math1090
Introduction to Logic for Computer Science
Lecture 6

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Some more semantics

Tautologies and contradictions

Claim

A formula α is a tautology if and only if $(\neg\alpha)$ is a contradiction.

Question:

- Is every contradiction β of the form $\beta = (\neg\alpha)$ for some tautology α?
- Is every tautology α of the form $\alpha = (\neg\beta)$ for some contradiction β?

Tautologies and contradictions–proof of claim

Claim
A formula α is a tautology if and only if $(\neg\alpha)$ is a contradiction.

Proof
Since this is an "if and only if" statement, we need to show two directions.

1. First we show that α being a tautology implies that $(\neg\alpha)$ is a contradiction. So let's assume that α is a tautology. Then, for every truth assignment v, we have $v(\alpha) = T$. Inspecting the truth table of the $\neg$ connective, we see that this means that for every truth assignment $v((\neg\alpha)) = F$. This, in turn, is exactly the definition of $(\neg\alpha)$ being a contradiction.
2. Now we assume that $(\neg\alpha)$ is a contradiction, and we need to show that then α is a tautology. If $(\neg\alpha)$ is a contradiction, then for every truth assignment v, we have $v((\neg\alpha)) = F$. Inspecting the truth table of the $\neg$ connective, we see that this means that for every truth assignment $v(\alpha) = T$. This means that α is a tautology, by definition.

Logical implication between formulas

Definition
We say that a formula α logically implies a formula β if for every truth assignment v for which $v(\alpha) = T$, we also have $v(\beta) = T$.

To denote that α logically implies β, we write

$$\alpha \models \beta$$

Logical implication between formulas–examples

Let $\alpha = (p \wedge q)$ and $\beta = (p \vee q)$.

To find our whether α logically implies β, we can inspect the truth table for these formulas:

p	q	$(p \wedge q)$	$(p \vee q)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

We see that in every row (that is, in every truth assignment) where α evaluates to true (this is only the first row), β also evaluates to true. Thus

$$\alpha \models \beta$$

Logical implication between formulas–examples

We now consider the (almost) same example with the roles of the two formulas reversed. Again, we let $\alpha = (p \wedge q)$ and $\beta = (p \vee q)$.

p	q	$(p \wedge q)$	$(p \vee q)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Looking at the same truth table, we see that β does not logically imply α, since there exists a a row (that is, truth assignment), for example the second one, where β is true but α is not.

$$\beta \not\models \alpha$$

Logical equivalence

Definition
We say that a formula α and a formula β are logically equivalent if each logically implies the other.

To denote that α and β are logically equivalent, we write

$$\alpha \equiv \beta$$

Logical equivalence–example

Let $\alpha = (p \to q)$ and $\beta = ((\neg q) \to (\neg p))$.

To find our whether α and β are logically equivalent, we draw a truth table

p	q	$(p \to q)$	$(\neg q)$	$(\neg q)$	$((\neg q) \to (\neg p))$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Since both formulas evaluate exactly to the same truth values for every truth assignment (the two columns for the formulas in questions are identical), these formulas α and β are logically equivalent.

This logical equivalence is often used in mathematical proofs. We will actually use it in the proof of the next claim.

Logical implication

Claim

For propositional formulas α and β, we have

$$\alpha \models \beta \text{ if and only if } (\alpha \rightarrow \beta) \text{ is a tautology}$$

Proof of claim

Since the claim is an "if and only if" statement we need to prove two directions. We now state the two directions before actually writing the proof. We need to show that

1. If $\alpha \models \beta$ then $(\alpha \to \beta)$ is a tautology, and
2. If $(\alpha \to \beta)$ is a tautology then $\alpha \models \beta$.

We now separately prove these two statements:

1. Instead of showing the first statement directly, we make use of the logical equivalence that we just learned about (namely that $(p \to q)$ is logically equivalent to $((\neg q) \to (\neg p))$). Applying this to our statement, we see that we may as well prove the following:

   - If$(\alpha \to \beta)$ is not a tautology then α does not logically imply β.

   So, let's assume that $(\alpha \to \beta)$ is not a tautology. Then there exists at least one truth assignment v, with $v((\alpha \to \beta)) = F$. Now looking at the truth table for the $\to$-connective, we see that this can only be the case if for this v, we have $v(\alpha) = T$ and $v(\beta) = F$. But now, the existence of this v shows that α does not logically imply β.

2. Now let's assume that $(\alpha \to \beta)$ is a tautology. We again inspect the truth table of the $\to$-connective:

α	β	$(\alpha \to \beta)$
T	T	T
T	F	F
F	T	T
F	F	T

   We see that the truth assignment that corresponds to the second row, can not happen, since $(\alpha \to \beta)$ is a tautology. Thus ,we are left with a smaller truth table:

α	β	$(\alpha \to \beta)$
T	T	T
F	T	T
F	F	T

   Now we see that here, for every truth assignment for which α is true, β is also true. This is exactly the definition of α logically implying β which is what we needed to show.