math1090
Introduction to Logic for Computer Science
Lecture 9

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Adequate sets of connectives

Truth function

Definition A truth function is a function from $\{T, F\}^n$ to $\{T, F\}$ for some natural number n.

Truth function–example

Every propositional well formed formula can be considered as a truth function. Here, we will avoid being overly formal and just illustrate this by an example.

Let's consider the truth table of the formula $\varphi = ((\neg p_1) \vee (\neg p_2))$:

p_1	p_2	$\varphi = ((\neg p_1) \vee (\neg p_2))$
T	T	F
T	F	T
F	T	T
F	F	T

We see that we can view the formula φ as a function with the following behavior:

$$\begin{gathered}\varphi(T, T) = F \\ \varphi(T, F) = T \\ \varphi(F, T) = T \\ \varphi(F, F) = T\end{gathered}$$

Adequate set of connectives

Definition A set of truth functions C is called adequate every truth function can be expressed as a propositional formula for over the set C.

Adequate set of connectives

Claim
The set of connectives $\{\neg, \wedge, \vee, \rightarrow\}$ is adequate.

Adequate set of connectives

Again, to avoid becoming overly formal, we will illustrate this by an example. We consider a truth table over $\{T, F\}^3$, that is a function that maps triples of truth values to T or F. The first three columns represent all possible triples (note that there are $2^3 = 8$ rows for all possible triples), and the last column represents the value of the function on these triples (recall that in class, we had a student flip a coin to generate some arbitrary function f and thus to illustrate that this construction does not depend on the particular function f chosen).

p_1	p_2	p_3	f
T	T	T	T
F	T	T	F
T	F	T	F
F	F	T	T
T	T	F	F
F	T	F	F
T	F	F	F
F	F	F	T

To find a formula that has this exact truth table, we first note which rows obtain value true (here in red). We then encode that, for the formula to be true, one of these (those rows) settings of the variables must be the case. That is, the formula for this truth table would say

"the variables need to be as in the first or in the forth or in the last row."

As a formula, this is encoded as:

$$((p_1 \wedge p_2 \wedge p_3) \vee ((\neg p_1) \wedge (\neg p_2) \wedge p_3) \vee ((\neg p_1) \wedge (\neg p_2) \wedge (\neg p_3)))$$

(Note that we are omitting some brackets here for readability, so that the rows are more easily identified). It is not hard to see that this construction of a formula for a given truth table would work for any arbitrary truth table.

Adequate set of connectives

Actually, we have even shown this stronger statement:

Claim

The set of connectives $\{\neg, \wedge, \vee\}$ is adequate.

Note that in the above construction, the connective $\rightarrow$ was not used. Thus, the smaller set $\{\neg, \wedge, \vee\}$ is adequate. This means that, in principle, we could have gotten away with less connectives and still have had the same expressive power of the the possible formulas. However, we still included $\rightarrow$ in our language (definition of well formed formula) to allow for a more intuitive translation between language (the way we think) and propositional formulas.

Adequate set of connectives

Claim

The set of connectives $\{\wedge, \vee, \rightarrow\}$ is not adequate.

Adequate set of connectives

Proof of claim
To show that a set of connectives is not adequate, we use a proof by induction to show that formulas over the set $\{\wedge, \vee, \rightarrow\}$ have a property that formulas over an adequate set can not have. We will prove the following property:

- Any formulas over $\{\wedge, \vee, \rightarrow\}$ will evaluate to true if all propositional variables used in the formula are set to true.

Note that for an adequate set of connectives, we need to be able to represent any function, also one that would map $(T, T, \ldots T)$ to false, when viewed as a function. Thus, proving the above property will show that the set in question here is not adequate.
We will for now denote the set of well formed formulas that use only the connectives $\{\wedge, \vee, \rightarrow\}$ by PF. Further, we denote the the truth assignment that maps every propositional variable to true by v_T.

Base case: Let $\alpha \in \mathrm{PF}$ be atomic. Then α consists of only one single propositional variable, and if this variable is set to T by a truth assignment α evaluates to T, that is $v_T(\alpha) = T$.

Induction hypothesis: We assume that for some $\alpha_1, \alpha_2 \in \mathrm{PF}$ we have

$$v_T(\alpha_1) = T \quad \text{and} \quad v_T(\alpha_2) = T$$

Induction step: Given the induction hypothesis, we get

$$v_T((\alpha_1 \wedge \alpha_2)) = T \quad \text{and} \quad v_T((\alpha_1 \vee \alpha_2)) = T \quad \text{and} \quad v_T((\alpha_1 \rightarrow \alpha_2)) = T$$

This concludes the proof. We have shown that for all formulas α in PF, we have $v_T(\alpha) = T$.

Adequate set of connectives

Claim

The set of connectives $\{\neg, \vee\}$ is adequate.

Adequate set of connectives

Proof of claim We have already seen that the set $\{\neg, \wedge, \vee\}$ is adequate. To show that even the smaller set $\{\neg, \vee\}$ is adequate, we need to express the $\wedge$-connective with these two other connectives. The following truth table shows that this is possible:

p	q	$(p \wedge q)$	$(\neg((\neg p) \vee (\neg q)))$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Task for you: verify that this truth table is correct by drawing the full truth table for the given formula.

Back to syntax–formal proofs

The Hilbert proof system

Axioms

AI $(\alpha \rightarrow (\beta \rightarrow \alpha))$.

AII $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

AIII $((\neg\alpha) \rightarrow (\neg\beta)) \rightarrow (\beta \rightarrow \alpha)$

Deduction rule

Modus ponens: $(\alpha, (\alpha \rightarrow \beta)) \mapsto \beta$