the "ZF" set theory, due to Zermelo and Fraenkel.

[‡]Actually, axiomatic set theory —in particular, its axioms are— is built upon the answers this group came up with. This story is told at an advanced level in [Tou03].

Bibliography

[Tou03] G. Tourlakis, *Lectures in Logic and Set Theory; Volume II: Set Theory*, Cambridge University Press, Cambridge, 2003.