math1090
Introduction to Logic for Computer Science

Lecture 5

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Some more syntax

Immediate predecessor of a formula

We define the immediate predecessors of a formula $\alpha \in \mathrm{WFF}$ as follows:

- If α is atomic, then it has no immediate predecessors.
- If $\alpha = o_{\neg}(\beta) = (\neg\beta)$, then β is an immediate predecessor of α.
- If $\alpha = o_{*}(\beta, \gamma) = (\beta * \gamma)$, then β and γ are immediate predecessors of α.

Immediate predecessor of a formula– examples

Let $\alpha = ((p \to q) \vee (q \to p))$. Then the immediate predecessors of α are:

- $(p \to q)$
- $(q \to p)$

Both $(p \to q)$ and $(q \to p)$ have as their immediate predecessors:

- p
- q

Unique readability of a formula

The next Theorem we will prove for WFF will explain, why we are putting so much emphasis on the correct placement of brackets for well formed formulas.
Imagine some curriculum rule that said, you need to take (course numbers are made up..)

$$\text{math1090} \ \wedge \ \text{eecs1019} \ \vee \ \text{eecs2018}$$

If the rule is stated like this, it is not clear whether you need to take

- both math1090 and eecs1019; or alternatively could take course eecs2018

OR you'd need to take

- math1090 and then also one of eecs1019 or eecs2018

We do not want such ambiguities in our well formed formulas (recall, that the goal was to develop a formal language with rules for how to get from assumptions to conclusions, and that we dismissed English, since it allowed for ambiguities and paradoxes..).
The next result shows that such ambiguities can indeed not happen the way we set up WFF, and this is due to the correct placement of brackets.

Unique readability of a formula

Theorem

• The immediate predecessors of a formula α are uniquely determined.

• Furthermore, if α has two immediate predecessors, then the connective between them (that is the last operation to obtain α) is uniquely determined.

Proof

The following proof is similar to the proof of Theorem 1.2.5 in the textbook. There are several cases for the formula α to consider:

1. If α is atomic, then α has no immediate predecessors. Thus, there is nothing to show. In all other cases, we know from one of the properties that we showed for WFF that α starts with an opening bracket "(". One can also (similarly) show, that α will end with a closing bracket ")" if it is not atomic.
2. If $\alpha = (\neg\beta)$ for some string β, then we note that the second symbol is $\neg$. If we also had $\alpha = o_*(\phi, \gamma)$ for some $\phi, \gamma \in \mathrm{WFF}$, then the second symbol would be an opening bracket (. Thus, we can exclude those cases. α must have been constructed by applying $o_\neg$ to some formula, which then must have been β. Thus the immediate predecessor is uniquely determined in this case and it is β. Further, the last operation was $o_\neg$.
3. We now consider the case that $\alpha = o_*(\beta, \gamma) = (\beta * \gamma)$ for some $* \in \{\wedge, \vee, \rightarrow\}$. We need to show that it can not be the case that $\alpha = o_\circ(\phi, \psi) = (\phi \circ \psi)$ for some $\circ \in \{\wedge, \vee, \rightarrow\}$ and $\circ \neq *$ being different symbols. Further, we need to argue that $\beta = \phi$ and $\gamma = \psi$.
   We assume for now β was not equal to ϕ[1]. Then one of them must be a proper initial segment of the other. We will assume β is a proper initial segment of ϕ[2]. Now, since β is a proper initial segment of $\phi \in \mathrm{WFF}$, β has more opening than closing brackets. But, on the other hand, since $\beta \in \mathrm{WFF}$, the number of left and right brackets in β is equal, a contradiction. Thus, since we arrived at a contradiction, the assumption that β was not equal to ϕ must have been wrong.
   In summary, we have shown that $\beta = \phi$. Then, by just counting symbols, we can also conclude that $* = \circ$ and $\gamma = \psi$. Thus, the immediate predecessors and the last operation are uniquely determined.

[1] This way of arguing in proofs is often termed "by way of contradiction", b.w.o.c. for short; the idea is to assume the opposite of what we are aiming to prove and show that this assumption leads to a contradiction; once we arrive at a contradiction, we know that assumption was wrong and the opposite, namely what we were aiming to prove, must be true.

[2] This way of arguing is typically termed "without loss of generality", since the other case, namely ϕ being a proper initial segment of β, can be treated in the exact same way.

Semantics

Semantics of propositional logic

The semantics of propositional logic use two truth values: true and false.
We will let T stand for true and F stand for false.

Recall the the purpose of logic is not to determine what is true and what is false, but rather two give rules on how to derive the truth value of some statement (eg some propositional formula) based on the truth value of assumptions (these are the propositional variables).

Truth assignments

Let P be some set of propositional variables.

- Eg $P = \{p, q, r\}$ or $P = \{p_i \mid i \in \mathbb{N}\} = \{p_1, p_2, p_3, \ldots\}$
- Recall that we can then consider the set $I(\Sigma^*, P, O)$ of well formed formulas using the propositional variables in P. If we want to stress the set P used, we will refer to this set as WFF_P. Otherwise, we will simply use the shorter notation WFF, and assume that the set of propositional variables used is clear from context.

A truth assignment (also called a valuation) is a function that maps every propositional variable to either true or false, that is

$$v : P \to \{T, F\}$$

Extending truth assignments to WFF

Given a truth assignment to the propositional variables, we extend it to the set of all well formed formulas using their inductive definition.

- If α is atomic, eg $\alpha = p$, then $v(\alpha) = v(p)$.
- If $\alpha = (\neg\beta)$, then $v(\alpha) = T$ if $v(\beta) = F$ and vice versa.
- If $\alpha = (\beta * \gamma)$ for $* \in \{\wedge, \vee, \rightarrow\}$, look up $v(\alpha)$ in truth table (on next slide) for the correct connective.

Truth tables

The following table defines the truth value of formulas obtain by applying one of the connectives:

α	β	$(\neg\alpha)$	$(\alpha \vee \beta)$	$(\alpha \wedge \beta)$	$(\alpha \to \beta)$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

Extending truth assignments to WFF

Note that the truth value of a formula α as defined above is uniquely determined, since we have shown that the predecessors of α are uniquely determined.

Extending truth assignments to WFF – examples

Say, we have $P = \{p, q\}$, and a formula $\alpha = ((p \to q) \vee (q \to p))$, and a truth assignment v that sets both p to T and q to F. To figure out the truth value of α under this assignment, we build a truthtable with one column for every element in a construction sequence of α as follows:

	p	q	$(p \to q)$	$(q \to p)$	$((p \to q) \vee (q \to p))$
v	T	F	F	T	T

We start with filling the first columns using the truth assignment v_1 and then successively fill the other columns using the truth table for the connectives.

Extending truth assignments to WFF – examples

We can use the same method to find out which values α can take under all possible truth assignments over the variables $P = \{p, q\}$. For this, we add new rows to the above truth table, once for every truth assignment:

	p	q	$(p \to q)$	$(q \to p)$	$((p \to q) \vee (q \to p))$
v_1	T	T	T	T	T
v_2	T	F	F	T	T
v_3	F	T	T	F	T
v_4	F	F	T	T	T

It turns out that this particular formula is true under every possible truth assignment. Such formulas are also called tautologies. See more on such notions on the next slide.

Important semantic notions

A formula $\alpha \in \mathrm{WFF}_P$ is

- satisfiable if there exists a truth assignment $v : P \to \{T, F\}$ such that $v(\alpha) = T$.
- a tautology if $v(\alpha) = T$ for all truth assignments $v : P \to \{T, F\}$
- unsatisfiable (or a contradiction) if $v(\alpha) = F$ for all truth assignments $v : P \to \{T, F\}$

Important semantic notions – examples

To show that a formula is satisfiable, there needs to be at least one truth assignment (corresponding to at least one row in the full truth table for the formula) that sets α to true. Consider $\alpha = ((p \vee q) \rightarrow (p \wedge (\neg q)))$.

	p	q	$(\neg q)$	$(p \vee q)$	$(p \wedge (\neg q))$	$((p \vee q) \rightarrow (p \wedge (\neg q)))$
v_1	T	T	F	T	F	F
v_2	T	F	T	T	T	T
v_3	F	T	F	T	F	F
v_4	F	F	T	F	F	T

The truth assignment v_2 is a certificate for α being satisfiable, since $v_2(\alpha) = T$.
(And so is v_4 in this case, but one of them suffices.)

Important semantic notions – examples

We have seen above that the formula $\alpha = ((p \to q) \vee (q \to p))$ is a tautology. Another simple example for a tautology is $\beta = (p \vee (\neg p))$.

A simple example for a contradiction is $\gamma = (p \wedge (\neg p))$. This formula will not evaluate to true under any truth assignment.