math1090
Introduction to Logic for Computer Science
Lecture 7 and 8

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Some more semantics

Logical implication

Claim

If α is a contradiction, then for every formula β, we have

$$\alpha \models \beta$$

Logical implication

Proof of claim

This is an example of a claim that is vacuously true.

To explain this in some more detail: Let's assume that α is a contradiction and β some arbitrary well formed formula. We need to show that every truth assignment v, which satisfies α, that is for which $v(\alpha) = T$, also satisfies β, that is $v(\beta) = T$. However, since α is a contradiction, there do not exist any truth assignments v for which $v(\alpha) = T$. Thus, every truth assignment that satisfies α also satisfies β.

Satisfiability of a set of formulas

Definition

Let Γ be a set of formulas. We say that Γ is satisfiable if there exists a truth assignment v such that

$$v(\gamma) = T$$

for every $\gamma \in \Gamma$.

Satisfiability of a set of formulas–examples

- Let $\Gamma = \{(p \wedge (\neg q)), (q \rightarrow p), ((\neg q) \vee (\neg p))\}$.
  This set is satisfiable. All formulas evaluate to true for the truth assignment v with $v(p) = T$ and $v(q) = F$.
  One option to check whether a (finite) set of formulas is satisfiable, is to draw the full truth table for all formulas, and check whether there exists at least one row for which all formulas evaluate to T.

p	q	$(p \wedge (\neg q))$	$(q \rightarrow p)$	$((\neg q) \vee (\neg p))$
T	T	F	T	F
T	F	T	T	T
F	T	F	F	T
F	F	F	T	T

We call a truth assignment that satisfies all formulas in Γ also a certificate for the satisfiability of Γ.

Satisfiability of a set of formulas–examples

- Now we consider $\Gamma = \{(p_1 \to p_2), (p_2 \to p_3), (p_3 \to p_4)\}$.
  This set is also satisfiable. There are several truth assignments that satisfy all formulas in Γ and can therefore serve as certificates for the satisfiability of Γ. Examples are:
  - v_T the truth assignment that sets every variable to T.
  - v_F the truth assignment that sets every variable to F.
  - v_1 that sets $v_1(p_1) = F$ and $v_1(p_j) = T$ for $j = 2, 3, 4$.

Satisfiability of a set of formulas–examples

- So far, we have only looked at finite sets Γ (that is sets, that contain only a finite number of formulas). Let's now consider an infinite set, for example $\Gamma = \{(p_i \to p_{i+1}) \mid i \in \mathbb{N}\}$.
  This set is satisfiable as well. There are infinitely many truth assignments that satisfy Γ and are therefore certificates of its satisfiability (note that to show that Γ is satisfiable the existence of one such certificate is enough). Γ is satisfied by:
  - v_T the truth assignment that sets every variable to T.
  - v_F the truth assignment that sets every variable to F.
  - v_k that sets $v_1(p_j) = F$ for $j \leq k$ and $v_1(p_j) = T$ for $j > k$.
    (Note that there are infinitely many such assignments v_k, one for every natural number k.)

Logical implication for a set of formulas

Definition

Let Γ be a set of formulas. We say that the set Γ logically implies a formula β if for every truth assignment v for which $v(\gamma) = T$ for every $\gamma \in \Gamma$, we also have $v(\beta) = T$.

To denote that Γ logically implies β, we write

$$\Gamma \models \beta$$

Logical implication for a set of formulas–examples

- Let's start with the same example as above. Let
  $\Gamma = \{(p \wedge (\neg q)), (q \to p), ((\neg q) \vee (\neg p))\}$.
  We saw that the only truth assignment that satisfies all formulas in Γ is the truth assignment v with $v(p) = T$ and $v(q) = F$. This truth assignment also satisfies $\beta = (p \vee q)$.

p	q	$(p \wedge (\neg q))$	$(q \to p)$	$((\neg q) \vee (\neg p))$	$\beta = (p \vee q)$
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	F	T	T	F

Thus, every truth assignment that satisfies all of Γ (there is only one such truth assignment in this case), also satisfies β. That is, we established that Γ logically implies β:

$$\Gamma \models \beta$$

Logical implication for a set of formulas–examples

- Let's again consider the infinite set from above, $\Gamma = \{(p_i \rightarrow p_{i+1}) \mid i \in \mathbb{N}\}$.
  It is not difficult to see (try to prove this yourself!) that Γ is exactly satisfied by the truth assignments:
  - v_T the truth assignment that sets every variable to T.
  - v_F the truth assignment that sets every variable to F.
  - v_k that sets $v_1(p_j) = F$ for $j \leq k$ and $v_1(p_j) = T$ for $j > k$.

  I.e., these truth assignments satisfy Γ and no other truth assignment satisfies Γ.

  Now we analyze various example formulas:
  - $\beta_1 = (p_1 \wedge p_2)$. Since v_F satisfies Γ but not β_1, $\Gamma \not\models \beta_1$.
  - $\beta_2 = (p_2 \rightarrow p_5)$. We see that β_2 is satisfied by v_T, v_F and all v_k, hence $\Gamma \models \beta_2$.
  - $\beta_3 = (p_5 \rightarrow p_2)$. Since v_4 satisfies Γ but not β_3, $\Gamma \not\models \beta_3$.

Logical implication for a set of formulas

Claim

If a set of formulas Γ contains only one formula γ (ie. $\Gamma = \{\gamma\}$), then

$$\Gamma \models \beta \text{ if and only if } \gamma \models \beta$$

Task: Make it clear to yourself why this is the case!

Logical implication for a set of formulas

Claim
If a set of formulas Γ contains a contradiction, then for every formula $\beta \in \mathrm{WFF}$, we have

$$\Gamma \models \beta$$

Task: This is again an example of a statement that is vacuously true. Make it clear to yourself why this is the case!

Logical implication for a set of formulas

Claim

If Γ is a finite set of formulas, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n\}$, then

$$\Gamma \models \beta \text{ if and only if } ((\ldots(\gamma_1 \wedge \gamma_2) \wedge \gamma_3) \ldots \wedge \gamma_n) \models \beta$$

Task: Make it clear to yourself why this is the case!