COSC 4111/5111 —Winter 2008

Posted: March 14, 2008
Due: April 7, 2008

Problem Set No. 3

NB. *Each problem will receive a letter grade which will be used to find the problem set's "GPA".* The problem set list for grad students is the entire list here. Undergrads should omit the problems marked "Grad". If they wish to do some of those for extra credit the extra credit will be applied on an "all or nothing" basis. That is, **no part marks/grades will be given** for a "Grad" problem attempted by undergrads.

This is not a course on *formal* recursion theory. Your proofs should be informal (but $\neq$ sloppy), correct, and informative (and if possible short). Please do not trade length for correctness or readability.

(1) **(Grad)** "W_i" is the symbol Rogers uses for "the i-th semi-recursive set", that is,

$$x \in W_i \equiv (\exists z)T(i, x, z)$$

Question: Is there a partial recursive function $\lambda x.f(x)$ such that for all i

$$W_i \neq \emptyset \Rightarrow f(i) \downarrow \wedge f(i) = \min\{y : y \in W_i\}$$

If you think that "yes", then you **must** give a proof.

If you think that "no", then you **must** give a definitive counterexample.

(2) **Without using Rice's theorem or lemma**, explore/prove

(a) the set $A = \{x : \mathrm{ran}(\phi_x)$ has exactly five distinct elements$\}$ is not recursive. (I.e., "$x \in A$ is unsolvable"). Is it r.e.? **Why?**

(b) the set $D = \{x : \phi_x$ is the characteristic function of some set$\}$ is not recursive. Is it r.e.? **Why?**.

(c) the set $E = \{x : \mathrm{ran}(\phi_x)$ contains only odd numbers$\}$ is not recursive. Is it r.e.? **Why?**

(3) Prove that if the graph of a function is r.e. then the function is in $\mathcal{P}$.

(4) Is the "proof" given below for the above question correct? If not, **where exactly** does it go wrong?

Proof. Let $y = f(\vec{x}_n)$ be r.e. Then $y = f(\vec{x}_n) \equiv \psi(y, \vec{x}_n) = 0$ for some $\psi \in \mathcal{P}$. Thus $g = \lambda\vec{x}_n.(\mu y)\psi(y, \vec{x}_n)$ is in $\mathcal{P}$. But $g = f$, since the unbounded search finds the y that makes $y = f(\vec{x}_n)$ true, if $f(\vec{x}_n) \downarrow$. Thus, $f \in \mathcal{P}$. $\Box$

(5) Let

$$f = \lambda x.\text{if } f_R(x) = 0 \text{ then } g(x) \text{ else if } f_Q(x) = 0 \text{ then } h(x) \text{ else } \uparrow$$

where R, Q are r.e. (and mutually exclusive), and g, h, f_R, f_Q are partial recursive, and $R(x) \equiv f_R(x) = 0$ and $Q(x) \equiv f_Q(x) = 0$.

Is f partial recursive? **Why?**

Is f' below the same as f? **Why?**

$$f'(x) = \begin{cases} g(x) & \text{if } R(x) \\ h(x) & \text{if } Q(x) \\ \uparrow & \text{otherwise} \end{cases}$$

If you answered **no**, is f' partial recursive? **Why?**

(6) **Chapter 13** problems 23, 26, (**Grad, 30**), (**Grad, 45**).

(7) Prove that $T \in \mathcal{E}_*^3$.

Hint. Systematically scan the proof that $T \in \mathcal{PR}_*$ contained in the Kleene Notes and modify it to obtain this sharper result.