math1090
Introduction to Logic for Computer Science

Lecture 4

Ruth Urner

September 17, 2018

Announcement

Evaluation

Assignments:

- 4 Assignments
- Roughly every two weeks starting next week

Tests:

- In-class midterm (expect mid/end October)
- Final exam (December)

Mark composition:

- 1/3 Assignments
- 1/3 Midterm
- 1/3 Final exam

Tutorial/Supervised practice session

Wednesdays, noon-1pm, ACE 009

It will be announced on moodle when the tutorial takes place.

(Structural) Induction

Structural induction–general definition

Consider some inductively defined set $\mathcal{A} = I(U, C, O)$. To show that all elements of $\mathcal{A}$ satisfy property $\mathcal{P}$ we prove the following:

Base case Show that all elements $c \in C$ of the core set satisfy the property

Induction hypothesis Assume that some $a_1, a_2, \ldots a_n \in I(U, C, O)$ satisfy the property
(here n needs to be the largest arity of the operations in O)

Induction step Show that (if the induction hypothesis holds) for all operation $o_i \in O$, the property also holds for

$$o_i(a_1, a_2, \ldots a_{r_i}).$$

Structural induction–example

Game with cups

We consider three cups placed on a table as follows:

$$\cup \; \cap \; \cup$$

(That is, two upright and the middle one upside down.)

• We can now play with the cups by, at each step, flipping exactly two of them

• Eg, flipping the two left ones results in $\cap \; \cup \; \cup$

Question: Can we, by repeatedly flipping two cups, end up with all cups upright $\cup \; \cup \; \cup$?

Structural induction–example

First, we note that we can define the set of all reachable cup-configurations as an inductively defined set:

- **Universe:** U_c =All ways to place three cups on the table.
  (**Question for you:** How big is this universe?)
- **Coreset:** The initial configuration, $C_c = \{\bigcup\bigcap\bigcup\}$
- **Operations:** $O_c = \{\text{flip-left-two, flip-outer-two, flip-right-two}\}$

Question: Is $\bigcup\bigcup\bigcup \in I(U_c, C_c, O_c)$?

Structural induction–example

Conjecture: It is not possible to get all cups upright..

We will prove the following property by induction:
In all reachable states, the number of upright cups is even.

Since $\bigcup \ \bigcup \ \bigcup$ has an odd number of upright cups, this will imply that this state is not reachable.

Structural induction–example

Property: The number of upright cups is even.

Proof by in induction:

Base case In the initial configuration $\bigcup\bigcap\bigcup$, the property holds (2 cups are up, which is even).

Induction hypothesis Assume that for some configuration $XYZ \in I(U_c, C_c, O_c)$ the number of up-cups is even.

Structural induction–example

Induction step If the number of up-cups in XYZ is even, it is either 0 or 2.

Case 1: It is 0 Then flipping two cups results in 2 up-cups, which is even again.

Case 2: It is 2 Then we either flip the two up-cups in XYZ or we flip one up-cup and one down-cup. In the first case, we end up with 0 up-cups, which is even, in the second case, we maintain 2 up-cups.

Thus in all cases, the number of up-cups in flip-left-two(XYZ), flip-outer-two(XYZ), flip-right-two(XYZ) is even again.

Question for you: Where did we use the induction hypothesis?

Back to well formed formulas..

The set WFF of well formed propositional formulas is the inductively defined set $I(\Sigma^*, P, O)$, where the three components are defined as follows:

1. **Universe:** Σ^*, the set of all strings over the alphabet of propositional logic
2. **Core set:** The set P of all propositional variables
3. **Operations:** The set $O = \{o_\neg, o_\wedge, o_\vee, o_\rightarrow\}$, defined as follows:
   - $o_\neg : \varphi \mapsto (\neg\varphi)$
   - $o_\wedge : \varphi, \psi \mapsto (\varphi \wedge \psi)$
   - $o_\vee : \varphi, \psi \mapsto (\varphi \vee \psi)$
   - $o_\rightarrow : \varphi, \psi \mapsto (\varphi \rightarrow \psi)$

Properties of WFF

Now we will show the following properties by by structural induction:

1. Every well formed formula is either atomic (that is, an element of the core set) or starts with the symbol (.

2. In every well formed formula the number of left brackets (is equal to the number of right brackets).

3. In every proper initial segment of a well formed formula, the number of left brackets (is strictly larger than the number of right brackets.

Atomic formulas

We call a well formed formula α atomic if it is a member of the coreset, that is, if α consists of a single propositional variable.

Examples:

- p
- q
- p_1
- q_{16}

are all atomic formulas.

Proper initial segments of WFF

Let $\alpha = a_1 a_2 \ldots a_l$ be a well formed formula. A string $\beta = b_1 b_2 \ldots b_k$ is a **proper initial segment of α**

- if $k < l$ and
- for all $i \leq k$ we have $b_i = a_i$

Proof of property 1 of WFF

Property 1
Every well formed formula is either atomic (that is, an element of the core set) or starts with the symbol (.

Base case Let $\alpha \in \mathrm{WFF}$ be a member of the coreset, that is atomic. (E.g. $\alpha = p$). Then the property holds.

Induction hypothesis We assume α_1 and α_2 are in WFF and that property 1 holds for them. That is, both are either atomic or start with (.

Induction step If we apply $o_\neg$ to α_1, we get

$$o_\neg(\alpha_1) = (\neg\alpha_1),$$

which starts with (, thus, the property holds for $o_\neg(\alpha_1)$.
Let $*$ be a placeholder symbol for any symbol in $\{\wedge, \vee, \rightarrow\}$. Then, by applying o_* we get

$$o_*(\alpha_1, \alpha_2) = (\alpha_1 * \alpha_2),$$

which again starts with (, thus, the property holds $o_*(\alpha_1, \alpha_2)$.

Proof of property 2 of WFF

Property 2
In every well formed formula the number of left brackets (is equal to the number of right brackets).

Notation:
For a string α, we let $l(\alpha)$ denote the number of occurrences of (in α and $r(\alpha)$ the number of occurrences of).

Base case: Let $\alpha \in \mathrm{WFF}$ be atomic. Then $l(\alpha) = 0 = r(\alpha)$. Thus, property 2 holds.

Induction hypothesis: We assume that for some $\alpha_1, \alpha_2 \in \mathrm{WFF}$ we have

$$l(\alpha_1) = r(\alpha_1) \quad \text{and} \quad l(\alpha_2) = r(\alpha_2)$$

Induction step: If we apply $o_\neg$ to α_1, we get

$$l(o_\neg(\alpha_1)) = l((\neg\alpha_1)) = 1 + l(\alpha_1) = 1 + r(\alpha_1) = r((\neg\alpha_1)) = r(o_\neg(\alpha_1))$$

Thus, $\neg(\alpha_1)$ has the same number of left and right brackets (and the red equality sign indicates where the hypothesis was used).
Again, we let $* \in \{\wedge, \vee, \rightarrow\}$. Then, by applying o_* we get

$$l(o_*(\alpha_1, \alpha_2)) = l((\alpha_1 * \alpha_2)) = 1 + l(\alpha_1) + l(\alpha_2) = 1 + r(\alpha_1) + r(\alpha_2) = r((\alpha_1 * \alpha_2)) = r(o_*(\alpha_1, \alpha_2))$$

Thus the number of left and right brackets in each of $o_\wedge(\alpha_1, \alpha_2)$, $o_\vee(\alpha_1, \alpha_2)$, and $o_\rightarrow(\alpha_1, \alpha_2)$ are equal.

Proof of property 3 of WFF

Property 3
In every proper initial segment of a well formed formula, the number of left brackets (is strictly larger than the number of right brackets.

Notation:
As before, we use $l(\alpha)$ for the number of (in α and $r(\alpha)$ for the number of).

Base case Let $\alpha \in \mathrm{WFF}$ be a member of the coreset. Then α has no proper initial segments. Therefore, the property is vacuously true.

Induction hypothesis We assume α_1 and α_2 are in WFF and that the property holds for them. That is, for every proper initial segment β_1 of α_1 and every proper initial segment β_2 of α_2, we have

$$l(\beta_1) > r(\beta_1) \quad \text{and} \quad l(\beta_2) > r(\beta_2)$$

Induction step Consider $o_\neg(\alpha_1) = (\neg\alpha_1)$, and let β be a proper initial segment of $o_\neg(\alpha_1)$. We now need to distinguish several cases:

Case 1: $\beta = ($ Then $l(\beta) = 1 > 0 = r(\beta)$.

Case 2: $\beta = (\neg$ Then $l(\beta) = 1 > 0 = r(\beta)$.

Case 3: $\beta = (\neg\beta_1$ where β_1 is a proper initial segment of α_1. Then

$$l(\beta) = 1 + l(\beta_1) > 1 + r(\beta_1) > r(\beta)$$

Case 4: $\beta = (\neg\alpha_1$ Then

$$l(\beta) = 1 + l(\alpha_1) = 1 + r(\alpha_1) > r(\alpha_1) = r(\beta)$$

Here, for the second equality, we used property 2, namely that $l(\alpha_1) = r(\alpha_1)$ since $\alpha_1 \in \mathrm{WFF}$.

Proof of property 3 of WFF

Step continued Again, we let $* \in \{\wedge, \vee, \rightarrow\}$ and consider $o_*(\alpha_1, \alpha_2) = (\alpha_1 * \alpha_2)$. We let β be a proper initial segment of $o_*(\alpha_1, \alpha_2)$. We again need to distinguish several cases:

Case 1: $\beta = ($ Then $l(\beta) = 1 > 0 = r(\beta)$.

Case 2: $\beta = (\beta_1$ where β_1 is a proper initial segment of α_1. Then

$$l(\beta) = 1 + l(\beta_1) > 1 + r(\beta_1) > r(\beta)$$

Case 3: $\beta = (\neg\alpha_1$ Then

$$l(\beta) = 1 + l(\alpha_1) = 1 + r(\alpha_1) > r(\alpha_1) = r(\beta)$$

Case 4: $\beta = (\neg\alpha_1 *$ Then

$$l(\beta) = 1 + l(\alpha_1) = 1 + r(\alpha_1) > r(\alpha_1) = r(\beta)$$

Case 5: $\beta = (\neg\alpha_1 * \beta_2$ where β_2 is a proper initial segment of α_2. Then

$$l(\beta) = 1 + l(\alpha_1) + l(\beta_1) > 1 + r(\alpha_1) + r(\beta_1) > r(\alpha_1) + r(\beta_1) = r(\beta)$$

Case 6: $\beta = (\neg\alpha_1 * \alpha_2$ Then

$$l(\beta) = 1 + l(\alpha_1) + l(\alpha_2) = 1 + r(\alpha_1) + r(\alpha_2) > r(\alpha_1) + r(\alpha_2) = r(\beta)$$

Task for you: Justify each step (each equality and each inequality sign) to yourself! Where are we using the induction hypothesis? Where property 2? How do you justify the remaining steps?