COSC 4111 3.0/5111 3.0—Fall 2004

Posted: Nov. 19, 2004
Due: End of term [Exact date TBA]

Problem Set No. 3

(1) "W_i" is the symbol Rogers uses for "the i-th semi-recursive set", that is,

$$x \in W_i \equiv (\exists z) T(i, x, z)$$

Question: Is there a partial recursive function $\lambda x.f(x)$ such that for all i

$$W_i \neq \emptyset \Rightarrow f(i) \downarrow \wedge f(i) = \min\{y : y \in W_i\}$$

If you think that "yes", then you **must** give a proof.

If you think that "no", then you **must** give a definitive counterexample.

(2) **Without using Rice's theorem**, prove that

(a) the set $A = \{x : \mathrm{ran}(\phi_x) \text{ has exactly five distinct elements}\}$ is not recursive. (I.e., "$x \in A$ is unsolvable").

(b) the set $B = \{x : \phi \text{ is 1-1}\}$ is not recursive. (I.e., "$x \in B$ is unsolvable").

(c) the set $C = \{x : \phi_x \text{ is onto}\}$ is not recursive. (I.e., "$x \in C$ is unsolvable").

(d) the set $D = \{x : \phi_x \text{ is the characteristic function of some set}\}$ is not r.e.

(e) the set $\mathbb{N} - D$ (i.e., $\overline{D}$) is not r.e. either.

(f) the set $E = \{x : \mathrm{ran}(\phi_x) \text{ contains only prime numbers}\}$ is not r.e.

(3) (a) (Eaaasyyy!) Give a **careful and complete** proof (no hand-waiving!) that if $\lambda y\vec{x}_n.y = f(\vec{x}_n)$ is in $\mathcal{P}_*$ then $f \in \mathcal{P}$

(b) (A bit tricky) Is the "proof" given below for the above question correct? If not, **where exactly** does it go wrong?

Proof. Let $y = f(\vec{x}_n)$ be r.e. Then $y = f(\vec{x}_n) \equiv \psi(y, \vec{x}_n) = 0$ for some $\psi \in \mathcal{P}$. Thus $g = \lambda\vec{x}_n.(\mu y)\psi(y, \vec{x}_n)$ is in $\mathcal{P}$. But $g = f$, since the unbounded search finds the y that makes $y = f(\vec{x}_n)$ true, if $f(\vec{x}_n) \downarrow$. Thus, $f \in \mathcal{P}$. $\square$